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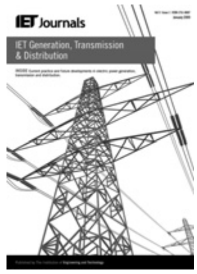


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# Revisiting the power flow problem based on a mixed complementarity formulation approach

*Mehrdad Pirnia, Claudio A. Cañizares, Kankar Bhattacharya*

*Electrical and Computer Engineering Department, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2 L 3G1*
*E-mail: mpirnia@uwaterloo.ca*

**Abstract:** A novel optimisation-based model of the power flow (PF) problem is proposed using complementarity conditions to properly represent generator bus voltage controls, including reactive power limits and voltage recovery processes. This model is then used to prove that the Newton–Raphson (NR) solution method for solving the PF problem is basically a step of the generalised reduced gradient algorithm applied to the proposed optimisation problem. To test the accuracy, flexibility and the numerical robustness of the proposed model, the IEEE 14-bus, 30-bus, 57-bus, 118-bus and 300-bus test systems and large real 1211-bus and 2975-bus systems are used, benchmarking the results of the proposed PF model against the standard NR method. It is shown that the proposed model yields adequate solutions, even in the case when the NR method fails to converge.

## Nomenclature

### *Indices and sets*

| | |
|---|---|
| gen | a subset of generator buses |
| $i, j$ | index of buses |

### *Variables*

| | |
|---|---|
| $\boldsymbol{\delta}$ | vector of bus angle variables (radians) |
| $\boldsymbol{\varepsilon}$ | vector of mismatch variables, pu |
| $\boldsymbol{\varepsilon_p}$ | vector of real power mismatch variables, pu |
| $\boldsymbol{\varepsilon_q}$ | vector of reactive power mismatch variables, pu |
| $P_i$ | real power injection at bus $i$, pu |
| $P_s$ | real power injection variable at the slack bus, pu |
| $\boldsymbol{Q}_G$ | vector of reactive power generation variables at $PV$ buses, pu |
| $Q_i$ | reactive power injection at bus $i$, pu |
| $\lvert\boldsymbol{V_D}\rvert$ | vector of bus voltage magnitude variable at $PQ$ buses, pu |
| $\lvert\boldsymbol{V_G}\rvert$ | vector of bus voltage magnitude variable at $PV$ buses, pu |
| $\lvert V_i\rvert$ | bus voltage magnitude at bus $i$, pu |
| $\boldsymbol{V_{Ga}}$, $\boldsymbol{V_{Gb}}$ | vector of auxiliary variables to track bus voltage magnitude variation, pu |
| $\boldsymbol{x}$ | vector of power flow (PF) variables |
| $\hat{\boldsymbol{x}}$ | vector of dependent variables |
| $\boldsymbol{y}$ | vector of independent auxiliary and $\lvert V_G\rvert$ variables |
| $\boldsymbol{z}$ | vector of optimisation variables |

### *Parameters and functions*

| | |
|---|---|
| $\alpha$ | correction factor in NR method |
| $\beta$ | scalar for step-size adjustment |
| $\boldsymbol{B}_{ij}$ | imaginary part of admittance bus matrix |
| $\delta_{ij}$ | the angle between bus $i$ and $j$, radians |
| $\boldsymbol{G}_{ij}$ | real part of admittance bus matrix |
| $\boldsymbol{J}$ | Jacobian matrix |
| $\boldsymbol{M}$ | matrix used to calculate GRG step |
| $N$ | number of buses |
| $N_G$ | number of generators |
| $\boldsymbol{Q}_G^{\max}$ | nector of maximum reactive power at generator buses, pu |
| $\boldsymbol{Q}_G^{\min}$ | nector of minimum reactive power at generator buses, pu |
| $s$ | GRG step |
| $\tau$ | tolerance level |
| $\lvert V_{G_{i0}}\rvert$ | set point value for bus voltage magnitude at $PV$ buses, pu |

### *Functions*

| | |
|---|---|
| $f(\cdot)$ | PF equations |
| $F(\cdot)$ | objective function for the PF optimisation model |
| $g(\cdot)$ | set of equality constraints |
| $\hat{g}(.)$ | set of inequality constraints |
| $h(\cdot)$ | set of complementarity and equality equations |
| $\Delta P(.)$ | non-linear function for real power mismatch at a bus |
| $\Delta Q(.)$ | non-linear function for reactive power mismatch at a bus |

## 1 Introduction

The power flow (PF) analysis problem is a widely used tool for power system operations and planning, since it provides network solutions such as bus voltage magnitudes and angles for a given set of operating conditions. It can also serve as a security analysis tool, providing guidelines on acceptable operating conditions in case of sudden disturbances and load changes.

Since the PF equations are non-linear, the solution methodologies for these problems have traditionally involved iterative procedures such as the Gauss–Seidel and the Newton–Raphson (NR) methods [1–3]. To address some of the numerical issues in the utilisation of the Gauss–Seidel method, the NR method implementation exploits its 'quadratic' convergence, when the initial solution is close to the final solution, and takes advantage of the admittance matrix sparsity to attain faster convergence [2]. Various improvements to the NR method are reported in the literature involving the selection of an effective starting point [3], reduction of the number of iterations [4, 5] and improving the algorithm robustness [6, 7].

In [8], an approach based on the Krylov subspace methodology is proposed to solve large-scale PF problems. The method uses an approximation of the Jacobian matrix without explicitly forming this matrix and eliminates the need for matrix factorisations. The Krylov subspace method uses the conjugate gradient (CG) method for linear systems to minimise the residuals in each iteration [9], which is improved in [10] to accommodate other types of matrices such as asymmetric, indefinite matrices and non-linear systems; since the objective of this algorithm is to minimise the residuals, it is also referred to as the generalised minimal residual method (GMRES) [9]. To improve the efficiency of the CG method, a preconditioning technique, based on the Chebyshev pre-conditioner, which does not need matrix ordering, is proposed in [11]. In [12], a fast Newton GMRES algorithm is presented to solve PF equations using three acceleration schemes: a hybrid scheme, a partial pre-conditioner update scheme, and an adaptive tolerance control scheme. In [13], a novel approach to formulating PF problems based on the vector continuous Newton's method is presented. The PF problem is classified in that paper into four possible categories: the well-conditioned case in which the solution can be reached from a flat start; the ill-conditioned case where the solution cannot be reached from a flat start; the bifurcation point case in which a solution exists, and can be associated with either a saddle-node bifurcation or a limit induced bifurcation; and finally the unsolvable case.

A large segment of the reported methods in the literature pertaining to PF analysis problems rely on a 'standard' NR-based approach to solve the non-linear system of PF equations. To our knowledge, there is no reported work in the relevant technical literature on optimisation approaches applied to the solution of PF problems that could be directly compared with the technique proposed in this paper. All existent iterative algorithms need an initial solution such as a 'flat-start', that is, setting the bus voltage angles to zero and load bus voltage magnitudes to 1 pu. However, most of the solution methods encounter convergence problems from flat-start initialisation when the size of the system is large (typically, more than 1000 buses) [3]. Thus, in the current paper, a novel formulation of the PF problem is proposed within an optimisation framework that includes complementarity constraints. Accordingly, the PF problem is formulated as a mixed complementarity problem (MCP), which can take advantage of state-of-the-art non-linear programming and complementarity problems using solvers such as COINIPOPT [14], MINOS [15] and PATHNLP [16]. The proposed MCP-based PF model, which by design always has a theoretical solution, is shown here to have increased robustness and flexibility with respect to the existent PF methods, which have convergence problems for large systems when using a flat-start and cannot yield solutions when the maximum loadability of the system is exceeded. Based on the proposed MCP formulation, it is also formally demonstrated that the NR solution of the PF problem is essentially a step of the traditional GRG algorithm. Finally, the solution of the proposed MCP model is compared with the 'standard' NR solution approach for a variety of small-, medium-, large-sized systems in order to examine the flexibility and robustness of this approach.

The main reasons to revisit the PF problem in the recent paper can be summarised as follows:

- Existing NR-based PF solution methods have convergence problems for large systems when using a flat-start or exceeding the maximum loadability of a system, and require an initial solution which is close to the final solution.
- The conventional PF solution methods require iterative *PV–PQ* bus switching when the reactive power at a generator bus violates the limits.

The proposed MCP-based method addresses the above issues by using complementarity constraints to properly represent reactive power generation limits and the voltage recovery process of voltage regulators. Furthermore, the proposed method is more flexible than standard NR-based PF solution approaches, as it readily allows using different forms of PF equations (polar and rectangular coordinates), explicitly imposing limits on certain variables such as bus voltage magnitude limits and yielding solutions to non-convergent PF problems.

The rest of the paper is organised as follows: in Section 2, a brief review of the PF problem and its solution using the NR method is presented. Section 3 presents and discusses the MCP formulation of the PF problem. Section 4 illustrates the application of the GRG method applied to the proposed MCP model, showing that the 'standard' NR solution technique is basically a step of this method applied to the MCP model. In Section 5, numerical results for a variety of systems with flat-start initialisation and for different non-linear optimisation and MCP solvers are presented and discussed, demonstrating the various advantages of the proposed PF formulation. Finally, in Section 6 the main conclusions and contributions of the paper are highlighted.

## 2 Background

The PF analysis problem is formulated as a set of simultaneous nonlinear equations for real and reactive power mismatches $\Delta P$ and $\Delta Q$, respectively, as follows

$$\Delta P(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, \boldsymbol{Q}_{\mathrm{G}}) = P_i - |V_i| \sum_{j=1}^{n} |V_j| \left(\boldsymbol{G}_{ij} \cos \delta_{ij} + \boldsymbol{B}_{ij} \sin \delta_{ij}\right) = 0 \quad \forall i \tag{1}$$

$$\Delta Q(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, \boldsymbol{Q}_{\mathrm{G}}) = Q_i - |V_i| \sum_{j=1}^{n} |V_j| \left(\boldsymbol{G}_{ij} \sin \delta_{ij} - \boldsymbol{B}_{ij} \cos \delta_{ij}\right) = 0 \quad \forall i \tag{2}$$

where all the variables are defined in the nomenclature. Equations (1) and (2) can be expressed in matrix-vector form as

$$f(x) = \begin{bmatrix} \Delta P(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, \boldsymbol{Q}_{\mathrm{G}}) \\ \Delta Q(\delta, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, \boldsymbol{Q}_{\mathrm{G}}) \end{bmatrix} = 0 \quad (3)$$

Since the solution to (3) cannot be expressed in closed form [17], these equations are solved iteratively, in which an initial guess $(\boldsymbol{\delta}^0, \boldsymbol{P}_{\mathrm{s}}^0, |\boldsymbol{V}_{\boldsymbol{D}}^0|, \boldsymbol{Q}_{\mathrm{G}}^0)$ is selected close to the 'desired' solution $(\boldsymbol{\delta}^*, \boldsymbol{P}_{\mathrm{s}}^*, |\boldsymbol{V}_{\boldsymbol{D}}^*|, \boldsymbol{Q}_{\mathrm{G}}^*)$, and it is iteratively updated to obtain a solution $\hat{\boldsymbol{x}}$, such that $f(\hat{\boldsymbol{x}}) \approx 0$.

### 2.1 NR solution method

Since 1980, significant research has been carried out on the development of solution methods for non-linear simultaneous equations, including pivoting and iterative methods. One of the well-established iterative methods, the NR method, has been found most suitable for solving the set of non-linear PF equations because of its fast convergence properties [18].

The NR method relies on non-linear approximation obtained from a Taylor series [19]. Thus, expanding the non-linear function $f(\boldsymbol{x}) = 0$ around a nominal value of $\boldsymbol{x} = \boldsymbol{x}^k$ results in the following expression for an iteration $k + 1$

$$\begin{aligned} f(\boldsymbol{x}^{k+1}) &\simeq f(\boldsymbol{x}^k) + D_x f(\boldsymbol{x}^k)(\boldsymbol{x}^{k+1} - \boldsymbol{x}^k) \\ &\simeq f(\boldsymbol{x}^k) + D_x f(\boldsymbol{x}^k)\Delta\boldsymbol{x}^{k+1} = 0 \end{aligned} \quad (4)$$

Therefore the solution update from the previous iteration can be expressed as follows, if the initial guess for the solution at $k = 0$ $(\boldsymbol{x}^0)$ is 'close' to the final solution $\boldsymbol{x}^*$

$$\Delta\boldsymbol{x}^{k+1} = -\alpha\left[D_x f(\boldsymbol{x}^k)\right]^{-1} f(\boldsymbol{x}^k) \quad (5)$$

where the scalar $\alpha > 0$ is a 'correction' factor to control the NR convergence [20]. This method, typically referred as 'robust' NR, converges when $|f(\boldsymbol{x}^{k+1}) - f(\boldsymbol{x}^k)| < \tau$, where $\tau$ is the convergence tolerance.

The NR method applied to the solution of the PF problem (1) and (2), yields the following linear equations

$$\begin{bmatrix} \Delta\boldsymbol{P} \\ \Delta\boldsymbol{Q} \end{bmatrix} = [\boldsymbol{J}] \begin{bmatrix} \Delta\boldsymbol{\delta} \\ \Delta\boldsymbol{P}_{\mathrm{s}} \\ |\Delta\boldsymbol{V_D}| \\ \Delta\boldsymbol{Q}_{\mathrm{G}} \end{bmatrix} \quad (6)$$

In this process, at the end of each iteration, the reactive power generation $\boldsymbol{Q}_{\mathrm{G}}$ from each generator is calculated. If $\boldsymbol{Q}_{\mathrm{G}}$ violates either of its limits at a bus, it is fixed at the corresponding limiting value and the particular 'generator' bus ($PV$ bus) is switched to a 'load' bus ($PQ$ bus). The bus is switched back to a $PV$ bus when $|\boldsymbol{V_G}|$ returns to its set-point value.

The NR method is much faster than other iterative methods since it converges quadratically when starting from an initial point 'close' to the observed solution [8]. One of the main disadvantages of this method is that, when the initial solution selected is 'too far' from the final solution, it has difficulties to converge. In such a case, the assumption that higher order terms of the Taylor series can be ignored is no longer valid, since these terms have a significant effect on the convergence process.

Although the NR method has a fast convergence with less iterations, each iteration is computationally expensive, since the computational burden of the Jacobian matrix is a function of $N^2$ [17]. In [21], a method is proposed to update the Jacobian matrix whenever the rate of convergence slows down, which is referred to as the 'dishonest' Newton method.

## 3 PF optimisation formulation

### 3.1 MCP model

Based on [22], the PF analysis problem is represented in this work, as an MCP problem as follows

$$\min \quad F(\boldsymbol{\varepsilon_p}, \boldsymbol{\varepsilon_q}) = \sum_i \left\{\boldsymbol{\varepsilon}_{p\,i}^2 + \boldsymbol{\varepsilon}_{q\,i}^2\right\} \quad (7)$$

$$\text{s.t.} \quad \Delta P_i(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, |\boldsymbol{V_G}|, \boldsymbol{Q}_{\mathrm{G}}) - \varepsilon_{p\,i} = 0 \quad \forall i \quad (8)$$

$$\Delta Q_i(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, |\boldsymbol{V_G}|, \boldsymbol{Q}_{\mathrm{G}}) - \varepsilon_{q\,i} = 0 \quad \forall i \quad (9)$$

$$|V_{G_i}| = |V_{Gi_0}| + V_{Ga_i} - V_{Gb_i} \quad \forall i \in \{\text{gen}\} \quad (10)$$

$$0 \le \left(Q_{G_i} - Q_{G_i}^{\min}\right) \perp V_{Ga_i} \ge 0 \quad \forall i \in \{\text{gen}\} \quad (11)$$

$$0 \le \left(Q_{G_i}^{\max} - Q_{G_i}\right) \perp V_{Gb_i} \ge 0 \quad \forall i \in \{\text{gen}\} \quad (12)$$

$$|V_{G_i}|, V_{Ga_i}, V_{Gb_i} \ge 0 \quad \forall i \in \{\text{gen}\} \quad (13)$$

where $\perp$ represents a complementarity condition, that is, for $0 \le a \perp b \ge 0$, $a = 0$, if $b \ge 0$, and $b = 0$, if $a \ge 0$, which can be presented by: $ab = 0$, $a \ge 0$, $b \ge 0$. This optimisation formulation comprises the set of non-linear constraints (8) and (9) representing the PF equations, where $|\boldsymbol{V_G}|$ is treated as a variable, and a set of complementarity constraints given by (11) and (12), and associated constraints (10) and (13). This model includes the auxiliary variables $\boldsymbol{V_{Ga}}$ and $\boldsymbol{V_{Gb}}$ to track bus voltage magnitude variations, at generator and slack buses, when reactive power generation reaches limits as explained in more detail in Section 3.2. The objective here is to minimise the total active and reactive power mismatches at all buses.

There may be multiple feasible solutions to the proposed optimisation model (7)–(13), as there are up to $2^N$ solutions to a PF problem [23], which may be found by varying the initial guess and utilising different solution algorithms. All solutions are mathematically acceptable, because they satisfy all the complementarity and PF constraints imposed in the model; however, only one is typically adequate in practice.

Expressing the PF problem as an optimisation model gives greater flexibility, because it allows finding 'partial' solutions and other types of constraints, such as voltage limits at buses, can be included to help find solutions to 'non-converging' PFs. Furthermore, this formulation allows finding 'critical' buses in the system, based on Lagrangian multipliers, for compensation purposes. Finally, the proposed MCP model always has a theoretically feasible solution since $\varepsilon$ may not necessarily be zero, which could be useful for studying non-convergent PF problems, as demonstrated in Section 5.




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### 3.2 Complementarity conditions to model reactive power limits

In [22], the following set of complementarity conditions is proposed in an optimal PF (OPF) framework, to model the relationship between the reactive power generation $\boldsymbol{Q}_{\mathrm{G}}$ and bus voltage magnitude $|\boldsymbol{V_G}|$ at each 'generator' bus, representing the effect of maximum and minimum limits in voltage control

$$0 \le \left(Q_{G_i} - Q_{G_i}^{\min}\right) \perp V_{Ga_i} \ge 0 \quad \forall i \in \{\text{gen}\} \tag{14}$$

$$0 \le \left(Q_{G_i}^{\max} - Q_{G_i}\right) \perp V_{Gb_i} \ge 0 \quad \forall i \in \{\text{gen}\} \tag{15}$$

Here, the operator $\perp$ denotes the following

$$\left(Q_{G_i} - Q_{G_i}^{\min}\right) V_{Ga_i} = 0 \quad \forall i \in \{\text{gen}\} \tag{16}$$

$$\left(Q_{G_i} - Q_{G_i}^{\min}\right) \ge 0 \quad \forall i \in \{\text{gen}\} \tag{17}$$

$$V_{Ga_i} \ge 0 \quad \forall i \in \{\text{gen}\} \tag{18}$$

$$\left(Q_{G_i}^{\max} - Q_{G_i}\right) V_{Gb_i} = 0 \quad \forall i \in \{\text{gen}\} \tag{19}$$

$$\left(Q_{G_i}^{\max} - Q_{G_i}\right) \ge 0 \quad \forall i \in \{\text{gen}\} \tag{20}$$

$$V_{Gb_i} \ge 0 \quad \forall i \in \{\text{gen}\} \tag{21}$$

Equations (16)–(18) state that when $\boldsymbol{Q}_{\mathrm{G}}$ is at its minimum limit, $\boldsymbol{V_{Ga}}$ takes a positive value and similarly for (19)–(21), so that when $\boldsymbol{Q}_{\mathrm{G}}$ is at its maximum limit, $\boldsymbol{V_{Gb}}$ takes a positive value. It is to be noted that (16) and (19) are complementarity conditions, and hence are not active simultaneously; therefore it is not possible for both $\boldsymbol{V_{Ga}}$ and $\boldsymbol{V_{Gb}}$ to have positive values at the same time. The auxiliary variables $\boldsymbol{V_{Ga}}$ and $\boldsymbol{V_{Gb}}$ are, accordingly, used to affect the changes of the bus voltage magnitudes at the generator buses as follows

$$|V_{G_i}| = |V_{Gi_0}| + V_{Ga_i} - V_{Gb_i} \quad \forall i \in \{\text{gen}\} \tag{22}$$

This complementarity model properly represents a generator's voltage control system, since if $V_{Ga_i}$ is positive and $V_{Gb_i} = 0$ for $Q_{G_i} = Q_{G_i}^{\min}$, the corresponding bus voltage increases; on the other hand, if $V_{Gb_i}$ is positive and $V_{Ga_i} = 0$ for $Q_{G_i} = Q_{G_i}^{\max}$, the corresponding bus voltage decreases. In (16)–(22), the $\boldsymbol{Q}_{\mathrm{G}}$ variables are independent of the auxiliary variables $\boldsymbol{V_{Ga}}$ and $\boldsymbol{V_{Gb}}$.

Based on (16)–(22), the proposed MCP model (7)–(13) can be represented as follows

$$\min \quad F\left(\boldsymbol{\varepsilon_p}, \boldsymbol{\varepsilon_q}\right) = \sum_i \left\{\varepsilon_{pi}^2 + \varepsilon_{qi}^2\right\} \tag{23}$$

$$\text{s.t.} \quad \Delta P_i\left(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, |\boldsymbol{V_G}|, \boldsymbol{Q}_{\mathrm{G}}\right) - \varepsilon_{pi} = 0 \quad \forall i \tag{24}$$

$$\Delta Q_i\left(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, |\boldsymbol{V_D}|, |\boldsymbol{V_G}|, \boldsymbol{Q}_{\mathrm{G}}\right) - \varepsilon_{qi} = 0 \quad \forall i \tag{25}$$

$$|V_{G_i}| - |V_{Gi_0}| - V_{Ga_i} + V_{Gb_i} = 0 \quad \forall i \in \{\text{gen}\} \tag{26}$$

$$\left(Q_{G_i} - Q_{G_i}^{\min}\right) V_{Ga_i} = 0 \quad \forall i \in \{\text{gen}\} \tag{27}$$

$$\left(Q_{G_i}^{\max} - Q_{G_i}\right) V_{Gb_i} = 0 \quad \forall i \in \{\text{gen}\} \tag{28}$$

$$\left(Q_{G_i} - Q_{G_i}^{\min}\right) \ge 0 \quad \forall i \in \{\text{gen}\} \tag{29}$$

$$\left(Q_{G_i}^{\max} - Q_{G_i}\right) \ge 0 \quad \forall i \in \{\text{gen}\} \tag{30}$$

$$|V_{G_i}|, V_{Ga_i}, V_{Gb_i} \ge 0 \quad \forall i \in \{\text{gen}\} \tag{31}$$

## 4 NR as an MCP solution step

This section presents an in-depth explanation of the differences between the proposed optimisation-based PF method and the standard NR-based PF solution approach. In [24], Carpentier discusses the use of NR to solve the corrector step equations of the GRG method applied to the 'classical' OPF problem. A similar approach is used here in order to demonstrate that the solution of the MCP PF model (7)–(13), which is 'not' a standard OPF problem, basically corresponds to the NR solution of the PF (8) and (9) for $\boldsymbol{\varepsilon_p} = \boldsymbol{\varepsilon_q} = 0$. Thus, this section explains why the PF solvers do not converge in some cases. To demonstrate this, first the GRG method is applied to the PF problem where the generator bus voltage magnitudes are fixed and the complementarity constraints are ignored, demonstrating that the standard NR-based PF solution method is just a particular step of the GRG approach applied to a simplified version of the proposed optimisation method. The complementarity constraints are subsequently included in the analysis to properly represent generator voltage controls and reactive power limits, proposing a possible extension to the existing NR-based approach to better compare the PF solution process for practical applications. Observe that such a perspective on PF problems is not available in the power system literature to the best of our knowledge.

First, the GRG method of solution of the optimisation model considering only constraints (8) and (9) is compared with the NR method. Thus, let the proposed PF model be written as follows, for $|V_G| = |V_{G_0}|$

$$\min \quad F(\boldsymbol{\varepsilon}) \tag{32}$$

s.t.

$$f(\boldsymbol{x}, \boldsymbol{\varepsilon}) = f(z) = \begin{bmatrix} \Delta P\left(\boldsymbol{\delta}, \boldsymbol{P}_{\mathrm{s}}, \left|\boldsymbol{V_D}\right|, \boldsymbol{Q}_{\mathrm{G}}, \boldsymbol{\varepsilon}\right) \\ \Delta Q\left(\delta, \boldsymbol{P}_{\mathrm{s}}, \left|\boldsymbol{V_D}\right|, \boldsymbol{Q}_{\mathrm{G}}, \varepsilon\right) \end{bmatrix} = 0 \tag{33}$$

where the optimisation variables $z$ are divided into PF variables $x$ and mismatch variables $\varepsilon$, as follows

$$x = \begin{bmatrix} \delta \\ P_S \\ |V_D| \\ Q_G \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_p \\ \varepsilon_q \end{bmatrix} \tag{34}$$

In order to solve (33) using the GRG method, the following two steps are required [25]

(1) *Predictor step*: Assuming that there is a set of values for $z$ satisfying the constraints $f(z) = 0$, say $z^m = (x^m, \varepsilon^m)$, this 'guess' can be improved by moving in the direction of the steepest descent, resulting in $z^{m+1} = (x^{m+1}, \varepsilon^{m+1})$ as follows

$$z^{m+1} = z^m + \beta s^m \tag{35}$$

where $s^m$ is the step calculated by the gradient of $F(\varepsilon)$ as follows

$$s^m = -MM^T \nabla_z F(\varepsilon^m) \tag{36}$$

$$M = \begin{bmatrix} -[D_x f(z^m)]^{-1} D_\varepsilon f(z^m) \\ I_{2N} \end{bmatrix}_{(2N+2N)\times 2N} \tag{37}$$

and $\beta > 0$ is a scalar for step-size adjustment such that $F(\varepsilon^{m+1}) < F(\varepsilon^m)$.

(2) *Corrector step:* The predicted value of $z^{m+1}$ should then be corrected to ensure it satisfies the constraints $f(z) = 0$. This can be done by the following robust NR procedure to obtain a $z^* = (x^*, \varepsilon^{m+1})$ such that $f(z^*) = 0$

$$x^{k+1} = x^k - \alpha[D_x f(x^k, \varepsilon^{m+1})]^{-1} f(x^k, \varepsilon^{m+1}) \tag{38}$$

where $x^k = x^{m+1}$ obtained from (35), and the scalar $\alpha > 0$ is used to ensure convergence. The iteration $k$ is repeated until convergence is obtained, that is, until $z^*$ is found. These predictor and corrector steps are repeated until $F(\varepsilon)$ is 'close' to zero.

Observe that (38) is exactly the same as (5) for $\varepsilon = 0$. Thus, it can be readily concluded that the NR method applied to the solution of PF (1) and (2), basically corresponds to the corrector step of the GRG method applied to the solution of the optimisation model (32) and (33).

Now, let rewrite the proposed MCP model as follows

$$\min \quad F(\varepsilon) \tag{39}$$

$$\text{s.t.} \quad g(\hat{x}, \varepsilon) = \begin{bmatrix} f(x, y, \varepsilon) \\ h(x, y, \varepsilon) \end{bmatrix} = 0 \tag{40}$$

$$\hat{g}(x, y) \geq 0 \tag{41}$$

where $f(x, y, \varepsilon)$ represents the equality constraints (24) and (25); $h(x, y, \varepsilon)$ represents the equality constraints (26)–(28); $\hat{g}(x, y)$ corresponds to the inequality constraints (29)–(31); and $\hat{x}$ is a vector of dependent variables defined as

$$\hat{x} = \begin{bmatrix} x \\ - \\ y \end{bmatrix} = \begin{bmatrix} \delta \\ P_S \\ |V_D| \\ Q_G \\ \hline |V_G| \\ V_{Ga} \\ V_{Gb} \end{bmatrix} \tag{42}$$

Therefore the predictor and corrector steps of the GRG method can be stated as follows, considering that $\varepsilon$ is a set of independent variables

(1) Predictor step

$$z^{m+1} = z^m + \beta s^m \tag{43}$$

$$s^m = -MM^T \nabla_z F(\varepsilon^m) \tag{44}$$

$$M = \begin{bmatrix} -[D_{\hat{x}}\, g(z^m)]^{-1} D_\varepsilon g(z^m) \\ I_{2N} \end{bmatrix}_{(2N+2N+3N_G)\times(2N+3N_G)} \tag{45}$$

where $z = (x, y, \varepsilon)$, and $\beta$ is a scalar chosen so that $F(\varepsilon^{m+1}) < F(\varepsilon^m)$ and $\hat{g}(\hat{x}^{m+1}) = \hat{g}(x^{m+1}, y^{m+1}) \geq 0$.

(2) Corrector step

$$\hat{x}^{k+1} = \hat{x}^k - \alpha[D_{\hat{x}} g(\hat{x}^k, \varepsilon^{m+1})]^{-1} g(\hat{x}^k, \varepsilon^{m+1}) \tag{46}$$

where $\alpha$ is chosen to ensure convergence, and guarantee that $\hat{g}(\hat{x}^{k+1}) = \hat{g}(x^{k+1}, y^{k+1}) \geq 0$.

Note that (46), for $\varepsilon = 0$, can be considered basically a 'new' NR solution procedure to solve the PF problem that properly models the generator voltage controls, since it accounts for the generator reactive power limits and its terminal voltage recovery.

Some commercially available solvers that use a variety of numerical techniques to solve optimisation problems, such as the NLP formulation (23)–(31), including the aforementioned GRG method, and their performance for various test-systems are discussed in the next section.

## 5 Results and discussions

### 5.1 Base model

The proposed mathematical model (23)–(31) is coded in the general algebraic modelling system (GAMS) programming platform [26]. The model is tested considering the IEEE 14-bus, 30-bus, 57-bus, 118-bus and 300-bus test systems and real 1211-bus and 2975-bus systems using the PF optimisation model (23)–(31). The 1211-bus system has 312 generators, 447 loads, 1143 lines, 622 fixed transformers; and the 2975-bus system has 374 generators, 874 loads, 273 shunts, 2146 lines and 1411 transformers. A flat start is used in all cases since this is known to yield convergence problems in 'standard' NR-based PF solvers as the system size increases, which was indeed the case when using the robust NR-based PF programs UWPFLOW [27] and DSAT [28] to solve the large practical systems from a flat start; these programs are based on a robust NR method to solve the PF equations.



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**Table 1** Execution time for polar form MCP PF model

| System | Interior-point method, s | Path method, s | GRG-based method, (s) |
|---|---|---|---|
| 14-bus | 0.125 | 0.083 | 0.078 |
| 30-bus | 0.063 | 0.145 | 0.188 |
| 57-bus | 0.275 | 0.109 | 0.516 |
| 118-bus | 0.615 | 0.625 | 8.02 |
| 300-bus | 5.187 | 5.5 | 4.047 |
| 1211-bus | 23.297 | 71.265 | non-convergent |
| 2975-bus | 162.301 | non-convergent | non-convergent |

**Table 2** Execution time for rectangular form MCP PF model

| System | Interior-point method, s | Path-based method, s | GRG-based method, s |
|---|---|---|---|
| 14-bus | 0.156 | 0.078 | 0.047 |
| 30-bus | 0.392 | 0.039 | 0.141 |
| 57-bus | 1.296 | 0.078 | 0.344 |
| 118-bus | 10.665 | 0.39 | 1.359 |
| 300-bus | 8.812 | 3.687 | non-convergent |
| 1211-bus | 94.907 | 28.39 | non-convergent |
| 2975-bus | 381.025 | non-convergent | non-convergent |

The solvers used for the studies shown here are: (i) MINOS [15], which is a GRG-based solver; (ii) PATH-NLP, a PATH-based solver; and (iii) COINIPOPT, an interior-point solver. All solvers have their parameters set at their respective default, off-the-shelf settings, so as not to bias their 'standard' performance. Major settings such as tolerance level or maximum number of iterations of the solver are by default the same for all solvers (e.g. feasibility tolerance is $10^{-6}$). The large execution times for the large systems in Tables 1–3 can be attributed in part to the overhead and non-optimality of the code used to solve the MCP model using GAMS.

The results presented in Table 1 shows the total execution time required to solve each test-system with various solvers in GAMS. Observe that for the small 14-bus, 30-bus and the 57-bus systems, and medium 118-bus and 300-bus systems, feasible and locally optimal solutions are attained in a few seconds, with all the different methods considered. However, the larger 1211-bus and 2975-bus systems are only solved by the IP method, which is a barrier method that generates a sequence of strictly feasible iterates lying in the interior of the feasible region. The GRG-based solver is only able to solve the test systems up to the 300-bus system, and does not converge for larger systems as expected, given the well-known poor convergence characteristics of this method. The PATH-based method shows reasonably good convergence for systems up to 1211-bus; however, it fails to arrive at an optimal solution for the 2975-bus system. In all the convergent cases, the objective function $F(\varepsilon)$ did not exceed a value of $10^{-7}$, thus showing that a proper solution of the PF equations was obtained.

Some of the NLP methods and associated solvers failed to yield a solution in some cases. This is because of the non-convex characteristics of the MCP model, as well as possible non-strict solutions of the complementarity constraints, which makes the problem numerically hard to solve.

The MCP solutions obtained for the IEEE test systems match closely the PF solutions reported by the standard NR-based PF solution approach, and the solutions for the real systems are basically the same as those obtained with commercial-grade PF solvers, UWPFLOW and DSAT. Hence, these solutions can be considered to be 'adequate in practice'.

### *5.2 Flexibility of the proposed model*

It is important to highlight the flexibility and adaptability of the proposed model, which easily accommodates other forms of PF representations. For example, PF analysis in rectangular coordinates, where the voltage phasor is represented as a complex number, can be carried out without the need for extensive software coding. Thus, Table 2 summarises the results of representing the PF equations in rectangular coordinates. Observe the faster convergence for the GRG-based and PATH-based solvers. For instance, for the IEEE 118-bus system, the GRG-based solver converges after 1.359 s, while it takes 8.02 s for the polar form model (Table 1). Furthermore, the PATH-based solver now converges for larger test systems, such as the 1211-bus system in 28.39 s, whereas it takes 71.265 s to converge in polar form. On the other hand, the interior-point-based solver shows worse performance with the rectangular form model, with a significant increase in the execution time.

Other advantage of the proposed MCP optimisation formulation of the PF problem is its ability to incorporate system constraints such as bus voltage limits to guarantee the quality of the solution. Thus, observe in Table 3, that imposing voltage limits [0.8, 1.2] and [0.9, 1.1] allows the solver to attain a feasible solution with the polar form. Considering the limits for the voltages reduces the feasible search area, resulting in a better search direction to find a feasible solution.

Finally, different loading conditions have been tested to demonstrate the flexibility of the model. For instance, when real power demand is increased to 91.6 MW from 21.7 MW at Bus 2 for the 30-bus test system, the standard NR method fails to obtain a solution, whereas the MCP model yields a PF solution under the same loading conditions while maintaining bus voltages within 0.95 and 1.05 at all buses, thus meeting the voltage constraints considered in this case. On another test for the same system, the standard NR solution approach fails to obtain a solution when reactive power demand is increased to 150 MVAR at Bus 21, whereas the MCP model yields a solution with a reactive power mismatch at this bus of $\varepsilon_q = 0.059$, being the largest mismatch of all PF mismatch equations, which signals to the operator that reactive power support is needed at that bus. These types of analyses are not feasible with a standard PF formulation.

**Table 3** Convergence time with different bus voltage limits, using interior-point method in polar form

| System | Bus voltage limits [0.8, 1.2], s | Bus voltage limits [0.9, 1.1], s |
|---|---|---|
| 14-bus | 0.156 | 0.109 |
| 30-bus | 0.079 | 0.074 |
| 57-bus | 2.301 | 0.625 |
| 118-bus | 4.451 | 1.764 |
| 300-bus | 17.547 | 5.797 |
| 1211-bus | 644.761 | 44.172 |
| 2975-bus | 225.016 | 419.922 |

**Table 4** Comparison of voltage quality index using different methods

| System | Proposed MCP based power flow, using IP | NR power flow solution approach |
|---|---|---|
| 14-bus | 0.04847 | 0.04864 |
| 30-bus | 0.02986 | 0.02983 |
| 57-bus | 0.02395 | 0.02370 |
| 118-bus | 0.02268 | 0.02277 |
| 300-bus | 0.02713 | 0.02555 |
| 1211-bus | 0.03597 | 0.03685 |
| 2975-bus | 0.00751 | 0.05981 |

In order to test the quality of the final converged solutions for bus voltage magnitudes, a parameter, 'voltage quality index' (VQI) is defined for each test system as follows

$$\text{VQI} = \frac{1}{N}\sum_{i=1}^{N}\left|\left|V_i\right| - V_0\right| \quad (47)$$

where $V_0$ is the desired bus voltage magnitude of 1.0 pu and $|V_i|$ is the converged bus voltage magnitude, using either the proposed MCP formulation or the standard NR-based PF solution approach. Note that a better value of this index means less loss and reactive PFs in the system, thus it is an adequate means for judging the quality of the solution. Table 4 depicts the VQI values for all test systems; observe that VQI is very small in all cases, but this index is smaller for large systems when using the proposed MCP method than when using the standard NR-based PF solution approach. Thus, it can be argued that the quality of the solutions obtained by the MCP formulation is somewhat superior to the standard NR-based approach for large systems.

### *5.3 Robustness*

The proposed model is said to be robust when a feasible and practical solution is obtained, regardless of the choice of initial guess. The robustness of the model is based on the premise that the proposed MCP formulation leads to converged PF solutions with flat-starts, when standard PF solution methods are not able to do so. Therefore since the proposed MCP leads to a converged solution with flat-starts for the large, real 1211-bus and 2975-bus systems, it is argued that the proposed method is robust.

To evaluate the convergence performance of the proposed MCP model in terms of iterations, a comparison is made between the PATH-based solver using the proposed MCP model and the robust NR-based PF solvers. As shown in Table 5, the performance of the MCP solver for small and medium sized systems is very close to the performance of the PF solvers. However, as shown in this table and Fig. 1, the proposed method converges to the solution from a flat start for large systems, whereas the PF solvers do not converge in these cases.

**Table 5** Comparison of the number of iterations between the proposed MCP model and the NR PF

| System | Proposed MCP-based power flow, using IP | NR power flow (UWPFLOW/DSAT) |
|---|---|---|
| IEEE 14-bus | 3 | 3 |
| IEEE 30-bus | 3 | 4 |
| IEEE 57-bus | 6 | 5 |
| IEEE 118-bus | 9 | 5 |
| IEEE 300-bus | 12 | 10 |
| real 1211-bus | 12 | non-convergent |
| real 2975-bus | 147 | non-convergent |

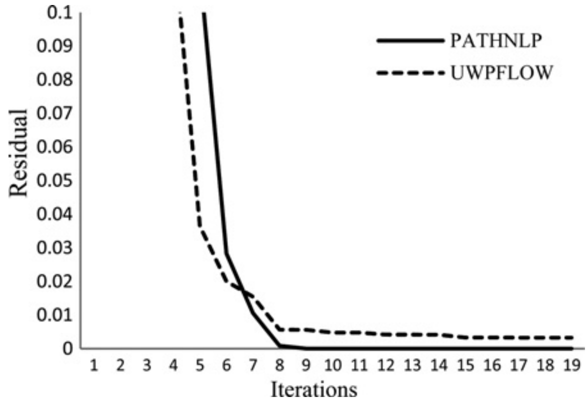


**Fig. 1** *Convergence performance of proposed MCP formulation compared to a robust NR-based solver*

It should be mentioned that restating the PF problem as the proposed optimisation model, allows the use of more sophisticated and robust optimisation solution approaches. For example, trust-region methods, which are shown to be quite robust for the solution of a variety of OPF-based problems in [29], can be used to obtain solutions of non-converging PF cases.

## 6 Conclusions

In this paper, a novel MCP model was proposed to solve the PF problem. This model was used to demonstrate that the NR-based iteration procedure is basically a step of the GRG method applied to the solution of the proposed MCP model. The optimisation was shown to have numerous benefits such as, ease of implementation, flexibility and more importantly, robustness. Thus numerical results showed that the proposed optimisation method converged when a robust NR-based PF solver failed to converge for large systems. The proposed model is now being applied to PF studies of systems with uncertain parameters, based on Affine Arithmetic techniques [30].

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