A Range Arithmetic based Optimization Model for Power Flow Analysis under Interval Uncertainty

Alfredo Vaccaro, *Senior Member, IEEE*, Claudio A. Cañizares, *Fellow, IEEE,* and K. Bhattacharya, *Senior Member, IEEE*

***Abstract*—This paper presents a novel framework based on Range Arithmetic for solving power flow problems whose input data are specified within real compact intervals. Reliable interval bounds are computed for the power flow problem, which is represented as an optimization model with complementary constraints to properly represent generator bus voltage controls, including reactive power limits and voltage recovery processes. It is demonstrated that the lower and upper bounds of the power flow solutions can be obtained by solving two determinate optimization problems. Several numerical results are presented and discussed, demonstrating the effectiveness of the proposed methodology and comparing it to a previously proposed Affine Arithmetic based solution approach.**

***Index Terms*—Power flow analysis, uncertainty studies, affine arithmetic, interval mathematics.**

I. INTRODUCTION

Power flow studies are typically used to determine the steady state operating conditions of power systems for a specified set of load and generation, and is one of the most intensely used tools in power engineering. The most common formulation of the power flow problem has all input data specified from a snapshot at a point in time associated with specific system conditions, or from a proper set of "crisp" values that the analyst defines under various assumptions for the system. The power flow solution is then deemed representative of a limited set of system conditions. However, when the input conditions are uncertain, numerous scenarios need to be analyzed to cover the required range of uncertainty.

Uncertainties in power flow analysis stem from several sources both internal and external to the power system. Many uncertainties are induced by the complex dynamics of the active and reactive load profiles that can vary in a fast and disordered manner due to many factors such as weather conditions and electricity price. A further source of uncertainty derives from the increasing number of non-dispatchable generators connected to the power system, particularly intermittent energy sources based on wind and solar power. Thus, solar radiation is subject to random coverage of clouds, which makes short-term variations of solar energy difficult to forecast, and similarly, wind speed variations may follow a generally well-known daily or seasonal pattern, but specific short-term, minute-to-minute and hourly changes are hard to predict [1]. The difficulties arising from the prediction and modeling of electricity market behavior, governed mainly by somewhat unpredictable economic dynamics, represents another source of uncertainty in power flow analysis [2]. Further uncertainties derive from variances in the model parameters of transmission system elements, such as resistance, reactance and/or capacitance values [3]. Under such conditions, reliable solution algorithms that incorporate the effect of data uncertainty into the power flow analysis are required. Reliable "interval" power flow algorithms would allow system operators to estimate both the data and the solution tolerance, i.e. uncertainty characterization and uncertainty propagation assessment, thus allowing them to evaluate the level of confidence of power flow studies.

The application of probabilistic methods has been reported in the literature for the aforementioned class of problems [4], which account for the variability and stochastic nature of the input data. In particular, uncertainty propagation studies using sampling based methods such as Monte Carlo simulations require several model runs with various combinations of input values. Since the number of simulations may be rather large, the needed computational resources for these types of studies could be prohibitively expensive [5]. Furthermore, as discussed in [6]-[9], these techniques present other shortcomings, such as the statistical dependence of the input data, and the problems associated with accurately identifying probability distributions for some input data, as in the case of power generated by wind or photovoltaic generators. These issues could lead to complex computations that may limit the use of these methods in practical applications, especially for the study of large networks.

In order to overcome some of the aforementioned limitations of sampling- and statistical-based methods, the application of self-validated computing for uncertainty representation in power flow studies has been proposed in the literature. The simplest and most popular of these models is Interval Mathematics (IM), which is basically a numerical

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A. Vaccaro is with the Department of Engineering, University of Sannio, Benevento, Italy (e-mail: vaccaro@unisannio.it).

C. Canizares and K. Bhattacharya are with the Department of Electrical and Computer Engineering, University of Waterloo, ON, Canada. (e-mail: ccanizar@uwaterloo.ca; kankar@ece.uwaterloo.ca)

computation technique where each quantity is represented by an interval of floating point numbers without the need to assume a probability structure [10]. The application of IM to power flow analysis has been investigated by various authors [8]-[12]. However, the adoption of this technique has many drawbacks derived mainly by the so called "dependency problem" of IM [13], [14]. In particular, the use of the Interval Gauss elimination in the power flow solution process leads to realistic solution bounds only for certain special classes of matrices (e.g. M-matrices, H-matrices, diagonally dominant matrices, tri-diagonal matrices) [12]. The problem of excessive conservatism in interval linear equation solving can be overcome by using Krawczyk's method or the Interval Gauss Seidel iteration procedure; nevertheless, in these cases, the linearized power flow equations should be preconditioned by an M-matrix in order to guarantee convergence. These drawbacks can make the application of IM to power flow studies very complex and time consuming.

In [15], these issues are addressed by proposing the application of an Affine Arithmetic (AA) based methodology for reliable power flow analysis in the presence of data uncertainty. AA is an enhanced model for self-validated numerical analysis in which the quantities of interest are represented as affine combinations (affine forms) of certain primitive variables, which stand for sources of uncertainty in the data or approximations made during the computation. Unlike IM, it keeps track of correlations between computed and input quantities, and is therefore robust with respect to the loss of precision often observed in long interval computations [14], [16].

In the current paper, an alternative formulation of the power flow problem is proposed within an interval optimization framework. This formulation presents greater flexibility, as it allows finding partial solutions and inclusion of other constraints, such as bus voltage limits, to help find solutions to non-converging power flows. In order to properly represent generator bus voltage controls, including reactive power limits and voltage recovery processes, a set of complementary constraints is used as proposed in [17]. The overall problem is then converted into a constrained nonlinear programming (NLP) problem under interval uncertainty. To solve this problem, a Range Arithmetic based solution methodology is proposed in this paper. The solution approach is based on the theory of direct interval matching and selection of the extreme value intervals [18]-[20]. The application of this approach allows solving the constrained interval power flow problem by reducing it to two determinate problems of the same type, namely, the lower boundary problem whose parameters are the low bounds of the parameters of the initial problem, and the upper boundary problem whose parameters are the upper bounds of the specified intervals, which can be solved by state-of-the-art NLP solvers.

The contributions of the present paper lie on the application of a new solution approach based on hybrid interval computing for power flow analysis with data uncertainties, considering that:

- New techniques that are computationally efficient and reliable, and yield better and accurate solutions, are needed for the analysis of power systems with uncertainties, especially in view of the increasingly wide deployment of highly variable power sources, as is the case of wind and solar power.
- Hybrid interval computing could play an important role in the development of efficient computational paradigms based on the integration of interval analysis [18] and affine arithmetic [15], which is an emerging trend in computational intelligence in the context of Granular Computing theory.
- Compared to the methodology described in [15], the proposed new formulation represents a different technique for solving uncertain power flow problems, based on hybrid interval computing and the more robust and flexible optimization-based power flow approach presented in [17] for solving deterministic power flow problems. Thus, the method proposed here allows to effectively manage multiple and heterogeneous constraints that are not possible with the formulation presented in [15]. This feature could be particularly useful in solving optimal power flow problems (e.g. economic dispatch) in the presence of data uncertainty, which is a problem currently under investigation by the authors.

The rest of the paper is organized as follows: Section II presents a brief review the power flow problem in the presence of data uncertainties. In Sections III and IV, the theoretical background of the proposed formulation and solution procedure are discussed in detail, respectively. To assess the effectiveness of the proposed methodology, detailed simulation studies for a realistic system, namely, the IEEE 57-bus test system, are presented and discussed in Section V, including a comparison with respect to the AA approach proposed in [15]. Finally, Section V summarizes the main conclusions and contributions of the paper.

II. Power Flow MCP Formulation

The equations typically used to solve the power flow problem are the real power balance equations at the generation and load buses, and the reactive power balance at the load buses. These equations can be written as:

$$\begin{aligned} P_i^{SP} &= V_i \sum_{j=1}^{N} V_j Y_{ij} \cos(\delta_i - \delta_j - \theta_{ij}) \quad \forall i \in nP \\ Q_j^{SP} &= V_j \sum_{k=1}^{N} V_k Y_{jk} \sin(\delta_j - \delta_k - \theta_{jk}) \quad \forall j \in nQ \end{aligned} \tag{1}$$

where N is the total bus number; nQ is the list of buses in which the reactive power is specified; nP is the list of the buses in which the active power is specified; P_i^{SP} and Q_j^{SP} are the real and reactive power injections specified at the *i*-th and *j*-th bus; $V_i\angle\delta_i$ is the *i*-th bus voltage in polar coordinates; and $Y_{ij}\angle\theta_{ij}$ is the *ij*-th element of the bus admittance matrix.

Because of the nonlinear nature of the power flow equations, numerical methods are employed to obtain a solution that is within an acceptable tolerance. This solution is referred to as an unconstrained solution since it has been obtained without taking into account the limits on the output variables, particularly reactive power limits at generation buses and voltages limits at load buses. Therefore, if the unconstrained solution is not feasible, a new solution satisfying the limits on the output variables, namely, a constrained feasible solution, should be computed. In this context, the feasibility of the reactive power limits at the generation buses is probably the most relevant issue that needs to be considered [21]. To solve this problem, the typical solution strategy is to carry out a bus-type switching, which involves converting a PV-bus into a PQ-bus for the buses where the reactive power limits are violated. If at any subsequent iteration, the voltage magnitude at that PQ-bus is below or above its original set point, depending on whether the generator is respectively under- or over-excited, the bus is then reverted back to a PV-bus [22]. Although the application of this technique is intrinsically straightforward, it presents some shortcomings concerning numerical blow-up and bus-type identification divergences. In order to overcome these limitations, more effective solution strategies based on nonlinear optimization techniques have been proposed in the literature [17], [23].

In [17], an optimization based model of the power flow problem integrating complementarity conditions is proposed to properly represent the generator bus voltage controls, including reactive power limits and voltage recovery processes. The proposed formulation of the power flow problem as a Mixed Complementarity Problem (MCP) is tested on several small and large, theoretical and real power networks under complex generator/load scenarios, exhibiting better convergence performances compared to other power flow algorithms. The power flow problem represented as an MCP is modeled as follows:

$$\min \quad F\left(\varepsilon_P, \varepsilon_Q\right) = \sum_i \left(\varepsilon_{Pi}^2 + \varepsilon_{Qi}^2\right) \tag{2}$$

$$\text{s.t.} \quad \varepsilon_{Pi} = P_i^{SP} - V_i \sum_{j=1}^{N} V_j Y_{ij} \cos(\delta_i - \delta_j - \theta_{ij}) \quad \forall i \in nP \tag{3}$$

$$\varepsilon_{Qj} = Q_j^{SP} - V_j \sum_{\backslash k=1}^{N} V_k Y_{jk} \sin(\delta_j - \delta_k - \theta_{jk}) \quad \forall j \in nQ \tag{4}$$

$$V_{Gi} - V_{Gi_o} - V_{Gai} + V_{Gbi} = 0 \quad \forall i \in nG \tag{5}$$

$$\left(Q_{Gi} - Q_{Gi}^{\min}\right) V_{Gai} = 0 \quad \forall i \in nG \tag{6}$$

$$\left(Q_{Gi}^{\max} - Q_{Gi}\right) V_{Gbi} = 0 \quad \forall i \in nG \tag{7}$$

$$\left(Q_{Gi} - Q_{Gi}^{\min}\right) \geq 0 \quad \forall i \in nG \tag{8}$$

$$\left(Q_{Gi}^{\max} - Q_{Gi}\right) \geq 0 \quad \forall i \in nG \tag{9}$$

$$V_{Gi}, V_{Gai}, V_{Gbi} \geq 0 \quad \forall i \in nG \tag{10}$$

where nG is the set of buses at which active power and voltage magnitude is specified (PV buses); ε_{Pi} and ε_{Qi} are the real and reactive power mismatch variables; Q_{Gi} is the reactive power generated at the PV buses; $Q_{Gi}^{\min}$ and $Q_{Gi}^{\max}$ are the minimum and maximum reactive power limits at PV buses; V_{Gi_o} is the set point value for bus voltage magnitude at PV buses; V_{Gi} is the voltage magnitude variable at PV buses; and V_{Gai} and V_{Gbi} are the auxiliary variables to track bus voltage magnitude variations.

The MCP formulation (2)-(10) aims at minimizing the total active and reactive power mismatch at all the buses subject to a set of equality and inequality constraints. The set of nonlinear constraints (3) and (4) represent the power flow equations, where V_{Gi} is treated as a variable, and (5)-(9) denotes the set of complementarity constraints. This model includes the auxiliary variables V_{Gai} and V_{Gbi} to represent bus voltage magnitude variations at generator buses when reactive power generation reaches its limits. These variables model the relationship between the reactive power generation Q_{Gi} and bus voltage magnitude V_{Gi} at each PV bus, representing the effect of maximum and minimum limits in voltage control, as explained in [24]. Thus, equations (6), (8) and (10) state that when Q_{Gi} reaches its minimum limit, V_{Gai} should take a positive value so that the power mismatch ε_{Qi} is minimized, as per (2), thus resulting in a generator terminal voltage V_{Gi} above its set point V_{Gi_o}, as per (5). On the other hand, equations (7), (9) and (10) state that when Q_{Gi} reaches its maximum limit, V_{Gbi} should take a positive value so that the power mismatch ε_{Qi} is minimized, as per (2), thus resulting in a generator terminal voltage V_{Gi} below its set point V_{Gi_o}, as per (5). It is to be noted that (6) and (7) are complementarity conditions, and hence they are not active simultaneously.

Expressing the power flow problem as an optimization problem presents greater flexibility, because it allows finding "partial" solutions and other constraints can be included to help find solutions to non-converging power flows. Furthermore, this formulation allows finding critical buses in the system, based on Lagrangian multipliers, for compensation purposes [17].

The MCP formulation of the power flow problem has all input data specified from the snapshot corresponding to a point in time or from a proper set of "crisp" values that the analyst constructs under certain assumptions for the system under study, such as the expected generation/load profiles for a given peak demand condition. The power flow solution is deemed representative of a limited set of system conditions; however, when the input conditions are uncertain, the precise conditions of the system become unknown and hence numerous scenarios need to be analyzed to cover the range of uncertainty.

III. Theoretical Background

An "uncertain" MCP formulation of the power flow can be expressed as a class of nonlinear interval optimization problems that can be defined as follows:

$$\min_{\hat{x}} \quad \hat{F}\left(\hat{x}\right) \tag{11}$$

s.t. $\quad \hat{g}_i(\hat{x}) \le 0 \quad \forall i \in [1, n_c]$

where the objective function $\hat{F}(\hat{x})$ and the n_c constrained functions $\hat{g}_i(\hat{x})$ are defined closed intervals of the interval $\hat{x}$ as follows:

$$\hat{F}(\hat{x}) = \left[F_{low}(\hat{x}), F_{up}(\hat{x})\right] \tag{12}$$

$$\hat{g}_i(\hat{x}) = \left[g_{i_{low}}(\hat{x}), g_{i_{up}}(\hat{x})\right] \quad \forall i \in [1, n_c] \tag{13}$$

Here, $F_{low}(\hat{x})$ and $g_{i_{low}}(\hat{x})$ are the lower boundary functions, while $F_{up}(\hat{x})$ and $g_{i_{up}}(\hat{x})$ are the corresponding upper boundary functions. Solution of the interval optimization problem (11) requires a reliable assessment of certain properties of a candidate solution $\hat{x}$ for all points in a given domain. Using standard real arithmetic is not a viable solution procedure since it can only sample a finite number of domain points in a finite number of steps. Range Arithmetic based models, on the other hand, are able to compute the properties of sets rather than points. A given domain can thus be covered by a finite number of subsets [10].

Interval Mathematics (IM) based methods allow for numerical computation where each quantity is represented by an interval $\hat{x}$, as a subset of $\Re$ of floating point numbers without a probability structure [14], as follows:

$$\hat{x} = [x_{low}, x_{up}] = \left\{x \in \Re : x_{low} \le x \le x_{up}\right\} \tag{14}$$

The intervals are added, subtracted, and/or multiplied in such a way that each computed interval is guaranteed to contain the unknown value of the quantity it represents. Besides, for every nonlinear function *f*, it is possible to define the corresponding interval extension $\hat{f}$. An important property of an interval extension is inclusion isotonicity:

$$x \in \hat{x} \Rightarrow f(x) \in \hat{f}(\hat{x}) \tag{15}$$

$$\hat{x} \subset \hat{y} \Rightarrow \hat{f}(\hat{x}) \subset \hat{f}(\hat{y}) \tag{16}$$

This guarantees that $\hat{f}(\hat{x})$ include all possible values of $f(x)$ for all $x \in \hat{x}$. The results of $\hat{f}$ may be wider than the exact result. This effect, referred to as over-estimation, affects most practical interval arithmetic calculations. The reason for over-estimation is the lack of information on the correlation of the intervals. To try and overcome this limitation, the AA method introduced in [14] and [16] can be used. This is a method for range analysis that allows the manipulation of the sources of error both external (e.g. imprecise or missing input data or uncertainty in mathematical modeling) and internal (e.g. round-off and truncation errors). While the AA method is similar to the standard IM method, it additionally keeps track of correlations between the input and computed quantities. This extra information provides much tighter bounds on the computing process, reducing the likelihood of error explosion problems observed in large IM computations [16]. Thus, in AA, a partially unknown quantity *x* is represented by an affine form $\hat{x}$ of the form [15]:

$$\hat{x} = x_0 + x_1\hat{\varepsilon}_1 + x_2\hat{\varepsilon}_2 + .. + x_n\hat{\varepsilon}_n \tag{17}$$

where x_0 is the central value, and the known real coefficients denoted by x_i represent various partial deviations. The variables $\hat{\varepsilon}_i$ stand for independent sources of uncertainty, each contributing to the total uncertainty of $\hat{x}$, and are assumed to lie in the interval [-1, 1]. In this context, the coefficients x_i represents the magnitude of the corresponding uncertainty. A key feature of the AA models is that the same noise $\hat{\varepsilon}_i$ may appear in different quantities, thus representing the same source of uncertainty in the given process.

In AA, each elementary real-number operation is replaced by a corresponding affine form operation, with the final result being in affine form as well. For the affine operations $\hat{x} \pm \hat{y}$, $a \pm \hat{x}$, $a\hat{x}$ ($a \in \Re$) the resulting affine forms are easily obtained by applying (17). Because of the linearity of affine operations no over-estimation occurs. On the other hand, if f is a non-affine operation, it cannot be expressed exactly as an affine combination of the noise symbols ε_i, and an affine function f^a that approximates the function reasonably well over its domain should be identified, as follows:

$$\hat{z} = \hat{f}^a(\hat{\varepsilon}_1, \hat{\varepsilon}_2, .., \hat{\varepsilon}_n) + z_k\hat{\varepsilon}_k \tag{18}$$

where the term $z_k\hat{\varepsilon}_k$ represents the residual or approximation error. The affine approximation function $\hat{f}^a$ can assume different structures, depending on the desired degree of accuracy and available computational resources. A good approximation function that is reasonably accurate and not very expensive to compute can be identified by applying the Chebyshev's approximation theory for univariate functions [16], which furnishes an algorithm for finding the optimum coefficients of the affine approximation function.

The inclusion isotonicity property of both IM and AA represents a strategic tool in solving nonlinear optimization problems under interval uncertainty. This fundamental property leads to the following theorem:

Theorem 1 [18]: For the interval function $\hat{f}(\hat{x}) = \left[f_{low}(\hat{x}), f_{up}(\hat{x})\right]$ to take the minimum (maximum) value at x^* in its domain *G*, it is necessary and sufficient that $f_{low}(\hat{x})$ and $f_{up}(\hat{x})$ boundary functions take the minimum (maximum) value at the same point:

$$\left\{\begin{aligned} &\hat{f}(x^*) = \left[f_{low}(x^*), f_{up}(x^*)\right] \\ &= \min_{\hat{x}\in G}\left(\hat{f}(\hat{x}) = \left[f_{low}(\hat{x}), f_{up}(\hat{x})\right]\right) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} f_{low}(x^*) &= \min_{\hat{x}\in G} f_{low}(\hat{x}) \\ f_{up}(x^*) &= \min_{\hat{x}\in G} f_{up}(\hat{x}) \end{aligned}\right\} \tag{19}$$

$$\left\{\begin{aligned} &\hat{f}(x^*) = \left[f_{low}(x^*), f_{up}(x^*)\right] \\ &= \max_{\hat{x}\in G}\left(\hat{f}(\hat{x}) = \left[f_{low}(\hat{x}), f_{up}(\hat{x})\right]\right) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} f_{low}(x^*) &= \max_{\hat{x}\in G} f_{low}(\hat{x}) \\ f_{up}(x^*) &= \max_{\hat{x}\in G} f_{up}(\hat{x}) \end{aligned}\right\} \tag{20}$$

This theorem reduces the search for the extremum of an interval function in a given domain to the search for the extrema of its lower and upper boundary functions in the same domain. In other words, the interval problem associated with (11) is reduced to two ordinary optimization problems, namely, the lower and upper boundary problems [20], [25]. Therefore, the solution to (11) can be restated as the solution of the following two NLP problems:

$$\begin{aligned} \min_{\hat{x}} \quad & F_{low}(x) \\ \text{s.t.} \quad & g_{iup}(x) \le 0 \quad \forall i \in [1, n_c] \end{aligned} \tag{21}$$

$$\begin{aligned} \min_{\hat{x}} \quad & F_{up}(x) \\ \text{s.t.} \quad & g_{iup}(x) \le 0 \quad \forall i \in [1, n_c] \end{aligned} \tag{22}$$

Thus, Theorem 1 yields the formulae (21) and (22) for determining the lower and upper bounds of the interval objective function $\hat{F}(\hat{x}) = [F_{low}(\hat{x}), F_{up}(\hat{x})]$ of the initial problem (11), with the respective constraints obtained from the upper bound of the system of constraints of the interval problem $\hat{g}_i(\hat{x}) = [g_{i_{low}}(\hat{x}), g_{i_{up}}(\hat{x})]$. These formulae are determinate optimization problems with point (non-interval) data, and significantly simplify the solution to the interval problem (11).

IV. Proposed Technique

Power flow analysis in the presence of data uncertainty and complementarity constraints requires the solution of the equations (1) with parameters that are non-determinate (uncertain) quantities as discussed in [15]. Based on the MCP formulation of the power flow (2)-(10), this problem is basically transformed into solving the following interval optimization problem assuming interval uncertainty in the input powers (generators and/or loads) as in [15]:

$$\min \qquad \hat{F}(\hat{\varepsilon}_P, \hat{\varepsilon}_Q) = \begin{bmatrix} \sum_i \left(\inf\left(\hat{\varepsilon}_{Pi}^2\right) + \inf\left(\hat{\varepsilon}_{Qi}^2\right) \right), \\ \sum_i \left(\sup\left(\hat{\varepsilon}_{Pi}^2\right) + \sup\left(\hat{\varepsilon}_{Qi}^2\right) \right) \end{bmatrix}$$

$$\begin{aligned} \text{s.t.} \quad & \hat{\varepsilon}_{Pi} = \hat{P}_i^{SP} - \hat{V}_i \sum_{j=1}^{N} \hat{V}_j Y_{ij} \cos(\hat{\delta}_i - \hat{\delta}_j - \theta_{ij}) \quad \forall i \in nP \\ & \hat{\varepsilon}_{Qi} = \hat{Q}_i^{SP} - \hat{V}_j \sum_{\backslash k=1}^{N} \hat{V}_k Y_{jk} \sin(\hat{\delta}_j - \hat{\delta}_k - \theta_{jk}) \quad \forall j \in nQ \\ & \hat{V}_{Gi} - V_{Gi_o} - \hat{V}_{Gai} + \hat{V}_{Gbi} = 0 \quad \forall i \in nG \\ & -Tol \le \left(\hat{Q}_{Gi} - Q_{Gi}^{\min}\right)\hat{V}_{Gai} \le Tol \quad \forall i \in nG \\ & -Tol \le \left(Q_{Gi}^{\max} - \hat{Q}_{Gi}\right)\hat{V}_{Gbi} \le Tol \quad \forall i \in nG \\ & \left(\hat{Q}_{Gi} - Q_{Gi}^{\min}\right) \ge 0 \quad \forall i \in nG \\ & \left(Q_{Gi}^{\max} - \hat{Q}_{Gi}\right) \ge 0 \quad \forall i \in nG \\ & \hat{V}_{Gi}, \hat{V}_{Gai}, \hat{V}_{Gbi} \ge 0 \quad \forall i \in nG \end{aligned} \tag{23}$$

where Tol is the tolerance control to relax the equality constraints associated with the complementary conditions, since these present computational challenges that may require the use of specialized solvers [17]. All interval variables are defined as $\hat{x} = \left[\inf(\hat{x}), \sup(\hat{x})\right]$, with $\sup(\hat{x})$ and $\inf(\hat{x})$ representing functions that define the upper and lower bound, respectively, of the interval variable. It is assumed, without loss of generality, that both the active and reactive power inputs at all buses are affected by interval uncertainty; however, other sources of uncertainty can be readily integrated in the model. In particular, line impedance uncertainties can be represented by an interval admittance matrix, affecting the upper/lower bounds $\hat{\varepsilon}_{Pi}$ and $\hat{\varepsilon}_{Qi}$ in (23) as follows:

$$\begin{aligned} & \hat{\varepsilon}_{Pi} = \hat{P}_i^{SP} - \hat{V}_i \sum_{j=1}^{N} \hat{V}_j \left|\hat{Y}_{ij}\right| \cos(\hat{\delta}_i - \hat{\delta}_j - \hat{\theta}_{ij}) \quad \forall i \in nP \\ & \hat{\varepsilon}_{Qj} = \hat{Q}_j^{SP} - \hat{V}_j \sum_{k=1}^{N} \hat{V}_k \left|\hat{Y}_{jk}\right| \sin(\hat{\delta}_j - \hat{\delta}_k - \hat{\theta}_{jk}) \quad \forall j \in nQ \end{aligned} \tag{24}$$

where $\hat{Y}_{ij} = [G_{ij\_\min}, G_{ij\_\max}] + j[B_{ij\_\min}, B_{ij\_\max}] = \left|\hat{Y}_{ij}\right| \angle \hat{\theta}_{ij}$. In this case, further non-affine operations, i.e. multiplication among affine forms, need to be considered.

Solving (23) requires (i) the concept of extremum of a function be generalized, (ii) the extremum conditions related to the non-determinacy of the parameters of the function be established, and (iii) special methods for the search of extrema of such functions be developed [19]. However, the Range Arithmetic based methodology discussed in Section III can be used to solve the MCP power flow formulation with undetermined parameters modeled as intervals. This solution approach is based on the theory of direct interval matching and selection of the interval that is extreme in value [19], [20], [25], and, as previously explained, it consists on solving the interval MCP problem (23) by reducing it to two determinate problems of the same type, namely, the lower boundary problem (21), whose parameters are the low bounds of the parameters of the initial problem, and the upper boundary problem (22), whose parameters are the upper bounds of the specified intervals [19]. Thus, to find the power flow solution interval associated with (23), based on Theorem 1, the following solution algorithm for is proposed:

1. Solve the lower boundary problem (21) using any appropriate solver for determinate nonlinear programming problems, obtaining a solution x_{low}^*.
2. Solve the upper boundary problem (22) using the same solver as in Step 1, obtaining a solution x_{up}^*.
3. Compute the solution set as $\hat{x} = [x_{low}^*, x_{up}^*]$. If $\hat{x} = \varnothing$, then the problem has no solution.

V. CASE STUDY

The application of the proposed methodology to the power flow problem with uncertainties for the IEEE 57-bus test system in [26] is discussed in this section; this test system was chosen since it is the same one used in the AA-based studies presented in [15], so that proper comparisons can be made. Thus, the power flow solution bounds obtained by the proposed Range Arithmetic based technique are compared to those calculated using the AA based methodology in [15] and a Monte Carlo simulation with a uniform distribution, which is typically assumed to yield the "correct" solution intervals. For the latter, 5000 different values of the input variables within the assumed input bounds were randomly selected (additional Monte Carlo simulations did not yield any significant changes to the solution intervals), and an MCP power flow solution was obtained for each; the desired interval solutions is then defined by the largest and the smallest values of the obtained bus voltage magnitudes and angles and line flows. All computational tasks were performed using Matlab.

Without loss of generality, a ±20% tolerance on load and generator power was assumed; the resulting input power profiles considered are shown in Fig. 1. Observe that this defines an interval wide enough to properly evaluate the proposed method. Since the system data does not contain information regarding reactive power limits at PV buses, these are assumed to be ±0.8 p.u. so as to test this particular feature of the proposed algorithm.

The interval profiles obtained are shown in Figs. 2 to 5 for the proposed method with respect to those obtained with the Monte Carlo and AA approaches[1]. Figure 2 depicts the bounds for the bus voltage magnitudes, Fig. 3 shows the bounds for the bus voltage angle, Fig. 4 presents the bounds of active power line flows, and Fig. 5 depicts the bounds for reactive power generated at the PV buses. Observe that the Range Arithmetic-based methodology arrives at fairly good approximations of the power flow solution bounds when compared to the benchmark intervals obtained using the Monte Carlo approach. Note that the solution bounds are slightly conservative and very similar to the AA-base method described in [15]; this is due to the fact that the proposed methodology, like AA, yields "worst case" bounds, which take into account any uncertainties in input data as well as all internal truncation and round-off errors. This is to be expected, since, as stated in [27], the random, uniformly distributed variation of parameters (with zero mean) assumed in the Monte Carlo approach tends to underestimate the worst case variations. This can be considered an advantage of the proposed approach, since no assumptions regarding the probability distribution of load and generator power variations are required.

Computationally, the presented Range Arithmetic based technique presents the following advantages with respect to the Monte Carlo and AA based techniques:

- It is significantly cheaper than the Monte Carlo approach, since it only requires the solution of 2 NLP problems, as opposed to computing thousands of power flow solutions.
- As opposed to the AA-based technique described in [15], it does not require assuming and computing the relationship of the intervals with the input uncertainties depicted in (17), which is based on sensitivity calculations that present several shortcomings, especially if linearization approaches are utilized, since these do not properly capture the nonlinear behavior of the system.
- In terms of computational resources, the proposed Range Arithmetic solution approach is comparable (slightly greater) to the iterative solution procedure required in the AA based methodology. This has been confirmed by comparing the simulation times required by the two solution techniques for interval power flow analysis of various power networks.

These conclusions have been confirmed by several simulation studies carried out for different power networks under multiple uncertainty scenarios.

VI. CONCLUSIONS

A new optimization framework for power flow solution with undetermined parameters of elementary type (intervals) was proposed in this paper. The proposed model considers complementarity constraints to properly represent generator bus voltage controls, including reactive power limits and voltage recovery processes. To solve this constrained nonlinear interval optimization programming problem, a solution strategy based on a fundamental theorem of Range Arithmetic theory was proposed. This allowed the computation of the range of power flow solutions associated with input interval uncertainties by solving two determinate problems of the same type, namely, the lower and the upper boundary problems, which can be readily solved using state-of-the-art NLP solvers. The main benefits of the proposed technique were assessed on a realistic power system, demonstrating that the proposed approach is better suited than other existing techniques for the assessment of uncertainty propagation in power flow solutions, independent of the types and levels of uncertainties in the input data.

It is important to highlight the fact that the intervals obtained with the proposed affine arithmetic based power flow can be used by system operators to plan for proper control and corrective actions. For example, voltage intervals would give operators a sense of whether bus voltage limits may be violated due to changes in active and reactive power injections at certain buses, so that actions such as activating/adding shunt compensators at buses with large voltage intervals can be taken. Furthermore, active power injection intervals obtained at dispatchable generator buses would give the operator a good sense of the size and location of reserves required to compensate for variations at non-dispatchable sources.

[1] To avoid the use of tables, the interval values for each variable are depicted graphically in these figures for better visual representation. The connecting lines between consecutive interval values are not meant to depict interpolations.

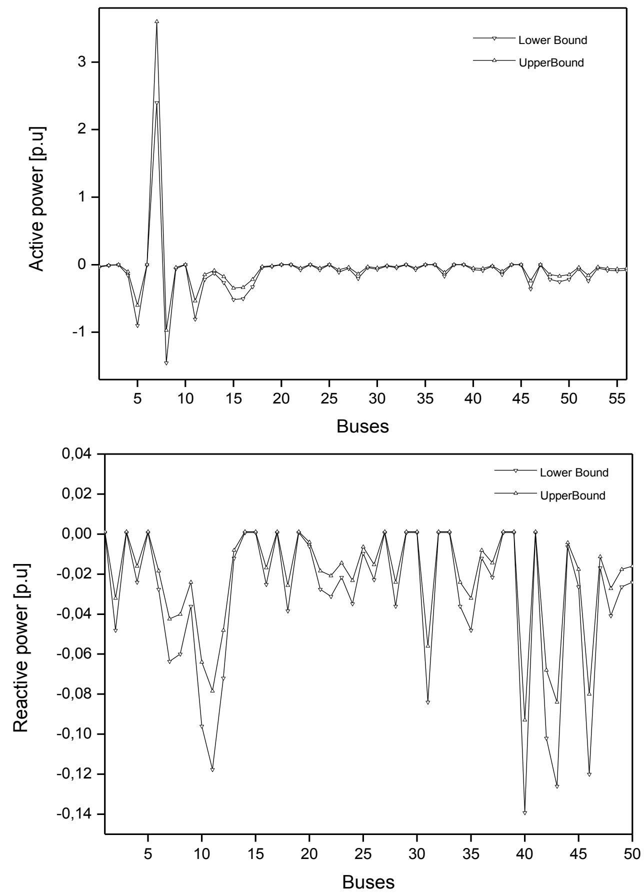

Fig. 1. Assumed active and reactive power input variations.

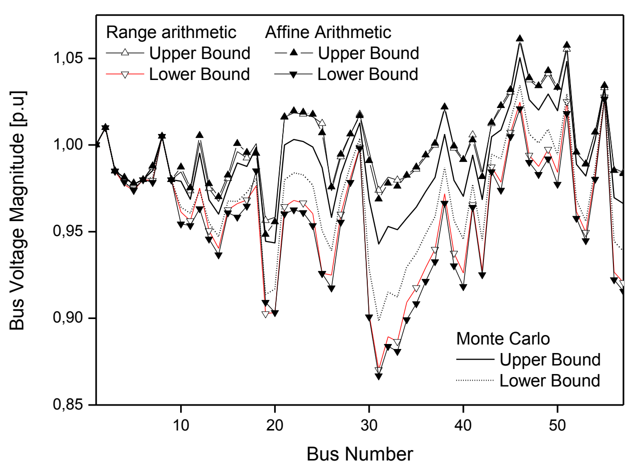

Fig. 2. Computed bus voltage magnitude intervals.

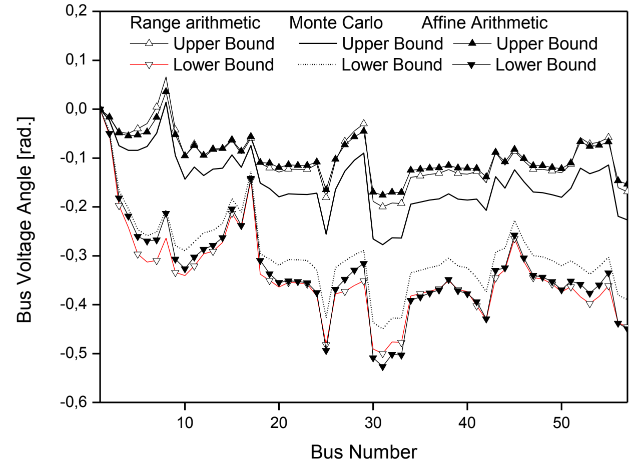

Fig. 3: Computed bus voltage angle intervals.

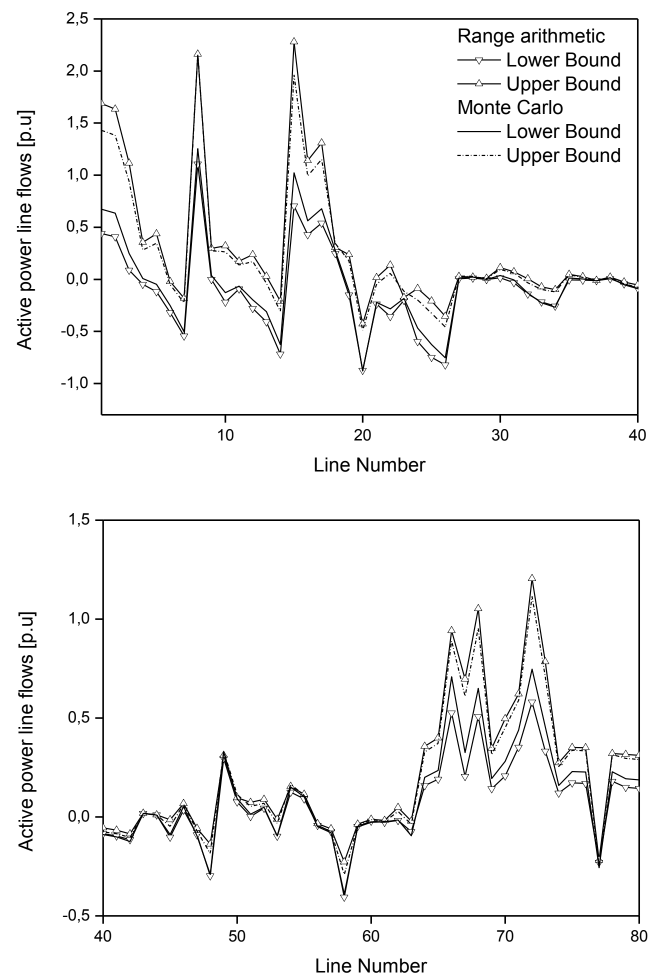

Fig. 4. Calculated active power line flow intervals

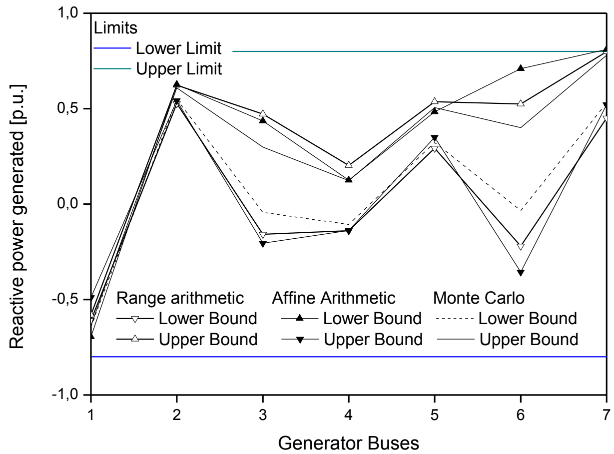

Fig. 5. Calculated reactive power intervals at generator buses with respect to their limits.

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Alfredo Vaccaro (M'01, SM'09) received the M.Sc. degree with honors in Electronic Engineering in 1998 from the University of Salerno, Salerno, Italy. From 1999 to 2002, he was an Assistant Researcher at the University of Salerno, Department of Electrical and Electronic Engineering. Since March 2002, he has been an Assistant Professor in electric power systems at the Department of Engineering of the University of Sannio, Benevento, Italy. His special fields of interest include soft computing and interval-based method applied to power system analysis and advanced control architectures for diagnostic and protection of distribution networks. Prof. Vaccaro is an Associate Editor and member of the Editorial Boards of IET Renewable Power Generation, the International Journal of Electrical and Power Engineering, the International Journal of Reliability and Safety, the International Journal on Power System Optimization and the International Journal of Soft Computing.

Claudio Cañizares (S'86, M'91, SM'00, F'07) received the Electrical Engineer degree from Escuela Politécnica Nacional (EPN), Quito-Ecuador, in 1984 where he held different teaching and administrative positions from 1983 to 1993. His MSc (1988) and PhD (1991) degrees in Electrical Engineering are from University of Wisconsin-Madison. He has been with the E&CE Department, University of Waterloo since 1993, where he has held various academic and administrative appointments and is currently a full Professor and the Associate Director of the Waterloo Institute for Sustainable Energy (WISE). His research activities concentrate in the study of stability, modeling, simulation, control and computational issues in power systems within the context of competitive electricity markets.

Kankar Bhattacharya (M'95, SM'01) received the Ph.D. degree in Electrical Engineering from the Indian Institute of Technology, New Delhi, India in 1993. He was in the faculty of Indira Gandhi Institute of Development Research, Mumbai, India, during 1993-1998, and then the Department of Electric Power Engineering, Chalmers University of Technology, Gothenburg, Sweden, during 1998-2002. He joined the E&CE Department of the University of Waterloo, Canada, in 2003 where he is currently a Full Professor. His research interests are in power system economics and operational aspects.