10.1 Introduction

→ Whereas electrostatics has stationary charges, magnetostatics has steady magnetic fields

Magnetostatics: the study of magnetic fields in systems where electric charges move at uniform velocity (doesn't change with time)

→ We use $\vec{B}$ to represent magnetic flux density (magnetic field)

↳ units of Wb/m^2

→ Let's suppose a charge q is placed in the region where both the electric field $\vec{E}$ and the magnetic field $\vec{B}$ exist

→ if the charge is at rest... only electric force exists

$$\boxed{\vec{F}_e = q\vec{E}}$$ ELECTRIC FORCE

→ if the charge is in motion with a velocity of $\vec{u}$, it will experience an additional magnetic force $\vec{F}_m$

$$\boxed{\vec{F}_m = q\vec{u} \times \vec{B}}$$ MAGNETIC FORCE

→ if your test charge is moving, it will experience a total force

$$\boxed{\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})}$$ LORENTZ FORCE

→ Unit analysis!
→ $\vec{B}$ is measured in Teslas [T]
→ Φ_B is measured in webers [Wb]

$$1\,T = 1\,\frac{Wb}{m^2} = \frac{N \cdot s}{C \cdot m}$$

10.2 The Cross Product

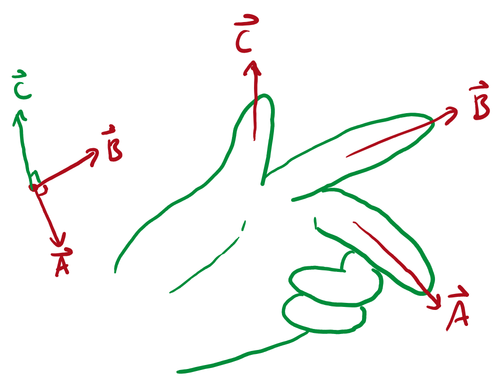
The cross product is crucial for the mathematical study of magnetic fields! If you have vectors $\vec{A}$ and $\vec{B}$ at an angle of θ to each other, then we define the cross product as:

$$\boxed{\vec{A}\times\vec{B} = \underbrace{|\vec{A}||\vec{B}|\sin\theta}_{\text{magnitude of the cross product}}}$$

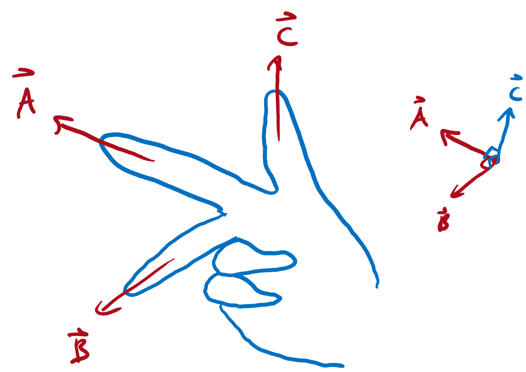
→ the direction of $\vec{A}\times\vec{B}$ is $\perp$ to the two vectors (normal to the plane)

direction can be determined easily using Fleming's right- or left-hand rules

① Fleming's Left Hand Rule



② Fleming's Right Hand Rule



Note: $\vec{C}$ is the direction of the cross product of $\vec{A}$ and $\vec{B}$

How do we know if the direction of $\vec{A}\times\vec{B} = \vec{C}$ is positive/negative?

↳ Let's follow this convention for each coordinate system

Cartesian	Cylindrical	Spherical
$\hat{x}\to\hat{y}\to\hat{z}$	$\hat{r}\to\hat{\phi}\to\hat{z}$	$\hat{r}\to\hat{\theta}\to\hat{\phi}$

↳ When we go with the arrows, the cross-product is in the +ve direction

$\hat{x}\times\hat{y} = +\hat{z}$ $\quad$ $\hat{\phi}\times\hat{z} = +\hat{r}$ $\quad$ $\hat{\phi}\times\hat{r} = +\hat{\theta}$

↳ When we go [illegible] ,

Ex: $\hat{x} \times \hat{y} = +\hat{z}$ $\qquad$ $\hat{\phi} \times \hat{z} = +\hat{r}$ $\qquad$ $\hat{\phi} \times \hat{r} = +\hat{\theta}$

↳ When we go against the arrows, the cross-product is in the -ve direction

Ex: $\hat{y} \times \hat{x} = -\hat{z}$ $\qquad$ $\hat{z} \times \hat{\phi} = -\hat{r}$ $\qquad$ $\hat{r} \times \hat{\phi} = -\hat{\theta}$

10.3 Gauss Law for Magnetism

→ Magnets will always have a north and south pole

↳ the magnetic field lines leave the north pole and into the south pole

↳ the magnetic field lines must make a closed loop

↳ this means the flux on a closed surface is zero!

$$\boxed{\oint_S \vec{B} \cdot \vec{dS} = 0}$$

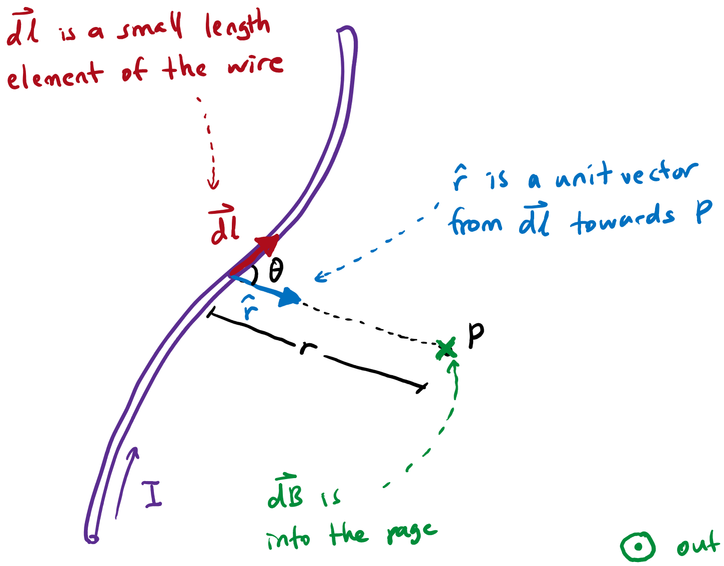
GAUSS LAW FOR MAGNETISM

(2nd Maxwell Equation)

↳ only true in the classical world

10.4 Biot-Savart Law due to Current

Biot-Savart Law: gives the contribution $d\vec{B}$ to the magnetic field at a point P due to a small element of current in the direction $d\vec{l}$



The differential magnetic field $d\vec{B}$ due to the differential current element $Id\vec{l}$ is:

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}}$$

BIOT-SAVART LAW

$\mu_0 = 4\pi \times 10^{-7}\ \frac{N}{A^2}$ or $\frac{T \cdot m}{A}$

↳ permeability of free space

⊙ out of page ⊗ into the page

To determine the TOTAL MAGNETIC FIELD $\vec{B}$:

$$\boxed{\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{l}\times\hat{r}}{r^2}}$$

TOTAL MAGNETIC FIELD FROM INTEGRATING THE BIOT-SAVART LAW

Steps to Evaluate $\vec{B}$

1. Pick an appropriate coordinate system
2. Define arbitrary point A on circuit and $d\vec{l}$ in terms of coordinate parameters
3. Evaluate $d\vec{l}\times\hat{r}$ (magnitude and direction)
4. Look for symmetry → decompose $d\vec{B}$ if it exists only in one specific direction
5. Integrate to find $\vec{B}$

10.5 Biot-Savart Law for Moving Charges

We can modify the Biot-Savart Law equation to be written in terms of charge and velocity (useful for moving charges @ a certain speed)

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^2} \quad \text{but} \quad I = \frac{dq}{dt} \quad \text{(current is charge per unit time)}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{dq}{dt} \frac{d\vec{l} \times \hat{r}}{r^2} \quad \text{but} \quad \vec{v} = \frac{d\vec{l}}{dt}$$

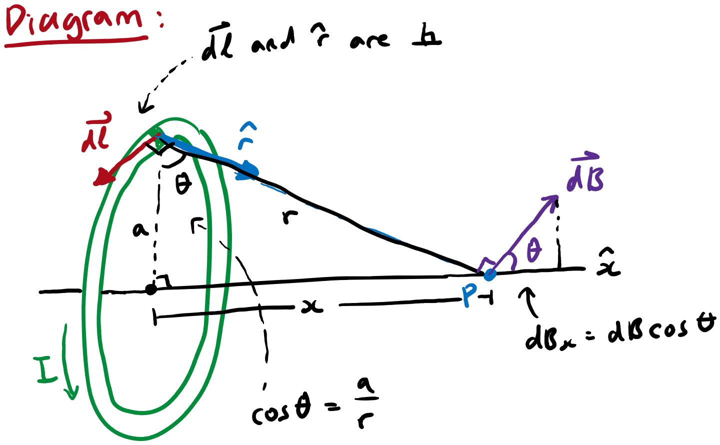
$$\boxed{d\vec{B} = \frac{\mu_0\, dq\, \vec{v} \times \hat{r}}{4\pi r^2}}$$

BIOT-SAVART LAW FOR MOVING CHARGES

10.6a Example #1

Find the magnetic field at an arbitrary point P on the axis of a circular loop of radius a carrying a current I

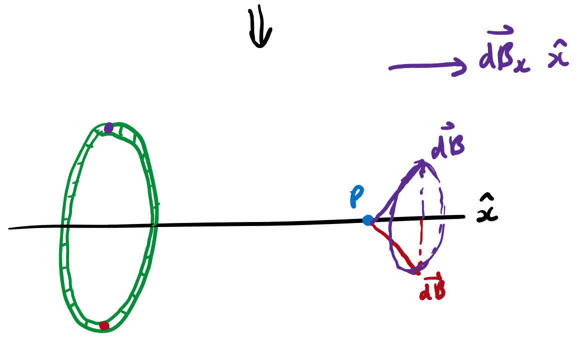
Diagram:



→ The Biot-Savart Law determines $\vec{dB}$ at P, so we must identify vectors $\underline{\vec{dl}}$ and $\underline{\hat{r}}$

→ When we go around the ring, the $\vec{dB}$ components that are perpendicular to the axis will cancel, giving a net magnetic field along the x-axis

→ We just need to find $\vec{dB}_x$ and integrate!



Biot-Savart Law:

$$\vec{dB} = \frac{\mu_0}{4\pi}\frac{I\,\vec{dl}\times\hat{r}}{r^2}$$

$$\vec{dB} = \frac{\mu_0 I\,dl}{4\pi r^2}$$

Aside: $dB_x = dB\cos\theta$; $\cos\theta = \frac{a}{r}$; $r = \sqrt{a^2+x^2}$

→ $\vec{dl}\times\hat{r}$ → we know $\vec{dl} \perp \hat{r}$ so $\phi = 90°$ and $|\hat{r}| = 1$

$= dl(r)\sin 90°$

$= dl$

⇓

$$\vec{dB}_x = \frac{\mu_0 I\,dl}{4\pi r^2}\cos\theta = \frac{\mu_0 I\,dl\,a}{4\pi r^3} = \frac{\mu_0 I\,dl\,a}{4\pi(a^2+x^2)^{3/2}}$$

⇓

$\int dl = \underline{\underline{2\pi a}}$

$$\vec{B} = \int d\vec{B}_x\,\hat{x} = \frac{\mu_0 I a}{4\pi}\int\frac{dl}{(x^2+a^2)^{3/2}} = \frac{\mu_0 I a}{4\pi(x^2+a^2)^{3/2}}\int dl$$

$$\vec{B} = \frac{\mu_0 I a(\cancel{2}\cancel{\pi}a)}{{}_{2}\cancel{4}\cancel{\pi}(x^2+a^2)^{3/2}}\hat{x}$$

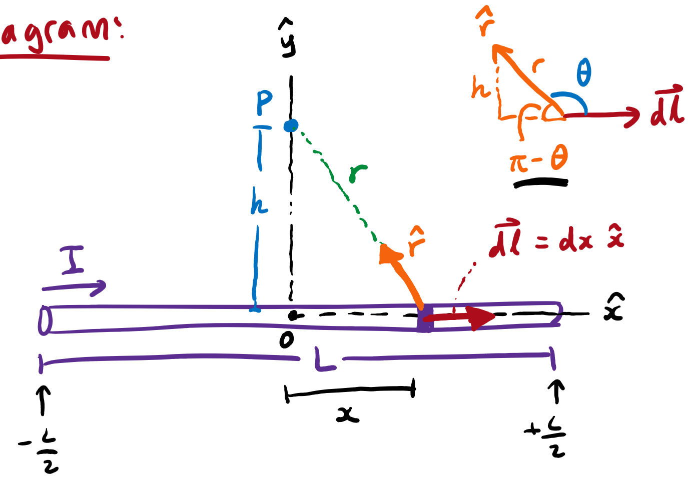
$$\vec{B} = \frac{\mu_0 I a^2}{\ \ }\hat{x}$$

$$\vec{B} = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}}\,\hat{x}$$

10.6b Example #2

A thin, straight wire is carrying current I. Calculate the magnetic field at point P on an axis going halfway through the wire. Assume the leads to the ends of the wire cancel the field at point P (eg. You twist the wires)

Diagram:



Aside: $|d\vec{l} \times \hat{r}| = |d\vec{l}||\hat{r}| \sin\theta$

$= dx\,(1)\sin\theta \leftarrow \sin\theta = \sin(\pi-\theta)$

$= dx \sin(\pi-\theta)$ but $\sin(\pi-\theta) = h/r$

$= dx\left(\frac{h}{r}\right)$ $(+\hat{z})$ by inspection

$\Rightarrow r = \sqrt{x^2+h^2}$

$$\Rightarrow \int \frac{dx}{(x^2+h^2)^{3/2}} = \frac{x}{h^2\sqrt{h^2+x^2}}$$

Biot-Savart Law:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,dx\,h}{r^3}\,\hat{z}$$

$$d\vec{B} = \frac{\mu_0 I h\,dx}{4\pi(x^2+h^2)^{3/2}}\,\hat{z}$$

$$\vec{B} = \int d\vec{B}\ \hat{z}$$

$$\vec{B} = \frac{\mu_0 I h}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx}{(x^2+h^2)^{3/2}}\,\hat{z}$$

$$\vec{B} = \frac{\mu_0 I h}{4\pi}\left[\frac{x}{h^2\sqrt{h^2+x^2}}\right]_{-L/2}^{L/2}\hat{z}$$

$$\vec{B} = \frac{\mu_0 I h\,(2L)}{4\pi h^2\sqrt{4h^2+L^2}}\,\hat{z}$$

$$\vec{B} = \frac{\mu_0 I h L}{2\pi h^2\sqrt{4h^2+L^2}}\,\hat{z}$$