This tutorial will cover a detailed introduction to the rotational world. Similar to the linear world, the rotational world is a part of everyday life and we need to develop different ways of expressing and quantifying motion in the rotational world. We'll start off by introducing angular or rotational terminology, which will allow us to talk about rotational kinematics. Once we've familiarized ourselves with rotational kinematics, we can move on to rotational kinetic energy and moment of inertia, which will be used in the an updated law for the conservation of energy that we can apply to questions related to rolling motion.

ANGULAR POSITION, VELOCITY AND ACCELERATION

When objects were moving in a straight line or at an angle, it was useful to establish a coordinate system. You would usually have an origin point, and an x- or y-direction where the object could relate to so that we could measure the object's position, velocity and acceleration.

But are linear terms really useful in describing objects that are rotating? We probably want something more convenient. Therefore, we need to define **angular quantities** that are analogous to an object's linear position, velocity and acceleration. As a general summary, we used to use x for position, v for velocity, and a for acceleration. Now, we will use θ for angular position, ω for angular velocity, and α for angular acceleration. Let's start by drawing a circle and defining these various angular quantities.

Angular Position θ is the angle that the object is at from a reference line (where θ = 0° or 0 radians). By convention, **counterclockwise ($\theta < 0$) is positive** while **clockwise ($\theta > 0$) is negative**

- Units are in radians, which are dimensionless
  - A **radian** is the angle for which the arc length on a circle of radius r is equal to the radius of the circle
- What is the arc length?
  - The **arc length** is given by $s = r\theta$
  - If we do a full 360° revolution (1 rev) on the circle, the radius of the circle is r and the arc length is the circumference of the circle, or $2\pi r$
  - This means 1 rev = 360° = 2π radians
- Commonly, we want to know the **angular displacement $\Delta\theta$** (or a change in position) so that we can figure out angular velocity

$$\Delta\theta = \theta_f - \theta_i$$

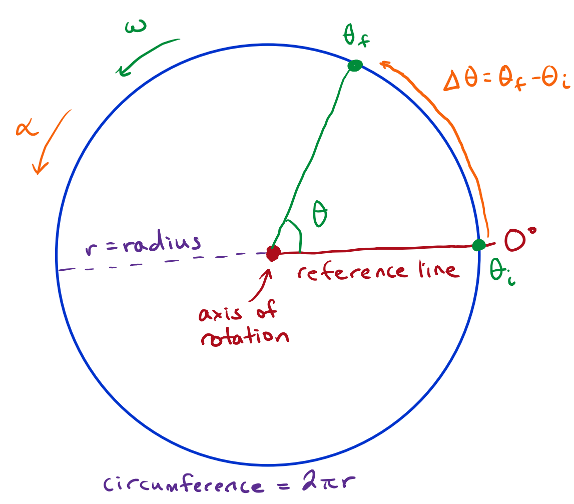

Figure 1. Angular Position, Velocity and Acceleration on an Axis of Rotation

Angular Velocity ω is defined as angular displacement over time, just like its linear counterpart. So now, we have an expression for the average angular velocity, in units of radians per second or s^{-1}:

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$$

- If the angular velocity is constant, then $\omega = \omega_{avg} = \Delta\theta/\Delta t$, which for one revolution means that $\Delta\theta = 2\pi$ and $\Delta t = T$, where T is the period. The **period for constant angular velocity** is then defined as:

$$T = \frac{2\pi}{\omega}$$

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Angular Acceleration α is defined as the change in angular velocity over time, also like its linear counterpart. We can then create an expression for the average angular acceleration, in units of radians per second squared, or s^{-2}:

$$\alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t}$$

ROTATIONAL KINEMATICS

In the linear world of kinematics, we know that the acceleration due to gravity is constant. From that, we get a set of kinematics equations that we can use. What about the rotational world of kinematics? Well, if we say that the angular acceleration is constant, couldn't we use the same set of kinematics equations with our angular quantities? Let's write these out, but first let's jot down what linear quantities relate to which rotational quantities.

Quantity	Linear World	Rotational World
Displacement	Δx	θ
Velocity	v	ω
Acceleration	a	α
Kinematics Equation 1	$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$
Kinematics Equation 2	$\Delta x = v_i t + \frac{1}{2}at^2$	$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$
Kinematics Equation 3	$\Delta x = v_f t - \frac{1}{2}at^2$	$\Delta\theta = \omega_f t - \frac{1}{2}\alpha t^2$
Kinematics Equation 4	$\Delta x = \frac{1}{2}(v_i + v_f)t$	$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t$
Kinematics Equation 5	$v_f^2 = v_i^2 + 2a\Delta x$	$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$

When we solve kinematic problems involving rotation, we simply apply these newly-defined angular kinematics equations in the same way we did with the linear ones. Let's go through a couple of problems.

PRACTICE PROBLEM #1

You are playing fetch with your dog. You throw a ball with an initial angular speed of 36.0 rad/s and your dog catches the ball 0.595 s later. When your dog catches the ball, its angular speed has decreased to 34.2 rad/s due to air resistance.

(a) What is the ball's angular acceleration, assuming it to be constant?
(b) How many revolutions does the ball make before being caught?

Make a Plan: This is our first venture into rotational kinematics. But, the approach is almost identical to linear kinematics. We just need three parameters in order to solve for a fourth using one of the kinematics equations. So, we have an initial angular speed (ω_i), a final angular speed (ω_f), and time (t). This means we can solve for α using a kinematics equation for part (a), and then determine the angular displacement ($\Delta\theta$) to answer part (b).

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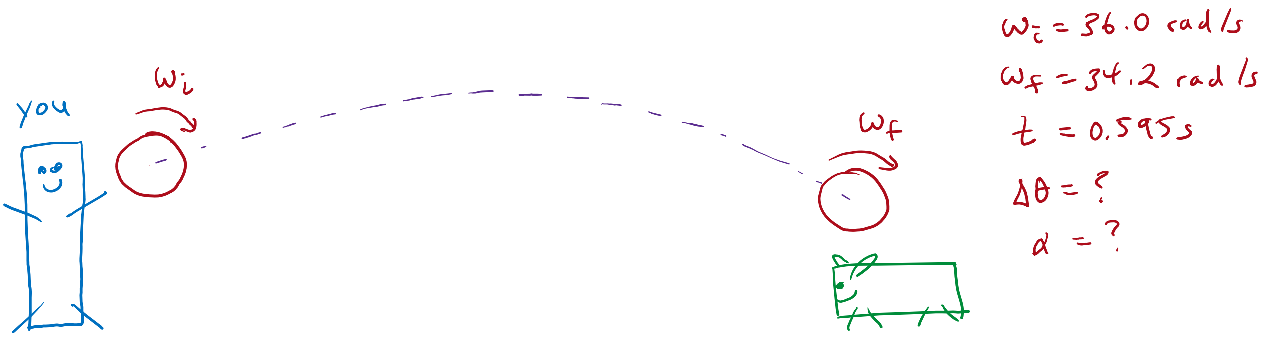
Step #1: Draw Diagram & State Known Values

Although a diagram isn't necessary for this question, we should probably get used to labelling some items in the rotational world. So let's draw a diagram for this scenario and state known values.



Step #2: Use Kinematics Equations to Solve

Part (a) is asking us to solve for the angular acceleration of this scenario. We are given ω_i, ω_f, t and asked to find α so we use this kinematics equation:

$$\omega_f = \omega_i + \alpha t$$

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{34.2 - 36.0}{0.595}$$

$$\alpha = -3.025\ \mathrm{s}^{-2}$$

Therefore the angular acceleration is -3.025 s^{-2}

Part (b) is asking us to solve for the number of revolutions that the ball makes before it is caught. We'll have to find the angular displacement $\Delta\theta$ and then convert that into revolutions. Let's use another kinematics equation to find $\Delta\theta$:

$$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)t = \frac{1}{2}(36.0 + 34.2)0.595$$

$$\Delta\theta = 20.8845\ \mathrm{rad}$$

We know that 1 rev = 360° = 2π rad so:

$$\Delta\theta = 20.8845\ \mathrm{rad}\left(\frac{1\ \mathrm{rev}}{2\pi\ \mathrm{rad}}\right)$$

$$\Delta\theta = 3.324\ \mathrm{rev}$$

Therefore the ball revolves 3.324 times before it is caught.

PRACTICE PROBLEM #2

A pulley rotating in the counterclockwise direction is attached to a mass suspended from a string. The mass causes the pulley's angular velocity to decrease with a constant angular acceleration α = -2.10 rad/s^2.

(a) If the pulley's initial angular velocity is 5.40 rad/s, how long does it take for the pulley to come to rest?

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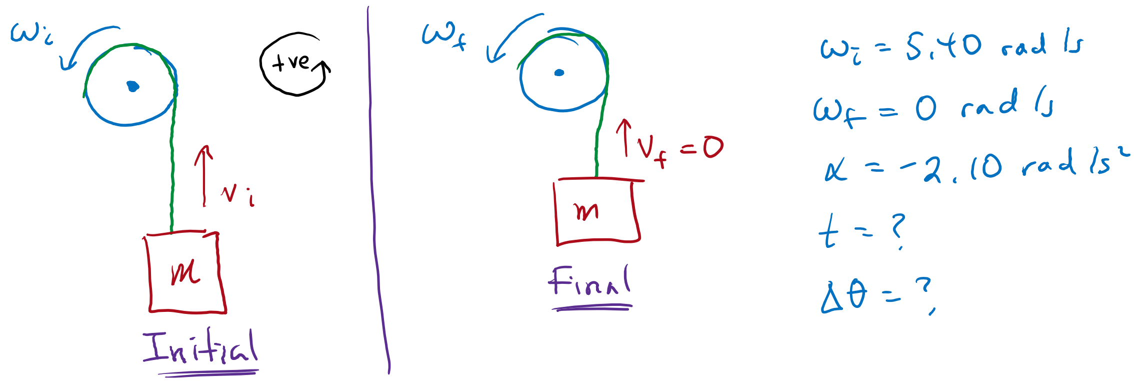
(b) Through what angle does the pulley turn during this time?

Make a Plan: Once again, a kinematics problem. We have angular acceleration, initial velocity, and final velocity (since the pulley comes to rest). This means we can use a kinematics equation to solve for time in part (a) and then displacement in part (b).

Step #1: Draw Diagram & State Known Values

We should probably draw a diagram to get used to rotational motion. Let's try that, and state our given values:



Step #2: Apply Kinematics Equations and Solve

In **Part (a)**, we're asked to solve for the time it takes for the pulley to come to rest. We have three parameters, we can pick the right kinematics equation to solve for t:

$$\omega_f = \omega_i + \alpha t$$

$$t = \frac{\omega_f - \omega_i}{\alpha} = \frac{0 - 5.40}{-2.10}$$

$$t = 2.571 \text{ s}$$

Therefore it takes 2.571 seconds for the pulley to come to rest.

In **Part (b)**, we're asked to solve for the angle the pulley turns during this time. So we want $\Delta\theta$:

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{\omega_f^2 - \omega_i^2}{2\alpha} = \frac{0^2 - 5.40^2}{2(-2.10)}$$

$$\Delta\theta = 6.943 \text{ rad}$$

Therefore, the pulley turns 6.943 radians to come to rest.

RELATING LINEAR AND ROTATIONAL QUANTITIES

With all this talk about linear and rotational quantities, we've kept the analyses separate. But what if there's a question that involves both? We'll have to figure out a way to connect the two worlds together. Let's try and see if there's a way to do this, and it all starts by looking at a rotating object that follows a circular path.

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When you take a snapshot of that circular, rotating motion, that object is moving in a direction tangential to the circle, with a linear speed and acceleration. So let's first take a look at **speed**. We know that the angular velocity of this object spinning around in a circle is ω. From our equation for the period, T, we know that:

$$T = \frac{2\pi}{\omega} \rightarrow \omega = \frac{2\pi}{T}$$

How can we relate this to tangential speed (v_t)? Well, similar to our experience with circular motion and centripetal acceleration, the tangential speed is going to be calculated by how much distance the object travels in a given about of time. Since the time to do one revolution around the circular motion is the period, T, that means the distance travelled will be the circumference of the circle, or $2\pi r$. This means we have:

$$v_t = \frac{2\pi r}{T} = r\left(\frac{2\pi}{T}\right) = r\omega$$

We now have a direct relationship between the linear and rotational worlds with the following equation:

$$\boldsymbol{v_t = r\omega}$$

Now, let's take a look at **acceleration**. We know that angular acceleration in the rotational world is given by:

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

If we look at tangential acceleration, we have:

$$a_t = \frac{\Delta v_t}{\Delta t} = \frac{\Delta r\omega}{\Delta t} = r\left(\frac{\Delta\omega}{\Delta t}\right) = r\alpha$$

This gives us a direct relationship between linear and rotational acceleration:

$$\boldsymbol{a_t = r\alpha}$$

But we also know that in the linear world, there is going to be centripetal acceleration, defined by:

$$a_{\text{cp}} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

So we have a third relationship that we can leverage when required:

$$\boldsymbol{a_{\text{cp}} = r\omega^2}$$

Now that we can relate the linear and rotational worlds, it's time to revisit and update some topics from the past, specifically with respect to rolling motion and rotational kinetic energy. Previously, we've only looked at objects moving from one point to another without any rotational motion. What if we combined both? We'll need additional tools to help analyze such scenarios.

ROTATIONAL KINETIC ENERGY AND MOMENT OF INERTIA

When we look at **rolling motion**, it is a combination of both **translational motion** and **rotational motion**. So, think of an object of radius r that rolls or rotates with an angular speed of ω, and translates or moves with a linear speed of v. That means we need to consider the kinetic energy in both the linear and rotational worlds and add them up to get the total kinetic energy in an object that rolls and moves at the same time.

We already know that **translational kinetic energy** can be calculated using our linear world equation:

$$\boldsymbol{KE_t = \frac{1}{2}mv^2}$$

But what about **rotational kinetic energy**? We need to come up with another equation to help define this. Since we already know that $v = r\omega$, why don't we just plug that in to KE_t to get KE_r?

$$KE_r = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$$

$$\boldsymbol{KE_r = \frac{1}{2}I\omega^2}$$

We now have an equation for rotational kinetic energy, also in units of joules. But, it may be more convenient to express (mr^2) as something else called the **moment of inertia I**, which is considered the rotational equivalent of mass. In other words, just like how mass determines how much force is required for a certain amount of acceleration… the moment of inertia determines how much rotational force (torque) is required for a certain amount of angular acceleration in the rotational world. The moment of inertia expresses an object's tendency to resist angular acceleration. So we define the moment of inertia as:

$$I = mr^2$$

But is this equation always true? No! Because the moment of inertia depends on the particular shape, or **mass distribution**, of an object. When we calculate the rotational kinetic energy (KE_r), we're trying to calculate the total rotational kinetic energy of the rotating object (a full 360° or 2π radian spin). That equation we've presented up there is just the moment of inertia at a particular point in the shape.

So how do we calculate the total moment of inertia I? We would need calculus. But long story short, let's look at a hoop of total mass M and radius R as an example. We would break up the entire object into infinitely small elements, and consider their individual contributions (or individual moments of inertia) to the total moment of inertia. We then have a new equation for the moment of inertia, where m_i is the mass of each individual element, and r_i is the distance the individual element is away from the axis of rotation:

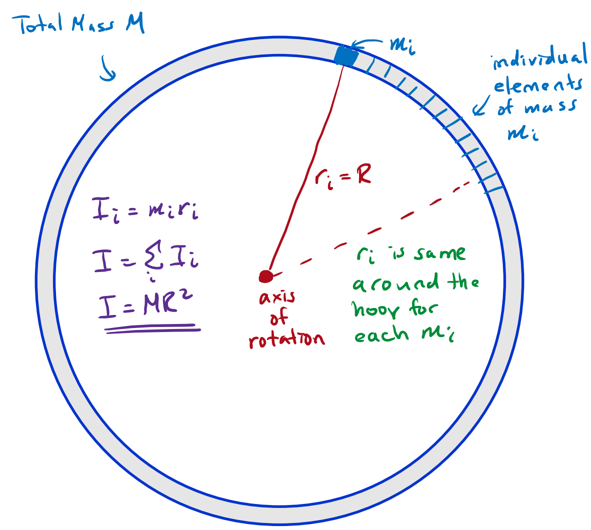

Figure 2. Moment of Inertia of a Hoop

$$I = \sum_i m_i r_i^2$$

For the hoop example here, we know that the total mass of the hoop is M so all the little m_i's should add up to M. We also know that the distance away from the axis of rotation (which is the center of the hoop) is R, so r_i is the same for each small element. So for a hoop, the moment of inertia is:

$$I_{\text{hoop}} = MR^2$$

But what about a solid disk? If we have the same total mass M and form a uniform disk of the same radius R, the moment of inertia, as expected, would be different. Why? Because $r_i \neq R$ for each individual element m_i. The elements at the outermost ring of the solid disk will have $r_i = R$, but the elements within the solid disk will have $r_i < R$, down to the central element which

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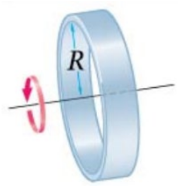
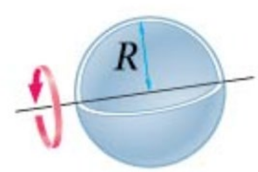
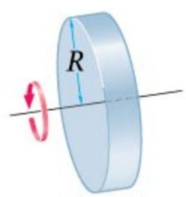
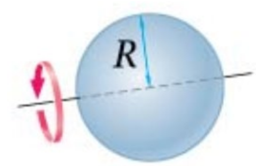
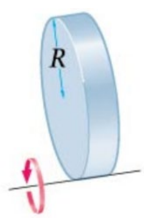
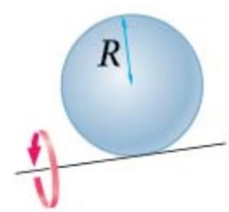
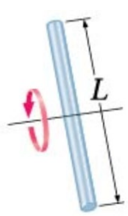
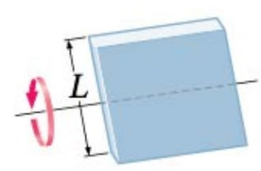
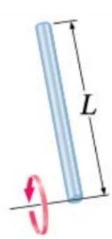
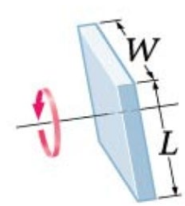
has $r_i = 0$. That means that the contribution from the small elements to the overall moment of inertia will get smaller and smaller as your element gets closer to the center of the disk, since $r_i \to 0$. It then makes sense that the overall moment of inertia for a solid disk will be less than the overall moment of inertia for a hoop. After a detailed calculation (with calculus) summing over all the mass elements, we get the following result:

$$I_{\text{disk}} = \frac{1}{2}MR^2$$

Here's a table of different moments of inertia for uniform, rigid objects of various shapes of mass M:

Shape	**Moment of Inertia**	**Shape**	**Moment of Inertia**
Hoop or Cylindrical Shell Axis: Center	$I = MR^2$ 	**Hollow Sphere** Axis: Center	$I = \frac{2}{3}MR^2$ 
Disk or Solid Cylinder Axis: Center	$I = \frac{1}{2}MR^2$ 	**Solid Sphere** Axis: Center	$I = \frac{2}{5}MR^2$ 
Disk or Solid Cylinder Axis: Edge	$I = \frac{3}{2}MR^2$ 	**Solid Sphere** Axis: Edge	$I = \frac{7}{5}MR^2$ 
Long Thin Rod Axis: Center	$I = \frac{1}{12}ML^2$ 	**Solid Plate** Axis: Center in Plane of Plate	$I = \frac{1}{12}ML^2$ 
Long Thin Rod Axis: End	$I = \frac{1}{3}ML^2$ 	**Solid Plate** Axis: Center Perpendicular to Plane of Plate	$I = \frac{1}{12}M(L^2 + W^2)$ 

Now, let's revisit the **Law of Conservation of Energy of Rolling Motion**. With objects that roll, we already came to the conclusion that we need to consider translational kinetic energy and rotational kinetic energy. Therefore, we define the **kinetic energy of rolling motion** as:

$$KE = KE_t + KE_r$$

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

If we do not have angular velocity handy to plug into this equation, we can redefine ω using $v = r\omega$ to get:

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2 = \frac{1}{2}mv^2 + \frac{\frac{1}{2}Iv^2}{r^2}$$

$$KE = \frac{1}{2}v^2\left(m + \frac{I}{r}\right)$$

Apart from this, we should also update our equation for **mechanical energy** to encompass everything we have covered so far in this course:

$$ME = KE + PE$$

$$ME = KE_t + KE_r + PE_g + PE_s$$

Now, let's go through a few practice problems related to moment of inertia and rotational kinetic energy.

PRACTICE PROBLEM #3

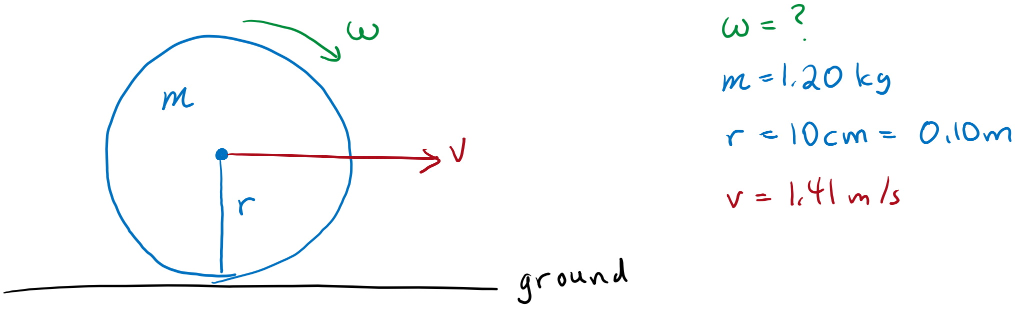
A 1.20 kg disk with a radius of 10 cm rolls without slipping. If the linear speed of the disk is 1.41 m/s, find:

(a) The translational kinetic energy of the disk
(b) The rotational kinetic energy of the disk
(c) The total kinetic energy of the disk

Make a Plan: We know that this is an energy problem, and we know that there is a disk that rolls and moves. We already have the equations for rotational and translation kinetic energy, so let's use those.

Step #1: Draw Diagram & State Known Values

Let's first draw a diagram of this system and state our known values.



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Step #2: Apply Energy Equations to Solve

To find the answer to **Part (a)**, we simply use the equation for translational kinetic energy:

$$KE_t = \frac{1}{2}mv^2 = \frac{1}{2}(1.20)(1.41)^2$$

$$KE_t = 1.193 \text{ J}$$

To find the answer to **Part (b)**, we just use the equation for rotational kinetic energy. But, that means we need moment of inertia, I, as well as angular velocity ω. Since this is a solid disk, we know that $I = \frac{1}{2}mr^2$ and we can relate $v = r\omega$ so we have:

$$KE_r = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{1}{4}mv^2 = \frac{1}{4}(1.20)(1.41)^2$$

$$KE_r = 0.596 \text{ J}$$

Now, to answer **Part (c)**, we just sum up the two types of kinetic energies:

$$KE = KE_t + KE_r = 1.193 + 0.596$$

$$KE = 1.789 \text{ J}$$

Therefore the total kinetic energy of the disk is 1.789 J.

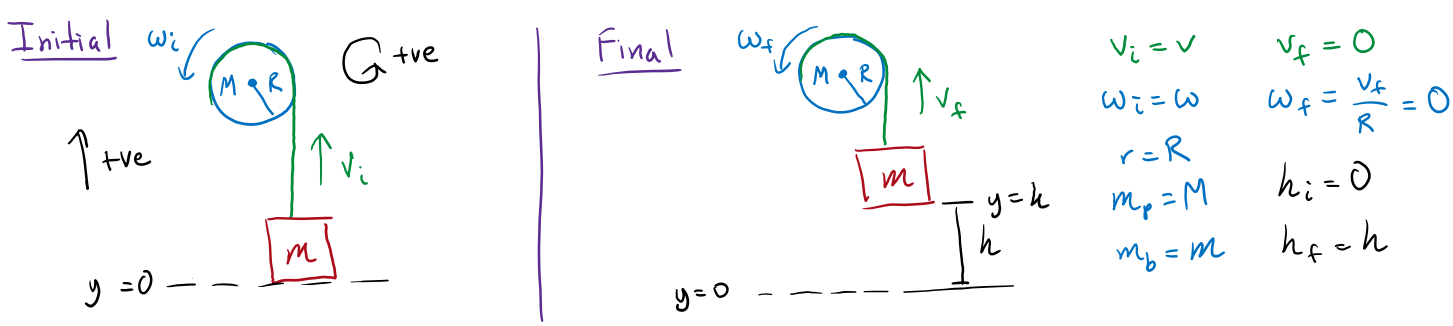
PRACTICE PROBLEM #4

A block of mass m is attached to a string that is wrapped around a wheel of radius R and mass M. The wheel rotates freely about its axis and the string wraps around its circumference without slipping. Initially, the wheel rotates with an angular speed ω, causing the block to rise with a linear speed of v. To what height does the block rise before coming to rest?

Make a Plan: This is a pulley question. But now, the pulley is no longer massless and we have to consider it as an object. The question gives us several key pieces of information. First, the pulley is a solid disk, so its moment of inertia is $I = \frac{1}{2}MR^2$. Second, the string wraps onto the disk without slipping, so we can relate linear to angular velocity: $v = R\omega$. Finally, the question states that the wheel rotates freely, so it means that the mechanical energy of the system is conserved. This means we can assume that when the block comes to rest, all of its kinetic energy should be converted to gravitational potential energy.

Step #1: Draw Diagram & State Known Values

Let's draw a diagram and label it properly, while stating known values and equations.



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Step #2: Apply Energy Equations to Solve

We can approach this like any other conservation of energy problem. Let's look at the initial and final energies, knowing that they are going to be conserved.

$$ME_i = ME_f$$

$$KE_{ti} + KE_{ri} + PE_{gi} = KE_{tf} + KE_{rf} + PE_{gf}$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 + mgh_i = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 + mgh_f$$

$$\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 + 0 = 0 + 0 + mgh$$

$$\frac{1}{2}mv^2 + \frac{1}{4}MR^2\omega^2 = mgh$$

$$h = \frac{1}{mg}\left[\frac{1}{2}mv^2 + \frac{1}{4}MR^2\left(\frac{v}{R}\right)^2\right]$$

$$h = \frac{v^2}{2g}\left(1 + \frac{M}{2m}\right)$$

PRACTICE PROBLEM #5

The moment of inertia of a 0.98 kg wheel rotating about its center is 0.13 kg·m^2. What is the radius of the wheel, assuming the weight of the spokes can be ignored?

Make a Plan: This is a moment of inertia problem. We are given the mass of the wheel, which is assumed to be a cylindrical shell or a ring. We are also given the value of I. So all we have to do is choose the correct equation for moment of inertia based on the shape, and solve for the radius.

Step #1: State Known Values and Solve

This question really doesn't require a diagram. We know that we're look at a cylindrical shell or hoop, so we know that:

$$I = MR^2$$

$$R = \sqrt{\frac{I}{M}} = \sqrt{\frac{0.13}{0.98}}$$

$$R = 0.36 \text{ m}$$

Therefore, the radius of the wheel is 0.36 m.

PRACTICE PROBLEM #6

Two masses (m_1 = 5.0 kg and m_2 = 3.0 kg) hang on each side of a pulley of mass M. This is called an Atwood's machine. The two masses are released from rest, with m_1 at a height of 0.75 m above the floor. When m_1 hits the ground, its speed is 1.8 m/s. Assuming that the pulley is a uniform disk with a radius of 12 cm, determine the pulley's mass.

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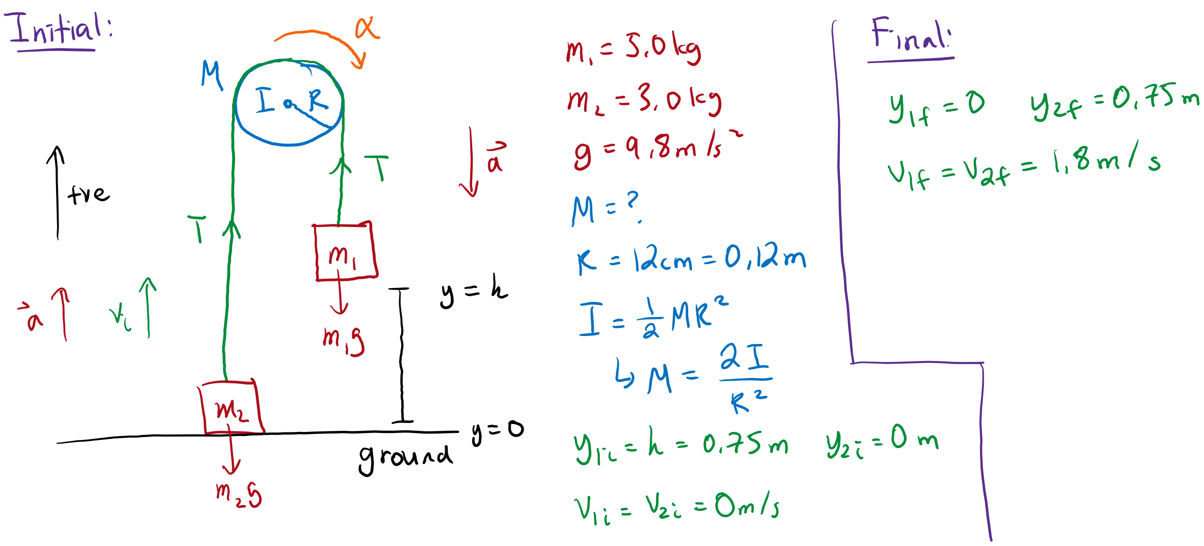
Make a Plan: We have quite a few things going on in this problem. We have up/down motion with the two masses, and rotational motion with the pulley. We should start with an initial vs. final diagram, labelling as much as we can. Then, we should probably figure out which energies to look at.

Step #1: Draw Diagram & State Known Values

Let's first draw a diagram of this system of objects and label what we know. We'll just draw the initial stage in its entirety, because there's a lot to draw and a lot of parameters to label.



Step #2: Use Energy Equations to Solve

We know that this is a conservation of energy equation. There are initial and final speeds, as well as heights. We also know that energy is conserved in this system so we have:

$$ME_i = ME_f$$

$$KE_{1i} + KE_{2i} + KE_{pi} + PE_{1i} + PE_{2i} = KE_{1f} + KE_{2f} + KE_{pf} + PE_{1f} + PE_{2f}$$

$$0 + 0 + 0 + m_1 g h_i + 0 = \frac{1}{2} m_1 v_f^2 + \frac{1}{2} m_2 v_f^2 + \frac{1}{2} I\omega^2 + 0 + m_2 g h_f$$

$$m_1 g h_{1i} = \frac{1}{2} v_f^2 (m_1 + m_2) + \frac{1}{2}\left(\frac{1}{2} M R^2\right)\left(\frac{v_f}{R}\right)^2 + m_2 g h_{2f}$$

$$\frac{1}{4} M v_f^2 = m_1 g h_{1i} - m_2 g h_{2f} - \frac{1}{2} v_f^2 (m_1 + m_2)$$

$$M = \frac{4}{v_f^2}\left(g\left(m_1 h_{1i} - m_2 h_{2f}\right) - \frac{1}{2} v_f^2 (m_1 + m_2)\right)$$

$$M = \frac{4g\left(m_1 h_{1i} - m_2 h_{2f}\right)}{v_f^2} - 2(m_1 + m_2)$$

$$M = \frac{4(9.8)\left(5(0.75) - 3(0.75)\right)}{(1.8)^2} - 2(5 + 3)$$

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$M = 2.148 \text{ kg}$

Therefore, the mass of the pulley is 2.148 kg.

PRACTICE PROBLEM #7

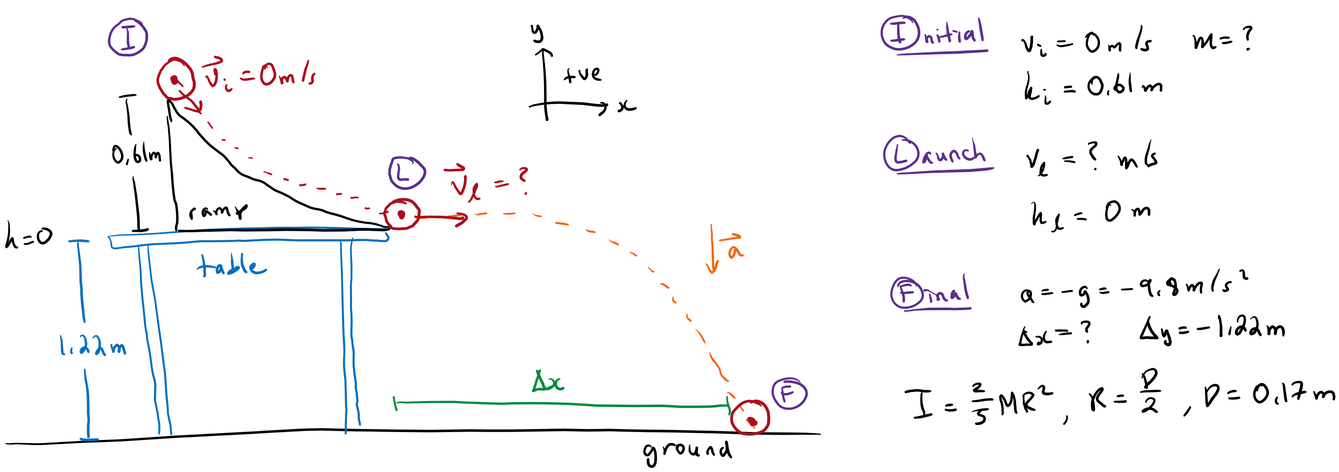
A solid sphere with a diameter of 0.17 m is released from rest. It then rolls (without slipping) down a ramp that is 0.61 m tall. The ball leaves the bottom of the ramp, which is located at the edge of a table 1.22 m above the ground. The ball leaves the bottom of the ramp horizontally.

(a) Through what horizontal distance does the ball move before landing?
(b) How many revolutions does the ball make during its fall?
(c) If the ramp were to be made frictionless, would the horizontal distance travelled increase, decrease, or stay the same?

Make a Plan: This is a multi-step problem. A sphere rolls without slipping down an incline and gains speed, then it's launched horizontally off the edge of the ramp. From this point onward, we're looking at a projectile motion problem. For the ramp portion, we can use conservation of energy to determine the horizontal launch speed of the ball at the bottom of the ramp. Then we can use linear kinematics to solve for the horizontal distance travelled.

Step #1: Draw Diagram & State Known Values

Let's draw a diagram of the entire scenario, and label it with everything we're given.



Step #2: Determine Horizontal Launch Speed Using Conservation of Energy

Looking at the Initial and Launch phases, we see that there is translational and rotational kinetic energy involved, plus gravitational potential energy. So we set up our energy equation:

$$ME_i = ME_f$$

$$KE_{ti} + KE_{ri} + PE_i = KE_{tl} + KE_{rl} + PE_l$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 + mgh_i = \frac{1}{2}mv_l^2 + \frac{1}{2}I\omega_l^2 + mgh_l$$

$$0 + 0 + mgh_i = \frac{1}{2}mv_l^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\left(\frac{v_l}{r}\right)^2 + 0$$

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$$mgh_i = \frac{1}{2}mv_l^2 + \frac{1}{5}mv_l^2$$

$$mgh_i = \frac{7}{10}mv_l^2$$

$$gh_i = \frac{7v_l^2}{10}$$

$$v_l = \sqrt{\frac{10gh_i}{7}}$$

$$v_l = \frac{\sqrt{(10)(9.8)(0.61)}}{7}$$

$$v_l = 2.922 \text{ m/s}$$

We now have the horizontal launch speed, and we can determine how far the ball travels before it reaches the ground.

Step #3: Determine Distance Travelled Using Linear Kinematics

To answer **Part (a)**, we need linear kinematics. So we need to split our analysis into the x- and y-directions.

x-direction

$$v_{lx} = \frac{\Delta x}{\Delta t}; \quad v_{lx} = v_l$$

$\Delta x = v_{lx}\Delta t$ (need to find Δt)

y-direction

$$\Delta y = -1.22; \quad a = -9.8; \quad v_{ly} = 0; \quad \Delta t = ?$$

$$\Delta y = v_{ly}t + \frac{1}{2}at^2$$

$$t = \sqrt{\frac{2\Delta y}{a}} = \sqrt{\frac{2(-1.22)}{9.8}}$$

$$t = 0.499 \text{ s}$$

Now that we've found time, we can solve for Δx:

$$\Delta x = v_l t = 2.922(0.499)$$

$$\Delta x = 1.458 \text{ m}$$

Therefore, the horizontal distance travelled is 1.458 m.

Step #4: Determine Number of Revolutions the Sphere Spins

In order to find the number of revolutions for **Part (b)**, we need angular speed. We can use $v_l = r\omega$ for this:

$$\omega = \frac{v_l}{r} = \frac{2.922}{\frac{0.17}{2}} = 34.376\frac{\text{rad}}{\text{s}}$$

And then we can find the angular displacement $\Delta\theta$:

$$\Delta\theta = \omega t = (34.376)(0.499) = 17.153 \text{ rad}$$

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$$\Delta\theta = 17.153 \text{ rad} \left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)$$

$$\Delta\theta = 2.730 \text{ rev}$$

Therefore 2.73 revolutions occur before the ball hits the ground.

Step #5: Thinking About a Frictionless Surface

If the ramp were to be made frictionless, **Part (c)** asks us whether the horizontal distance travelled would increase, decrease, or stay the same. With a frictionless ramp, the sphere would slide instead of rolling. It would therefore store no energy in its rotation, and all of its gravitational potential energy would become translational kinetic energy. This would then make the ball launch from the table edge with a higher speed (since rotational kinetic energy is zero) and therefore the landing distance would **increase**!

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