Calculating Optimal System Parameters to Maximize the Distance to Saddle-node Bifurcations

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***Abstract*—This paper presents a new methodology to calculate parameters of a nonlinear system, so that its distance to a saddle-node bifurcation is maximized with respect to the particular parameters that drive the system to bifurcation. The technique is thoroughly justified, specifying the conditions when it can be applied and the numeric mechanisms to obtain the desired solutions. A comparison is also carried out between the proposed method and a known methodology to determine closest saddle-node bifurcations in a particular power system model, showing that the new technique is a generalization of the previous method. Finally, applications to power systems are discussed, particularly regarding the design of some FACTS devices, and a simple generator-line-load example is studied to illustrate the use of the proposed technique to determine the optimal shunt and/or series compensation to maximize distances to voltage collapse. The effect of the optimal compensation on the stability of the sample system is also analyzed.**

I. Introduction

Bifurcations are well known phenomena in nonlinear systems, as many physical systems show their presence, which may drive the system to chaotic behavior in some cases [1, 2]. Of the different types of bifurcations, saddle-node bifurcations are of particular interest due to its catastrophic effects on the system [3]. In power systems (a certain class of nonlinear systems), these types of bifurcations have been shown as one of the primary causes for "static" voltage collapse problems, and hence have been thoroughly studied in many articles and reports written on the subject, e.g., [4, 5, 6, 7, 8, 9, 10, 11].

This section presents a brief review of the relevant technical literature related to the particular matter of interest to this paper, i.e., saddle-node bifurcation theory and its applications to power systems, and describes the content of the present paper.

In the power system literature, several papers have shown the direct relation between saddle-node bifurcations and voltage collapse problems, e.g., [12, 13, 14, 15]. In [12], the authors show the existence of saddle-node bifurcations in power systems, discussing their relation to voltage collapse. The authors in [13] thoroughly study how these voltage collapse problems occur in power systems by using center manifold theory of saddle-node bifurcations. In [14], the

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authors present several dynamic simulations of voltage collapse in realistic power systems, showing the relation of these events to particular singularities of the system Jacobian, i.e., saddle-node bifurcations. The author in [15] discusses the issue of load modeling and how this affects saddle-node bifurcations and its relation to voltage collapse through some academic and practical simulations.

Not all events of voltage collapse in power systems can be associated to saddle-node bifurcations, or other kinds of bifurcations for that matter [9]. Some voltage collapse problems may be caused by fast dynamic events that have nothing to do with bifurcation phenomena, such as large disturbances that push the system outside its stability region producing voltage problems. For these reasons, voltage instabilities directly related to bifurcations have been categorized as static voltage collapse problems, due to the specific way the system approaches the bifurcation point with system parameters changing in a quasi steady-state manner. On the other hand, not all static voltage collapse problems are caused by saddle-node bifurcations, as other bifurcations have also been shown to induce collapse, such as Hopf bifurcations [16] or chaotic blue sky bifurcations [17]. This paper deals only with locations of saddle-node bifurcations; thus, the scope of application in power systems of the proposed methodology is limited to this particular area of voltage stability analysis.

The idea of minimum distance to static voltage collapse or saddle-node bifurcation points in power systems is studied in [18]. The authors in that paper discuss how to calculate the distance to a closest saddle-node bifurcation in the particular parameter space of the system load, for a nonlinear system model of certain characteristics; practical implementations of this methodology are also discussed. The same concept is used by the authors in [19] to determine an optimal direction for unloading the system to avoid saddle-node bifurcation points. The idea is generalized in [20] for a variety of bifurcations of generic nonlinear systems. The current paper extends the concept of minimum or maximum distance to saddle-node bifurcations for any system model and parameter space, so that it can be used to calculate optimal parameters that maximize this distance.

The application of saddle-node bifurcation theory to determine optimal location of shunt compensation in a power system to increase the distance to static voltage collapse points is presented in [21, 22]. The authors in [23] analyze a similar problem with Static Var Compensators (SVCs [24]) applying the same methodology. In [25], the authors utilize also bifurcation theory to study the problem of locating, dimensioning, and controlling SVCs and Thyristor Control Series Capacitance (TCSCs [26, 27]) to increase the distance to a saddle-node bifurcation point. The latter paper poses the question of how to dimension these particular Flexible AC Transmission Systems (FACTS [28]) devices to increase the distance to collapse; the new methodologies proposed in this paper address that particular question.

Based on saddle-node bifurcation theory, the present paper concentrates on describing a new technique to control the location of these bifurcations by changing various system parameters, so that the distance from a particular operating point to a saddle-node bifurcation can be maximized, thus avoiding the related catastrophic events within the desired operating ranges. A simple generator-line-load example is used to illustrate the effect of shunt and series compensation on the distances to static voltage collapse, and to study the effect that changes on compensation have on system stability. The proposed technique is used to illustrate the computation of "optimal" compensation to maximize these distances in the sample system.

By increasing the distance to a saddle-node bifurcation from a given stable operating point, based on Lyapunov stability concepts for nonlinear systems [29], one could expect an improvement on the stability of the system for that operating point. This is particularly true in power systems if one observes profiles of Transient Energy Functions or TEFs (approximate Lyapunov functions [30]) for system parameter changes. In [31, 32], the authors show that, in general, the "energy" distance between an stable equilibrium point (s.e.p.) an its closest unstable equilibrium point (u.e.p.), which is a relative measure of the "size" of the stability region of the s.e.p., shows a quasi linear behavior as the loading in the system increases, reaching a value of zero at the saddle-node bifurcation point where the s.e.p. and u.e.p. merge. Hence, as the distance to a saddle-node increases by changing certain parameters in the system, one would expect a set of "parallel" TEF profiles for load changes, indicating that the size of the stability region at a given loading condition increases, thus making the system "more stable." Determining the parameter values that maximize the distance to a saddle-node bifurcation for a given system condition, may yield the maximum size of the stability region. However, it is also important to analyze the stability of the equilibrium point as the parameters of interest change to guarantee a s.e.p. at the given operating conditions, since the system may undergo Hopf bifurcations at certain parameter values. The example presented in this paper is used to illustrate these ideas by studying the stability of the test system using TEFs.

Section II presents briefly the most relevant issues of bifurcation theory for this paper, and discusses the main theorems and numerical techniques on which the new maximization methods are based. In this section, a known methodology to determine closest saddle-node bifurcations is compared to the proposed technique, showing that this is a specific case of the general method presented here. In section III, a simple power system model is used to illustrate the use of the maximization methods in the calculation of "optimal" shunt and series compensation, and to study the effect of these values on system stability with the help of TEFs. Finally, section IV highlights the new ideas presented in the paper and discusses practical applications of the new techniques to power systems, particularly in FACTS design.

II. Optimal Parameter Computation

This section discusses the basic theory behind the new proposed methodology, based on the main idea of maximizing or minimizing the distance to a saddle-node bifurcation by varying several system parameters. The theory presented here may be applied to any nonlinear dynamical system with generic saddle-node bifurcations in a known parameter space; power systems are certainly a class of these types of systems.

A brief discussion of saddle-node bifurcations and the corresponding transversality conditions is presented first, as the main results in this paper are based on these principles. The proposed techniques are then stated as theorems and formally demonstrated. A previously proposed method to directly compute a closest saddle-node bifurcation in a particular power system model is shown to be a particular case of the main general theorem. Practical implementation issues related to the use of the proposed methodology in real size systems are also discussed.

A. *Saddle-node Bifurcations*

Saddle-node bifurcations, also known as turning points or fold bifurcations, are generic codimension one local bifurcations of nonlinear dynamical systems of the form

$$\dot{x} = f(x, \lambda) \tag{1}$$

where $x \in \Re^n$ are the state variables, $\lambda \in \Re$ is a particular scalar parameter that drives the system to bifurcation in a quasi-static manner, and $f : \Re^n \times \Re \mapsto \Re^n$ is a nonlinear function [1, 2].

System (1) exhibits a saddle-node bifurcation at the equilibrium point (x_o, λ_o), i.e., $f(x_o, \lambda_o) = 0$, if the corresponding system Jacobian $D_x f|_o = D_x f(x_o, \lambda_o)$ has a unique zero eigenvalue, and the following transversality conditions hold at that particular equilibrium point [2]:

$$D_x f|_o \, v = D_x^T f|_o \, w \quad = \quad 0 \tag{2}$$

$$w^T \left. \frac{\partial f}{\partial \lambda} \right|_o \quad \neq \quad 0 \tag{3}$$

$$w^T [D_x^2 f|_o v] v \quad \neq \quad 0 \tag{4}$$

where v and w are properly normalized right and left "zero" eigenvectors in $\Re^n$ of $D_x f|_o$, respectively.

Transversality conditions (2), (3) and (4) are important as these allow, by means of center manifold theory, to describe the meaningful dynamic phenomena that control the system close to the bifurcation point [1, 2]. Thus, using the Lyapunov-Schmidt method [33], the relevant bifurcation dynamics of system (1) can be reduced, with the help of a Taylor series expansion about the bifurcation point (x_o, λ_o), to the one-dimensional center manifold

$$\dot{x}_c = \frac{1}{2} \, w^T [D_x^2 f|_o v] v \; x_c^2 + w^T \left. \frac{\partial f}{\partial \lambda} \right|_o (\lambda - \lambda_o) + o(x_c^2, \lambda - \lambda_o) \tag{5}$$

where x_c is a scalar variable resulting from a linear transformation of the original state variables x.

Saddle-node bifurcations are typically identified by a couple of equilibrium points merging at the bifurcation point as the slow varying parameter λ approaches a maximum local value λ_o. The way this phenomenon takes place depends on the approximate center manifold equation (5). Hence, the signs and magnitudes of conditions (3) and (4) determine locally how the system bifurcates.

Local saddle-node bifurcations are generic, i.e., they are expected to occur in nonlinear systems with one slow varying parameter, as opposed to other types of "singular" local bifurcations such as transcritical and pitchfork, which require certain specific symmetries in the system to occur [2]. In particular, systems that have a constant $\partial f / \partial \lambda$ vector, which is the typical case in power systems with constant active and reactive power or PQ load models, can be expected to bifurcate through a saddle-node, as condition (3) is generically met [16].

Typical power system models present an additional difficulty, as these are systems represented by a set of differential and algebraic equations (DAE) of the form

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} f(x, y, \lambda) \\ g(x, y, \lambda) \end{bmatrix} = F(x, y, \lambda) \tag{6}$$

where $x \in \Re^n$ and $\lambda \in \Re$ stand for the state variables and slow varying parameter, respectively, and $f : \Re^n \times \Re^m \times \Re \mapsto \Re^n$ corresponds to the nonlinear vector field; the vector $y \in \Re^m$ represents the set of algebraic variables defined by the nonlinear algebraic function $g : \Re^n \times \Re^m \times \Re \mapsto \Re^m$. The stability of these types of systems is thoroughly discussed in [34]. The main idea is that if $D_y g(x, y, \lambda)$ can be guaranteed to be nonsingular along system trajectories of interest, the behavior of system (6) along the given trajectories is primarily determined by the local reduction

$$\dot{x} = f(x, y^{-1}(x, \lambda), \lambda)$$

where $y = y^{-1}(x, \lambda)$ comes from applying the Implicit Function Theorem to the algebraic constraints $g(x, y, \lambda) = 0$ on the trajectories of interest. The reader is referred to [35, 36] for a full discussion on the cases where the nonsingularity of $D_y g(x, y, \lambda)$ cannot be guaranteed over the full range of operating conditions.

The nonsingularity assumption of the algebraic constraints Jacobian is used in [37] to demonstrate that the transversality conditions at a saddle-node bifurcation point (x_o, y_o, λ_o) for system (6) are equivalent to

$$\begin{aligned} D_{(x,y)} F|_o \, \hat{v} = D^T_{(x,y)} F|_o \, \hat{w} &= 0 \\ \hat{w}^T \left. \frac{\partial F}{\partial \lambda} \right|_o &\neq 0 \\ \hat{w}^T [D^2_{(x,y)} F|_o \hat{v}] \hat{v} &\neq 0 \end{aligned}$$

where $\hat{v}$ and $\hat{w}$ are the corresponding right and left zero eigenvectors of $D^T_{(x,y)} F|_o$, properly normalized. Thus, the methods described in this paper also apply to DAE systems under a nonsingularity assumption for $D_y g|_o$.

B. *Maximizing the Distance to Bifurcation*

As saddle-node bifurcations have catastrophic effects on the stability of nonlinear systems, mechanisms must be developed to avoid them. One possible way of avoiding these stability problems would be to maximize the distance to the saddle-node bifurcation with respect to the slow varying parameter λ. This maximization can be achieved by changing other system parameters, say p, in the nonlinear system

$$\dot{x} = f(x, p, \lambda) \tag{7}$$

where $x \in \Re^n$, $p \in \Re^m$ with $m \leq n$ (typically $m \ll n$), $\lambda \in \Re$, and $f : \Re^n \times \Re^m \times \Re \mapsto \Re^n$. The idea then is to calculate p so that the value of λ at a saddle-node bifurcation is maximized within certain operating conditions. The following lemma presents the necessary conditions to determine such local maximum of λ:

Lemma 2.1: Assume that for system (7), as defined above, (x_o, p_o, λ_o) is a saddle-node bifurcation. Then, the necessary conditions for this equilibrium point to locally maximize the distance $|\lambda_o - \lambda_*|$, where λ_* is an initial value of λ, are

$$\begin{aligned} f|_o = f(x_o, p_o, \lambda_o) &= 0 \\ D^T_x f|_o w_o &= 0 \\ D^T_p f|_o w_o &= 0 \\ w_o^T \left. \frac{\partial f}{\partial \lambda} \right|_o - (\lambda_o - \lambda_*) &= 0 \end{aligned} \tag{8}$$

The vector $w_o \in \Re^n$ is a properly normalized left eigenvector of the unique zero eigenvalue of the system Jacobian $D_x f|_o = D_x f(x_o, p_o, \lambda_o)$.

Proof: This lemma can be demonstrated using a similar approach to the one discussed in [18, 20], i.e., show that equations (8) are a solution to the following optimization problem:

$$
\begin{aligned}
\text{Min.} \quad & -\frac{1}{2}(\lambda - \lambda_*)^2 \\
\text{s.t.} \quad & f(x, p, \lambda) = 0 \\
& D_x^T f(x, p, \lambda)\; w = 0
\end{aligned}
\tag{9}
$$

where w is assumed to be a properly normalized nonzero vector. Observe that the optimization constraints in (9) force the solution to be a saddle-node bifurcation of system (7), as the first condition guarantees an equilibrium point, whereas the second one requires the system Jacobian to be singular for a nonzero left eigenvector w.

To solve optimization problem (9), the following Lagrangian function can be used:

$$
\mathcal{L}(x, p, \lambda, w, \mu, \gamma, \xi) = -\frac{1}{2}(\lambda - \lambda_*)^2 + \mu^T f(x, p, \lambda) + \gamma^T D_x^T f(x, p, \lambda) w \tag{10}
$$

where $\mu \in \Re^n$ and $\gamma \in \Re^n$ are known as the Lagrange multipliers. Hence, a necessary condition for a saddle-node bifurcation point (x_o, p_o, λ_o) to be a solution of (9) is [38]

$$
\nabla \mathcal{L}(x_o, p_o, \lambda_o, w_o, \mu_o, \gamma_o) = 0 \tag{11}
$$

which can be rewritten as

$$
\begin{aligned}
\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_o &= -(\lambda_o - \lambda_*) + \mu_o^T \left.\frac{\partial f}{\partial \lambda}\right|_o + \gamma_o^T {D_{x\lambda}^T}^2 f|_o w_o = 0 \\
D_x \mathcal{L}|_o &= \mu_o^T D_x f|_o + \gamma_o^T {D_x^T}^2 f|_o w_o = 0 \\
D_w \mathcal{L}|_o &= \gamma_o^T D_x^T f|_o = 0 \\
D_p \mathcal{L}|_o &= \mu_o^T D_p f|_o + \gamma_o^T {D_{xp}^T}^2 f|_o w_o = 0 \\
D_\mu \mathcal{L}|_o &= f|_o = 0 \\
D_\gamma \mathcal{L}|_o &= D_x^T f|_o w_o = 0
\end{aligned}
$$

Now, since by assumption (x_o, p_o, λ_o) is a saddle-node bifurcation point, conditions (2), (3) and (4) apply at this point. Therefore, *a solution* that satisfies the saddle-node bifurcation transversality conditions and optimality conditions (9) can be shown to be $\mu_o = w_o$, $\gamma_o = 0$, and hence

$$
\begin{aligned}
f|_o &= 0 \\
D_x^T f|_o w_o &= 0 \\
D_p^T f|_o w_o &= 0 \\
w_o^T \left.\frac{\partial f}{\partial \lambda}\right|_o - (\lambda_o - \lambda_*) &= 0
\end{aligned}
$$

Observe that the last equation ensures a proper normalization of w (transversality condition (3)). $\square$

Dobson et al. in [18] proposed a set of equations similar to the following set:

$$
\begin{aligned}
f(x_o, \lambda_o) &= 0 \\
D_x^T f(x_o)\, w_o &= 0 \\
(\lambda_o - \lambda_*)^T + w_o^T D_\lambda f &= 0
\end{aligned} \tag{12}
$$

to compute a closest bifurcation point (x_o, λ_o) in the load parameter space for a power system model with constant power loads of the form

$$
\dot{x} = f(x, \lambda) = \tilde{f}(x) + K\lambda \tag{13}
$$

where $x \in \Re^n$ are the state variables, $\lambda \in \Re^\ell$ $(\ell \leq n)$ represents the system loads with initial conditions λ_*, together with a constant matrix $K \in \Re^{n\times\ell}$. Hence, $f : \Re^n \times \Re^\ell \mapsto \Re^n$ has the particular property of $D_\lambda f = K$, with $D_x f(x)$ being only a function of x. Equations (12) can be simply shown to be a solution to the necessary optimality conditions of the following optimization problem:

$$
\begin{aligned}
Min. \quad & \frac{1}{2}(\lambda - \lambda_*)^T(\lambda - \lambda_*) \\
s.t. \quad & f(x, \lambda) = 0 \\
& D_x^T f(x)\, w = 0
\end{aligned} \tag{14}
$$

for a properly normalized left eigenvector w at the bifurcation point. In [18, 20], the proposed equations are actually a slight modified version of (12), as a scaling factor $k \in \Re$ affecting w_o is introduced to define $q = k(\lambda_o - \lambda_*) = D_\lambda^T f\, w_o$, which is shown to be a normal vector to the hypersurface formed in $\Re^\ell$ by the set of saddle-node bifurcation points of system (13). The equation $||q|| = 1$ is added to (12) in order to compute k, and to ensure that $||q|| \neq 0$ during the solution process.

The methodology proposed in the current paper is of different nature than Dobson's, as equations (8) are used to solve a different problem than (12). The latter set is based on a system where all parameters λ are capable of driving the system to bifurcation, i.e., no particular parameter can be identified as the one taking the system to a saddle-node bifurcation point, and hence the user is interested in addressing the issue of which particular values of all these parameters yield an "optimal" location of the saddle-node bifurcation point. Equations (8), on the other hand, are used to solve the problem of finding values of "non-bifurcating" parameters p that guarantee and optimal location of a saddle-node bifurcation in terms of one bifurcating parameter λ. Thus, (8) can be used to calculate optimal shunt and/or series compensation to maximize distance to a saddle-node bifurcation in a particular direction of load variation, as shown in the examples below. The differences are made clear by simply comparing the optimization problem that each technique tries to solve, i.e., (9) and (14).

From this comparison, Theorem 2.1 below is proposed. This theorem addresses the issue of calculating values of parameters $p \in \Re^m$ for an optimal saddle-node bifurcation location when several bifurcation parameters $\lambda \in \Re^\ell$ are of interest. In other words, equations (8) and (12) are integrated in one set.

Theorem 2.1: Consider the following nonlinear system:

$$\dot{x} = f(x, p, \lambda)$$

where $x \in \Re^n$ represents the state variables, and $p \in \Re^m$ ($m \leq n$) and $\lambda \in \Re^\ell$ ($\ell \leq n$) stand for the system parameters, with λ driving the nonlinear vector field $f : \Re^n \times \Re^m \times \Re^\ell \mapsto \Re^n$ to a saddle-node bifurcation at (x_o, p_o, λ_o) from the initial value λ_*. Then, the necessary conditions for (x_o, p_o, λ_o) to locally maximize the distance $||\lambda_o - \lambda_*||$ are

$$\begin{aligned} f|_o &= 0 \\ D_x^T f|_o w_o &= 0 \\ D_p^T f|_o w_o &= 0 \\ D_\lambda^T f|_o w_o - (\lambda_o - \lambda_*) &= 0 \end{aligned} \tag{15}$$

The last equation properly normalizes the vector $w_o \in \Re^n$, which is the left eigenvector of the system Jacobian $D_x f|_o = D_x f(x_o, p_o, \lambda_o)$ for the corresponding unique zero eigenvalue.

Proof: As in Lemma 2.1, the proof of this theorem is simply based on showing that equations (15) are the necessary optimality conditions for the following minimization problem:

$$\begin{aligned} Min. \quad & -\frac{1}{2}(\lambda - \lambda_*)^T(\lambda - \lambda_*) \\ s.t. \quad & f(x, p, \lambda) = 0 \\ & D_x^T f(x, p, \lambda)\, w = 0 \end{aligned}$$

with properly normalized "zero" eigenvector w. Thus, the following Lagrangian function can be defined:

$$\mathcal{L}(x, p, \lambda, w, \mu, \gamma, \xi) = -\frac{1}{2}(\lambda - \lambda_*)^T(\lambda - \lambda_*) + \mu^T f(x, p, \lambda) + \gamma^T D_x^T f(x, p, \lambda) w$$

Based on transversality conditions (2), (3) and (4) at the saddle-node bifurcation point (x_o, p_o, λ_o), the Lagrange multipliers that solve the necessary optimality conditions $\nabla \mathcal{L}|_o = 0$, can be shown to be once more $\mu_o = w_o$ and $\gamma_o = 0$. These values lead to equations (15), where the last equation ensures a nonzero w_o based on transversality condition (3). □

It is important to highlight the fact that the method presented in this paper, i.e., equations (15), pose no restrictions with regard to the system model and its corresponding set of parameters λ and p, and hence could be applied to any nonlinear system displaying saddle-node bifurcations where the bifurcating parameters are clearly identified. For example, equations (15) could be used to determine closest saddle-node bifurcations for power systems with voltage dependent load models.

Although equations (15) allow to compute the values of system parameters λ and p that maximize the distance to a saddle-node bifurcation point, one is especially interested in calculating the values of p that maximize the distance to the *closest* bifurcation point in λ space. To solve this particular problem with the help of these equations, the initial guesses for the Newton-Raphson based methods used to compute the corresponding numerical solutions have to be chosen with care. The next section discusses this issue with more detail, and an iterative method is presented so that the desired solutions may be obtained. Examples that thoroughly illustrate this particular problem are presented in section III.

C. *Practical Implementation Issues*

Equations (15) have the problem of requiring a solution of a $2n+m+\ell+1$ nonlinear problem, which for large values of n, m and ℓ, the typical case in the study of large power systems, could pose a significant difficulty. Furthermore, as numerical solutions are usually obtained through the use of the Newton-Raphson method or variations, "good" initial guesses are required.

The problem of adequate initial guesses can be addressed based on typical initial values of p, which can be calculated from standard, non-optimal design techniques. Hence, for the case of a scalar λ, continuation or direct methods can be used to find a saddle-node bifurcation for a given p, using the numerical techniques extensively described and tested in [22]. If λ is multidimensional, the methods described in [18, 39, 20] and tested in [40] can be used to obtain a closest bifurcation point for the given parameter vector p. These techniques yield somewhat proper initial values for x, λ and w, depending on how far the systems is to the optimal solution point. Thus, if the system is relatively close, equations (15) readily produce the desired solution. However, if the initial guess is far from the actual solution, inadequate solutions may be obtained or convergence problems may arise, particularly if operational limits are considered. Although interior point methods could be used to improve the handling of system limits (inequality constraints) when solving (15), as demonstrated in [41] for the computation of saddle-node bifurcations in reactive power dispatch problems, if the initial guesses are far from the optimal solution, the problem may converge slowly to the desired solution or simply converge to an undesirable solution. For these types of problems, alternative methods should be applied.

Continuation methods are iterative numerical techniques that may be used to help solve equations (15) when proper initial conditions are not available. These types of methods also reduce the dimension of the problem, making it more manageable. In [20, 39], the authors discuss an iterative procedure to solve (12), which is based on the idea that vector $q = D_\lambda^T f w$ is perpendicular to the hypersurface of saddle-node bifurcation points. In this technique, the parameters λ are iteratively changed by solving a standard bifurcation problem using the direction given by the vector q at any particular iteration; this method is shown to converge to the desired solution. This technique may be considered to try to solve equations (15) for specific values of the parameters p; however, a method must be devised to iteratively change the parameters p.

An alternative method is proposed here that addresses the problem of how to vary the parameters p and λ at the same time. The technique is based on the generalized reduced-gradient (GRG) method, which is a well-known nonlinear optimization technique [38, 42], and which could be viewed as a continuation method, since the basic principles of these two numerical methods are essentially the same, i.e., move using predictor-corrector steps in an "optimal" direction on a given constraint manifold based on tangent information at a known point on this manifold. Indeed, when only one system parameter is free to change, continuation methods can be readily and formally demonstrated to be equivalent to GRG methods by restating the bifurcation problem as an optimization one. The authors in [43] propose a similar method to solve equations (12), based on the special structure of nonlinear system (13). The idea is that, once an appropriate saddle-node bifurcation point is calculated, this point can be used to move in a particular direction on the hypersurface of bifurcation points using predictor and corrector steps. Based on this general principle, the following continuation algorithm is proposed to help solve the problem stated in equations (8):

1. *Problem statement*: In order to apply the GRG method, the problem has to be represented as an optimization problem, i.e.,

$$\begin{aligned} Min. \qquad & F(z) = -\frac{1}{2} \parallel \lambda - \lambda_* \parallel_2 \\ s.t. \qquad & c(z) = \begin{bmatrix} f(x,p,\lambda) \\ D_x^T f(x,p,\lambda)\, w \end{bmatrix} = 0 \end{aligned} \tag{16}$$

   where z stands for all the system variables and parameters x, w, p and λ. A nontrivial solution that satisfies the equality constraints is assumed to be available, so that $||w|| \neq 0$. This initial condition can be calculated based on the methodologies previously described, i.e., continuation or direct methods with fixed p and a scalar λ. Now, since $c : \Re^r \mapsto \Re^t$, with $r > t$ ($r = 2n + m + 1$ and $t = 2n$), the optimization variables have to be split into dependent variables z_1 and independent variables z_2, where $z_1 \in \Re^t$ and hence $z_2 \in \Re^{(r-t)}$. In (16), the choice is apparently obvious, as x and w are basically defined by the equality constraints, whereas p and λ are independent system parameters. However, in order to ensure that $||w|| \neq 0$ during the solution process, an entry $w_k \in w$ is added to z_2, and a parameter $p_l \in p$ is then moved to z_1, i.e.,

$$z_1 = \begin{bmatrix} x \\ \tilde{w} \\ p_l \end{bmatrix} \qquad z_2 = \begin{bmatrix} w_k \\ \tilde{p} \\ \lambda \end{bmatrix}$$

   where $\tilde{w}$ is the eigenvector w without the corresponding w_k entry, and $\tilde{p}$ is the parameter vector p with the entry p_l removed.

2. *Predictor step*: Assuming that a set of initial values for z satisfying the constraints is known, say z^k such that $c(z^k) = 0$, the initial vector z^k is changed by taking a step s^k in the direction of the steepest descent on the tangent plane to $c(z) = 0$ at z^k. Thus,

$$\begin{aligned} z^{k+1} &= z^k + \alpha s^k \\ s^k &= -ZZ^T \nabla_z F(z^k) \\ Z &= \begin{bmatrix} -D_{z_1}^{-1} c(z^k) D_{z_2} c(z^k) \\ I_{r-t} \end{bmatrix} \end{aligned} \tag{17}$$

   The scalar $\alpha > 0$ is used to change the size of the step so that $F(z^{k+1}) < F(z^k)$, and to guarantee that the corrector step yields a solution, as discussed below. The matrix $Z \in \Re^{r \times (r-t)}$ constitutes a basis for the null space of $D_z c(z^k) \in \Re^{t \times r}$, i.e., $D_z c(z^k) Z = 0$. Furthermore, based on the transversality conditions of a saddle-node bifurcation point, the matrix $D_{z_1} c(z^k) \in \Re^{t \times t}$ can be guaranteed to be nonsingular [44, 45].

3. *Corrector step*: The value z^{k+1} predicted by "linearized" step (17), must now be corrected to ensure that the constraints are satisfied. Thus, the following modified Newton-Raphson iteration is used to obtain a solution for z_1 so that $c(z) = c(z_1, z_2^{k+1}) = 0$, by using as an initial guess $z_1^1 = z_1^{k+1}$ and keeping z_2^{k+1} constant, i.e., w_k and all system parameters but one are kept fixed:

$$z_1^{i+1} = z_1^i - \beta D_{z_1}^{-1} c(z_1^i, z_2^{k+1})\, c(z_1^i, z_2^{k+1}) \tag{18}$$

where $i = 1, 2, \ldots$ stands for the iteration number. The scalar constant $\beta > 0$ is used to ensure a solution to $c(z) = 0$ [46]. Hence, if no solution is encountered as the value of β is reduced up to certain reasonable tolerance, the vector z^{k+1} generated by the predictor step (17) is changed by reducing the value of α. This process is repeated until convergence of (18) is attained.

The predictor step (17) and corrector step (18) are taken sequentially until $F(z)$ is minimized within an acceptable tolerance.

Observe that the proposed continuation method requires repetitive solutions of $c(z) = 0$, which basically translates into larger CPU times, as is the case with any continuation method when compared to its direct counterpart [22]. Nevertheless, the dimension of the problem is significantly reduced, from $2n + m + 1$ to $2n$; furthermore, inadequate initial guesses do not pose such a significant problem as in the case of direct methods.

In this type of technique is relatively simple to introduce operational limits, as these can be enforced "on-the-fly" as the solution is approached; however, one must be aware of the possibility of reaching certain limits that significantly change the "shape" of the saddle-node bifurcation manifold (as defined by $c(z) = 0$), which could generate nontrivial changes in the corresponding tangent surfaces, resulting on convergence and solution difficulties during the continuation process. A simple example of this type of problem in standard one-parameter bifurcation studies of power systems is the "Q-limit" problem [22] or "Q-limit instabilities" [47]. Finally, as with any GRG-based method, one must be also aware of the possibility of "zigzagging" during the computation process [38], which considerably slows down the convergence process as demonstrated on one of the examples below.

When λ is multidimensional, i.e., $\lambda \in \Re^\ell$ ($\ell > 1$), the proposed continuation method must be modified to allow for the tracing of the proper constraint manifold, so that the final solution meets the requirement of being the maximum, closest saddle-node bifurcation on λ space. This can be accomplished by changing the constraints $c(z) = 0$ in (16) to include the closest saddle-node bifurcation conditions, i.e.,

$$c(z) = \begin{bmatrix} f(x, p, \lambda) \\ D_x^T f(x, p, \lambda)\, w \\ D_\lambda^T f(x, p\lambda)\, w + (\lambda - \lambda_*) \end{bmatrix} = 0 \tag{19}$$

and then apply the same predictor and corrector steps with

$$z_1 = \begin{bmatrix} x \\ w \\ \lambda_1 \\ p_1 \end{bmatrix} \qquad z_2 = \begin{bmatrix} \lambda_2 \\ p_2 \end{bmatrix}$$

Observe that in this case one needs not to worry about w becoming zero, as the third constraint in (19) guarantees a properly normalized nonzero eigenvector according to transversality condition (3). Thus, some of the system parameters in λ and p are used as the independent variables $z_2 \in \Re^m$. The system parameters that are chosen for z_2 depend on the specific problem under study. From the problem set-up, a possible generic choice may be $z_2 = p$; however, the idea is to structure z_2 so that zigzagging is avoided. A particular set of independent parameters is suggested for the power system example discussed in the next section, based on several numerical tests and other heuristic criteria.

III. Examples

The simple generator-line-load model of Fig. 1 is used to illustrate the use of the proposed techniques, as thorough calculations and intuition can be utilized to justify the results. The dynamic equations used to represent this system correspond to a standard quasi-steady state phasor model typically employed in transient stability studies. Thus, the generator is modeled using a second order representation of the mechanical system; the transmission system, which includes the internal generator reactance, the transmission line and the series and shunt compensation, is represented by a Π impedance model; the load is modeled as a constant active and reactive power (PQ) load with dynamic frequency and voltage terms [15, 37]. These element models yield the following per-unit equation set:

$$\begin{aligned} \dot{\omega} &= \frac{1}{M}[P_m - P_{e1}(\delta, V) - D_G\omega] \\ \dot{\delta} &= \omega - \frac{1}{D_L}[P_{e2}(\delta, V) - P_d] \\ \dot{V} &= \frac{1}{\tau}[Q_e(\delta, V) - Q_d] \end{aligned} \tag{20}$$

where

$$\begin{aligned} \delta &= \delta_1 - \delta_2 \\ P_{e1}(\delta, V) &= G - V(G\cos\delta - B\sin\delta) \\ P_{e2}(\delta, V) &= -V^2G + V(G\cos\delta + B\sin\delta) \\ Q_e(\delta, V) &= -V^2(B - B_C) - V(G\sin\delta - B\cos\delta) \\ G &= \frac{R}{R^2 + (X_L - X_C)^2} \\ B &= \frac{X_L - X_C}{R^2 + (X_L - X_C)^2} \end{aligned}$$

The state variable ω stands for the angular speed deviation of the generator with respect to the reference speed (e.g., synchronous speed), and the mechanical power P_m is regularly changed to supply for system losses along system trajectories, i.e., $P_m = P_m(\delta, V)$, to guarantee the typical condition that ω is zero at all equilibrium points. The latter is not necessary if transmission losses are neglected, i.e., $P_m = Pd$ for $G = 0$, or if the generator is assumed to be an infinite bus, i.e., the generator inertia $M = \infty$ and ω is a constant, which is a typical assumption. The variables D_G, D_L and τ stand for various system constants that are of no particular interest for the studies presented in this paper. No operational limits, such as generator or transmission line current limits, are considered here, as these would only clutter the results with unnecessary information. Nevertheless, there is nothing particular about the proposed methodologies that would prevent users from applying them to systems with all limits considered; this is especially true for the proposed continuation technique.

In system (20), the state variables are $x = [\omega\ \delta\ V]^T$, and the system parameters that one is interested in computing to optimize the bifurcating parameters $\lambda = [P_d\ Q_d]^T$, are $p = [B_C\ X_C]^T$. In other words, the proposed techniques are used to calculate the optimal shunt and series compensation to maximize the minimum distance to collapse $\| \lambda \|_2 = \sqrt{P_d^2 + Q_d^2}$. If a particular direction of load increase is chosen by fixing the load power factor, so that $Q_d = k\, P_d$ for a given scalar value k, then the suggested methodologies can be used to compute the values of B_C and X_C that maximize the active power $\lambda = P_d$ delivered to the load.

A Transient Energy Function for a particular s.e.p. $(0, \delta^o, V^o)$ may be defined along any system trajectory as follows [30, 31]:

$$\vartheta(\omega, \delta, V) = \frac{1}{2} M\omega^2 - B(V\cos\delta - V^o\cos\delta^o) + \frac{1}{2}(B - B_C)(V^2 - V^{o2}) - P_d(\delta - \delta^o) + Q_d \ln\left(\frac{V}{V^o}\right) \quad (21)$$

This scalar function is used to determine the relative "size" of the stability area by evaluating its value at the corresponding u.e.p. (this sample system has only two valid equilibrium points with $V > 0$ for a given set of parameter values). Based on classical Lyapunov stability criteria applied to power systems, the larger the value of the "energy distance" between the s.e.p. and corresponding u.e.p., the more "stable" the system is [31]. Thus, in this paper, the effect that the system parameters P_d, Q_d, B_C, and X_C have in the stability of the system is studied by monitoring the profiles of this energy distance as these parameters change.

Observe that for $\lambda = [P_d\ Q_d]^T$, $D_\lambda f = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^T$, whereas for a scalar parameter $\lambda = P_d$, $\partial f/\partial\lambda = [0\ \ -1\ \ -k]^T$. Hence, since these partial derivatives are constant matrices, "singular" bifurcations of system (20) are expected to be of the saddle-node type, as $w^T D_\lambda f \neq 0$ in general. Furthermore, for this system, given the condition of $\omega \equiv 0$ at equilibrium points, saddle-node bifurcations can be detected by looking for singular solutions of the following set of equations (power flow equations):

$$F(\hat{x}, p, \lambda) = \begin{bmatrix} -V^2 G + V(G\cos\delta + B\sin\delta) - P_d \\ -V^2(B - B_C) - V(G\sin\delta - B\cos\delta) - Q_d \\ G - \dfrac{R}{R^2 + (X_L - X_C)^2} \\ B - \dfrac{X_L - X_C}{R^2 + (X_L - X_C)^2} \end{bmatrix} = 0 \quad (22)$$

where $\hat{x} = [\delta\ V\ G\ B]$. This can be easily justified by observing that equilibrium points of system (20) correspond to solutions of equations (22), and that singularities of $D_x f|_o$ correspond to singularities of $D_{\hat{x}} F|_o$.

For per-unit values of $R = 0.1$ and $X_L = 0.5$, and no compensation, i.e., $B_C = X_C = 0$, Fig. 2 depicts the bifurcation diagram obtained by solving (22) for several values of P_d when $Q_d = 0.25 P_d$ ($k = 0.25$). The corresponding saddle-node (SN) bifurcation obtained by a direct method is $P_{do} = 0.66614$ ($Q_{do} = 0.16654$), $\delta_o = 32.327°$, and $V_o = 0.59171$. This "initial" bifurcation point is depicted in (P_d, Q_d) space in Fig. 3, which illustrates the hypersurface of saddle-node bifurcation points when both P_d and Q_d are allowed to change, and is used to solve equations (12) to obtain the closest bifurcation point $P_{do}^c = 0.09615$, $Q_{do}^c = 0.48077$, $\delta_o^c = 0°$, and $V_o^c = 0.5$. Whether these bifurcation points are feasible or not, from the point of view of "reasonable" values of P_d, Q_d, δ and V, it is not an issue in this paper, as the sample system is mainly theoretical and only used to illustrate some of the possible applications of the proposed techniques.

The question now is to determine optimal values of B_C and/or X_C to maximize the value of $\lambda_o = P_{do}$ in the given load direction $Q_d = 0.25P_d$, or to maximize the "distance" $\| \lambda \|_2$ to a closest bifurcation point $\lambda_o^c = [P_{do}^c \; Q_{do}^c]^T$. The next sections discuss these issues in detail. Notice, however, that these compensation values are calculated here with the idea of "optimizing" λ, and no other considerations are made with regard to the actual feasibility of these values in a real system, as these values might be "too expensive" to actually implement or could create some additional dynamic problems such as sub-synchronous resonance. Some of these additional constraints may be readily included in the optimization problem as inequality constraints; however, some constraints may be difficult to consider during the optimization process and may require of additional steady-state and dynamic studies.

A. *Optimal Shunt Compensation*

First, an optimal value of B_C is computed for the given load direction $Q_d = 0.25P_d$, assuming that there is no series compensation, i.e., $X_C = 0$. Thus, for a lossless system ($R = 0$), one can see by inspection that if $B_C = B = 2.0$, the system has no limits on the amount of power P_d that can be transferred, i.e., the system does not have a saddle-node bifurcation ($P_{do} \rightarrow \infty$). This simple analysis yields an initial value for B_C to be used in the solution process of equations (15).

For the sample system, Fig. 4 depicts the effect of changing the parameter B_C on the saddle-node bifurcation values of P_d ($Q_d = 0.25P_d$), δ and V. This figure was obtained by a slight modification of the proposed continuation method, so that the hypersurface of saddle-node bifurcation points can be traced around the desired optimal value. This modification consists on simply detecting when the optimal value of the objective function has been "surpassed" by monitoring and comparing the values of the state variables and parameters at each step, so that the sign of the α in (17) can then be changed to move on the opposite direction to the optimal solution, thus generating the desired manifold. A maximum value of $P_{do} = 2.5$ ($Q_{do} = 0.625$), at $(\delta_o, V_o) = (78.690°, 2.5495)$, is calculated for $B_{Co} = 2.0192$ using equations (15), based on the initial guess obtained with $B_C = 2.0$. In this case, the direct method does not fail to converge for a "flat" start (saddle-node bifurcation point for $B_C = 0$); however, this is not the situation for all other examples depicted in the paper.

The continuation process of equations (17) and (18) takes about 25 steps from the flat start to converge within a 10^{-4} tolerance of the optimal solution $P_{do} = 2.5$, yielding the value $B_{Co} \approx 2.01468$. The same convergence tolerance of 10^{-4} for the objective function is used for all continuation examples in the paper. Observe that the steps taken are all part of the manifold of saddle-node bifurcations, and hence are useful as they give the user a sense of how the parameters p affect the location of these bifurcations.

If both P_d and Q_d are free to change, then one is interested in calculating the optimal value for B_C to maximize the distance to the closest bifurcation point. Figure 5 illustrates the problem; this figure depicts several hypersurfaces formed by the saddle-node bifurcation points in different parameter spaces. The plots clearly show how the bifurcation values of the parameters $\lambda_o = [P_{do}\ Q_{do}]^T$ are affected by the values of shunt compensation B_C, and in particular illustrate the effect of B_C on the closest bifurcation points λ_o^c. Notice that the maximum distance $\| \lambda_o^c \|_2$ occurs again at $B_C \approx 2$, as expected. This shunt compensation value is used to calculate an initial closest bifurcation point to solve the direct method equations (15), which yield the desired maximum closest bifurcation point $(P_{do}^c, Q_{do}^c) = (2.5, 0)$ at $(\delta_o^c, V_o^c) = (78.69°, 2.54951)$, for $B_{Co}^c = 1.92308$. This method fails to converge for a flat initial guess. The proposed continuation method with modified constraints (19), converges to $B_{Co}^c \approx 1.91878$ in 24 steps from a flat start.

B. *Optimal Series Compensation*

In this case, the optimal series compensation is computed assuming that $B_C = 0$. Once again, by simple inspection, a value of $X_C = X_L = 0.5$ (100% compensation) would impose no limits on the maximum amount of power P_d that can be transmitted from the generator to the load in a lossless system. This value is used to compute an initial guess for equations (15), for both $\lambda = P_d$ and $\lambda = [P_d\ Q_d]^T$; flat starts fail to converge.

Figure 6 shows the saddle-node bifurcation values of P_d, δ and V for $Q_d = 0.25P_d$ as the series compensation X_C is changed. The direct method yields the maximum value of $P_{do} = 2.5$ ($Q_{do} = 0.625$) at $X_{Co} = 0.5250$, $(\delta_o, V_o) = (-14.04°, 0.51539)$. The continuation technique yields $X_{Co} \approx 0.52423$ in approximately 46 steps from a flat start; the large number of steps are not due to zigzagging, but rather due to small steps taken close to the solution point, as the predictor step's α must be significantly reduced during the corrector step to obtain convergence.

For $\lambda = [P_d\ Q_d]^T$, the direct method yields $X_{Co}^c = 0.5$ for $P_{do}^c = 2.5$, $Q_{do}^c = 0$ at $(\delta_o, V_o) = (0°, 0.5)$. This point corresponds to the maximum closest bifurcation depicted in Fig. 7. In spite of zigzagging, the continuation method converges from a flat start to $X_{Co}^c \approx 0.50089$ in 18 steps.

C. *Optimal Shunt and Series Compensation*

If both B_C and X_C are allowed to change, the optimization problem has multiple solutions, and hence estimating the optimal solutions is not as simple as in the previous cases.

For $Q_d = 0.25P_d$, Fig. 8 depicts the saddle-node bifurcation values of P_d for several levels of shunt and series compensation. Observe that there is no unique solution to this problem, i.e., there are several values of B_C and X_C that yield the maximum value $P_{do} = 2.5$ ($Q_{do} = 0.625$). Thus the solution to the problem would depend on the initial conditions, with the direct and continuation methods yielding different results for the same starting point. For example, the continuation method yields in 12 steps the solution $B_{Co} \approx 6.2423$, for $X_{Co} = 0.4$, whereas the direct method, for the same starting point on the saddle-node bifurcation hypersurface, yields $B_{Co} = 6.4030$, $X_{Co} = 0.4204$ at $(\delta_o, V_o) = (38.52°, 0.6391)$. In this case, the best optimal values for B_C and X_C may be chosen based on their feasibility, in terms of costs, dynamic considerations, and the values of δ_o and V_o.

Finally, Fig. 9 depicts a hypersurface of closest bifurcation points for several values of shunt and series compensation. Once again, notice the multiple optimal solutions, all corresponding to $P_{do}^c = 2.5$ and $Q_{do}^c = 0$. The direct methods yields the optimal solution $B_{C_o}^c = 1.17424$, $X_{C_o}^c = 0.48809$ at $(\delta_o^c, V_o^c) = (6.791°, 0.50353)$, for initial conditions $X_C = 0.5$ and $B_C = 2$; this method diverges for a flat start. The continuation technique yields $B_{C_o}^c \approx 3.96528$, $X_{C_o}^c \approx 0.44890$ in 57 steps (due to zigzagging) for a flat start, and fails to converge to an optimal solution for the initial condition used in the direct method. The continuation method is implemented using $z_2 = [P_d \; X_C]^T$, which were chosen due to the sensitivity of the system to these two parameters.

D. *Stability Analysis for Shunt Compensation*

In order to study the effect of different compensation levels in system stability, the case of shunt compensation changes for a particular direction of load increase, i.e., $\lambda = P_d$, is analyzed here, as there is intuitive and detailed knowledge of the expected system behavior in this case. Thus, if the system resistance is ignored, one can expect a set of quasi parallel profiles of the TEF defined in (21) for different values of B_C, similar to what is depicted in Fig. 10 for $0 \leq B_C \leq 1.5$ The larger the value of B_C, the larger the stability region for a given value of P_d, and therefore the more stable the system, up to the maximum value $B_C = 2$. Furthermore, the system cannot undergo Hopf bifurcations in this case, as demonstrated in [48], since the system Jacobian presents a symmetric structure.

For the sample system, where the transmission system resistance has been chosen unusually large ($X_L/R = 5$) to allow for more interesting results and more complex system behavior, Fig. 10 depicts the effect of different B_C values on the TEF profiles. This figure clearly shows the effect of B_C on system stability; initially, as B_C increases, so does the stability region, as expected, until it reaches a maximum before the optimal compensation $B_{C_o} = 2.0192$. Furthermore, by monitoring the system eigenvalues, the system undergoes a Hopf bifurcation at $B_C = 1.923$, close to the optimal compensation value. Similar results are obtained for series compensation.

Although the presented results show that "maximum" system stability is not necessarily reached at optimal compensation for the sample system, one can expect closer results for more realistic values of R. Thus, the proposed methodologies, and particular the continuation process, certainly yield important information that allows to determine the "optimal" compensation for each system.

IV. Conclusions and Applications

A new technique to calculate system parameters to maximize the distance to saddle-node bifurcations has been presented. The technique has been thoroughly analyzed and justified for general nonlinear systems based on bifurcation theory, demonstrating that the new method is a generalization of a previously proposed technique to calculate closest saddle-node bifurcations. Practical numerical implementation issues have been discussed, and a continuation method has been proposed to obtain reasonable solutions when system size and/or initial conditions pose a problem. Finally, a simple power system has been thoroughly analyzed to illustrate the application of the new methods and study the effect of different compensation levels in system stability, as measure by the TEF method.

This paper presents a solution to the problem of how to calculate system parameters to move away from saddle-node bifurcations. In power systems, this is certainly an important issue, especially when determining the amount of shunt and series compensation needed in a system, since one may expect in general, as demonstrated for the sample system in the paper, that increasing compensation up to an "optimal" value leads to increased system stability. FACTS devices, especially SVCs and TCSCs, could be dimensioned using the proposed techniques. Moreover, one could envision the use of this new methodology in the design of "slow" controls that dynamically change the values of the SVCs and TCSCs reactance to move the system away from static voltage collapse problems; tests in larger systems should lead to a better understanding of this particular problem. Other possible areas of application in power systems could be in the determination of optimal, closest saddle-node bifurcations for diverse steady state load models and various system parameters.

The concepts introduced here present new ideas and related methodologies to determine optimal compensation levels in power systems. However, since power systems show rather complicated stability behavior, as demonstrated by the stability analysis of the sample system, one must use these tools together with other analysis techniques to be able to form a full picture of the behavior of the system, in a similar manner as what is currently done with any other analysis tools used to study, operate and design power systems.

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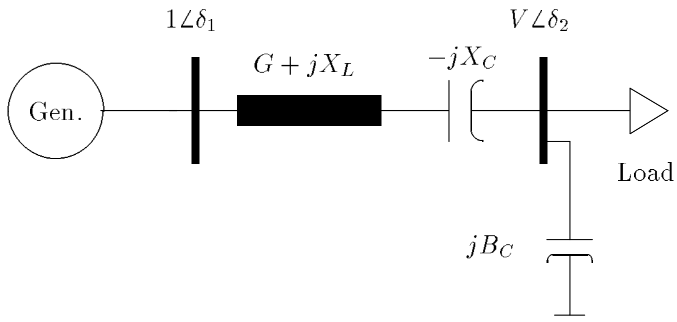


Fig. 1. Sample system.

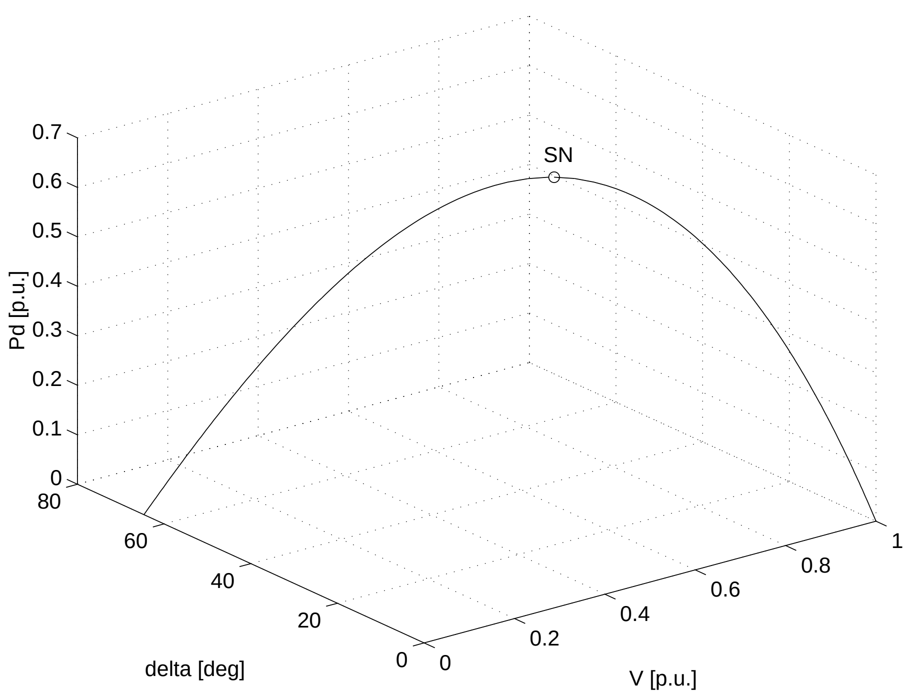


Fig. 2. Bifurcation manifold in state-parameter space when $\lambda = P_d$. SN stands for the saddle-node bifurcation point.

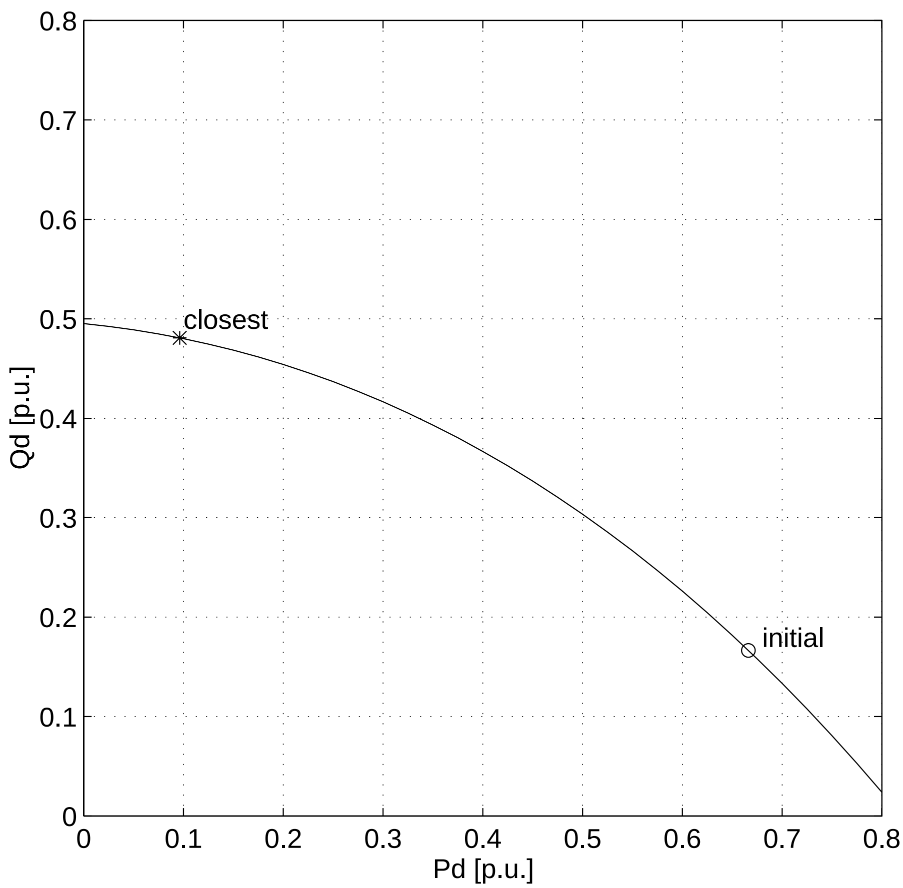
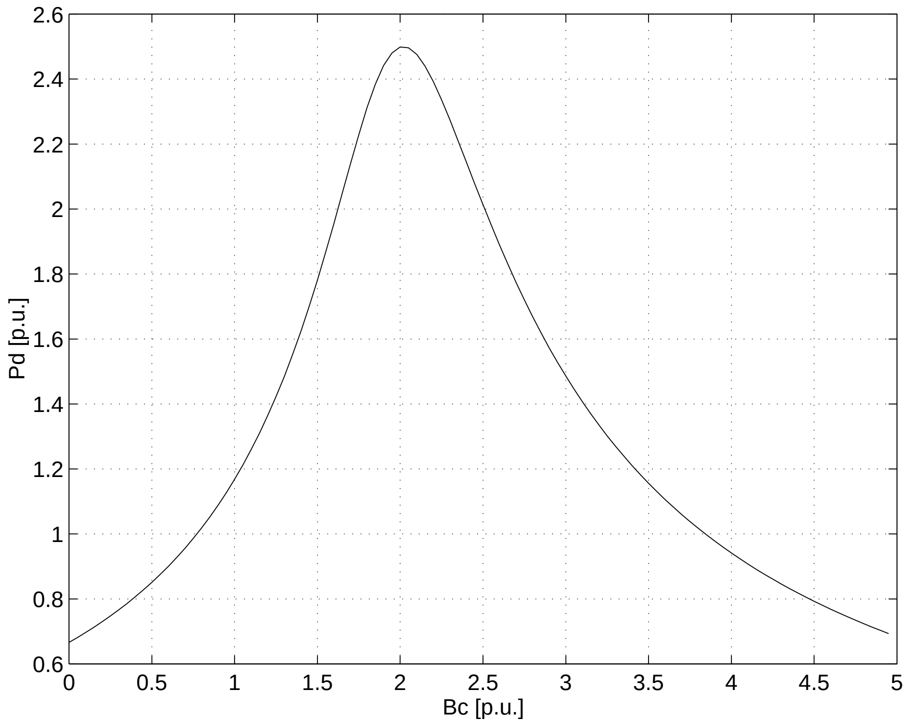
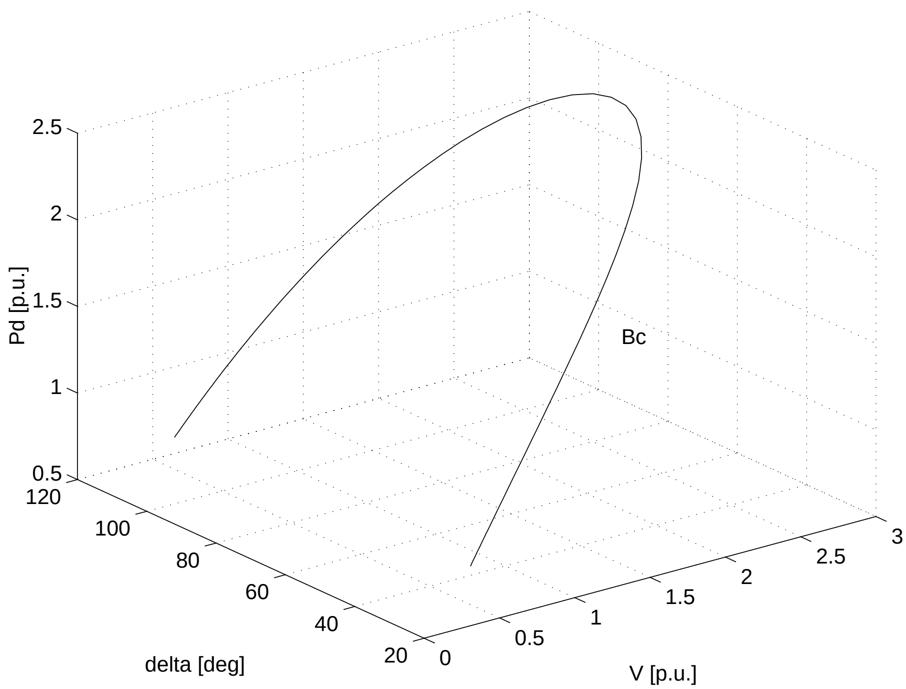


Fig. 3. Hypersurface of saddle-node bifurcations when $\lambda = [P_d \, Q_d]^T$.

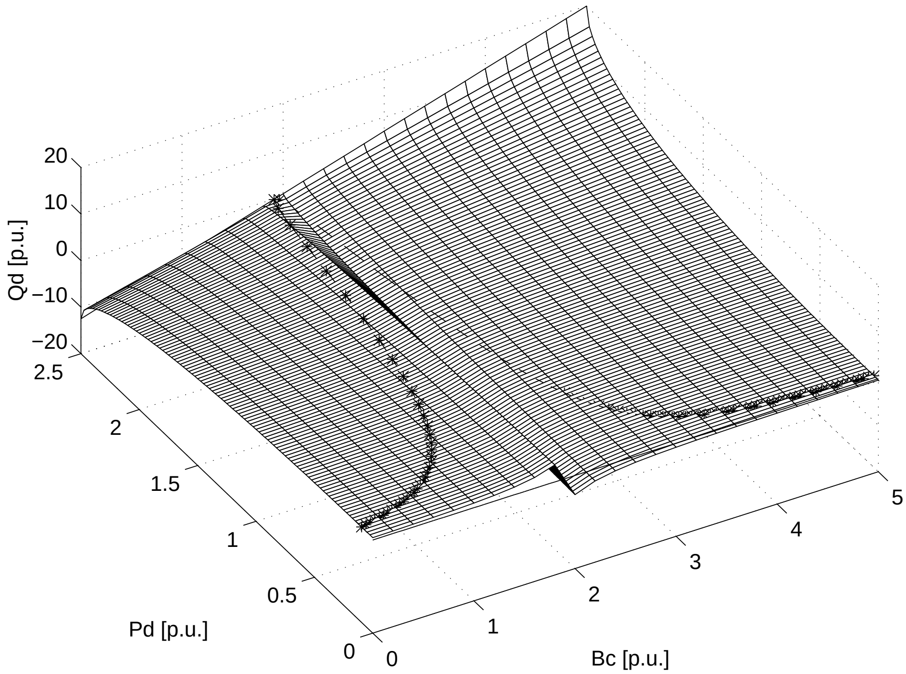


(a)



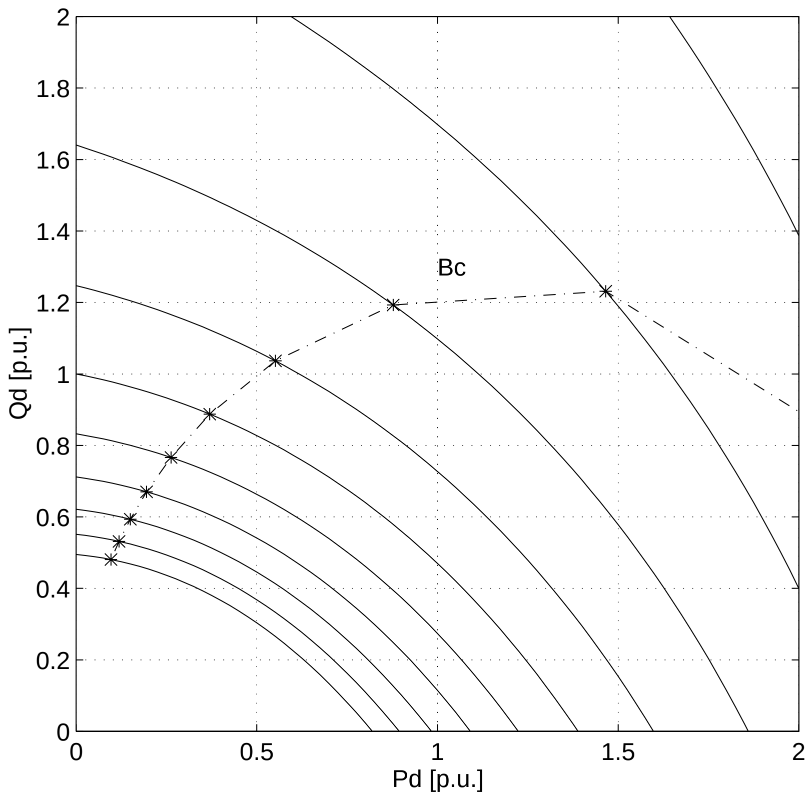
(b)

Fig. 4. Saddle-node bifurcations as shunt B_C compensation changes for $\lambda = P_d$: (a) Parameter values at bifurcation points; (b) bifurcation points in state-parameter space for values of B_C from 0 to 5 p.u.



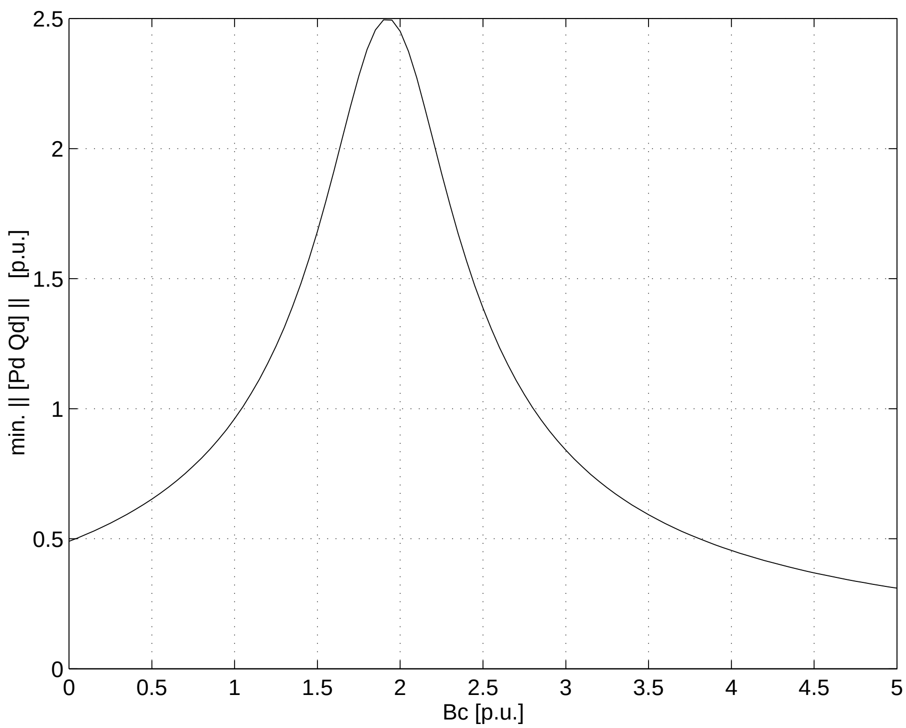
Qd [p.u.]
20
10
0
−10
−20
2.5
2
1.5
1
0.5
0
Pd [p.u.]
0
1
2
3
4
5
Bc [p.u.]

(a)



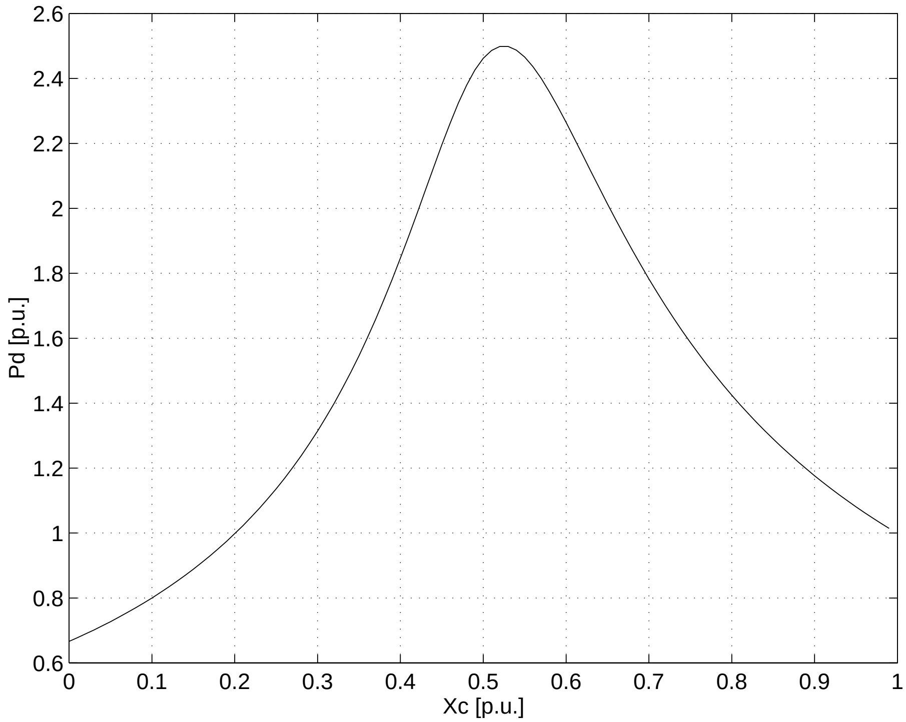
Qd [p.u.]
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Bc
0
0.5
1
1.5
2
Pd [p.u.]

(b)

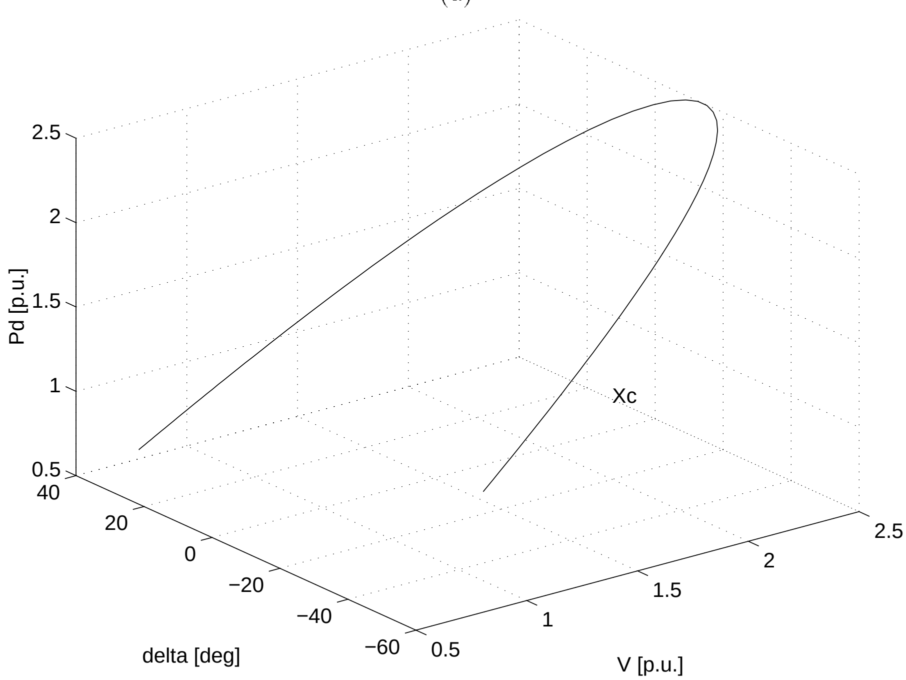


(c)

Fig. 5. Saddle-node bifurcation hypersurfaces for $\lambda = [P_d \; Q_d]^T$ with shunt B_C compensation changes: (a) Bifurcation points in parameter space with closest bifurcations marked with $*$; (b) bifurcation points in λ space with closest bifurcations marked with $*$ for values of B_C from 0 to 1.6 p.u.; (b) minimum value of $\| \lambda \|_2$ (objective function).



(a)



(b)

Fig. 6. Saddle-node bifurcations as series X_C compensation changes for $\lambda = P_d$: (a) Parameter values at bifurcation points; (b) bifurcation points in state-parameter space for values of X_C from 0 to 1 p.u.

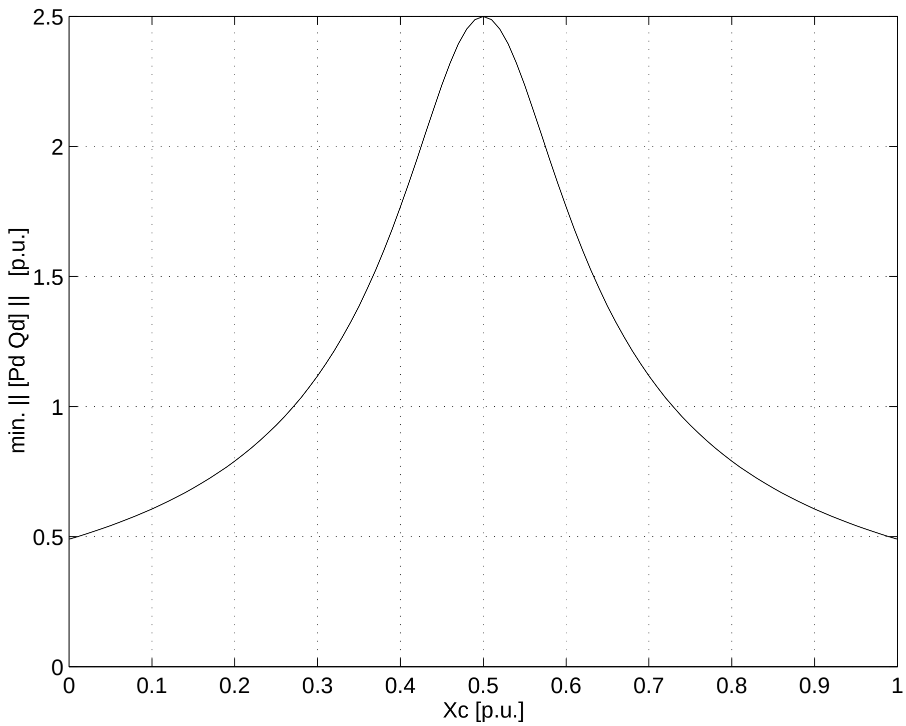


Fig. 7. Minimum $\| \lambda \|_2$ as series X_C compensation changes.

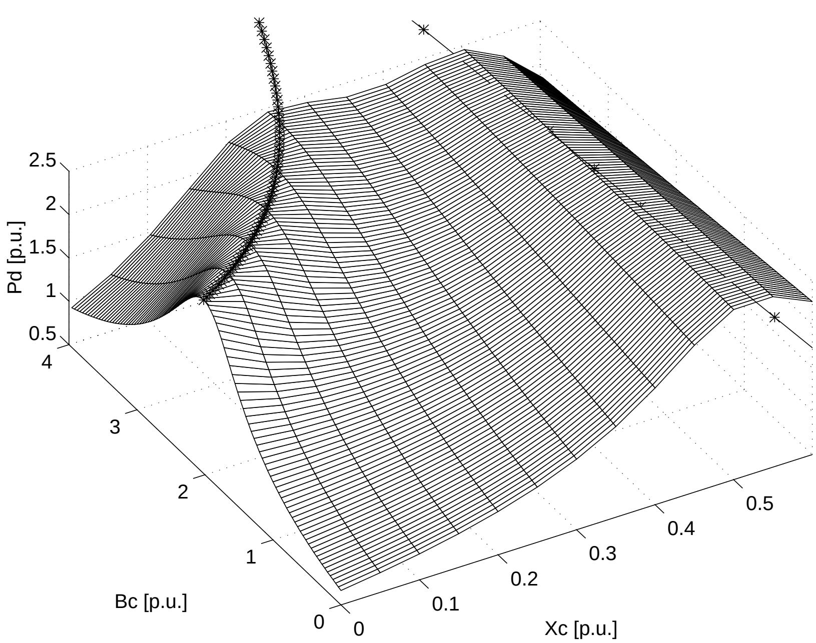


Fig. 8. Hypersurface of saddle-node bifurcations in parameter space. Optimal bifurcations are marked with $*$.

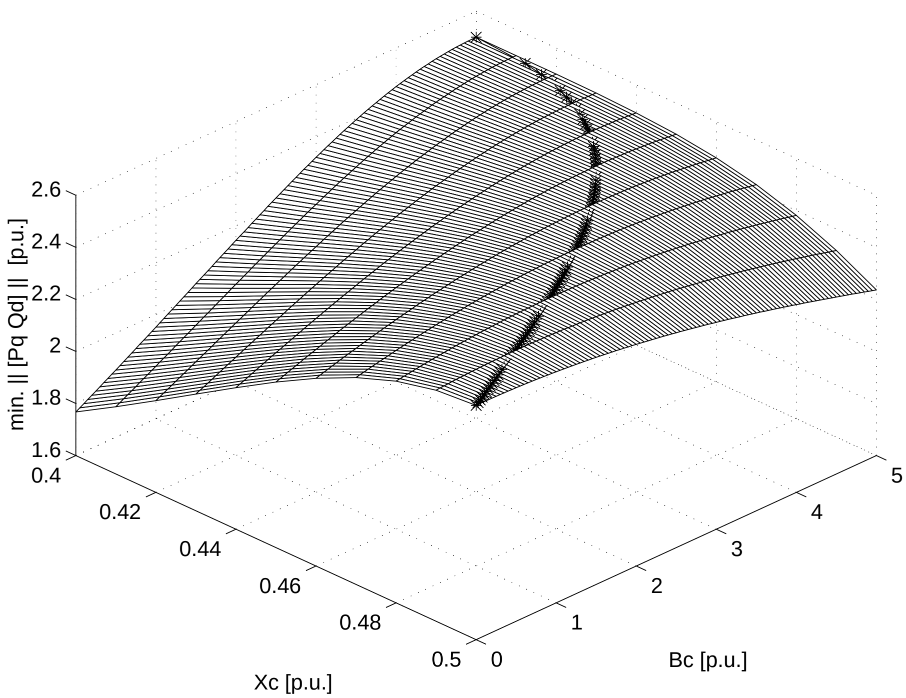


Fig. 9. Minimum $\| \lambda \|_2$ as series X_C and shunt B_C compensation changes. Optimal closest bifurcation points are marked with $*$.

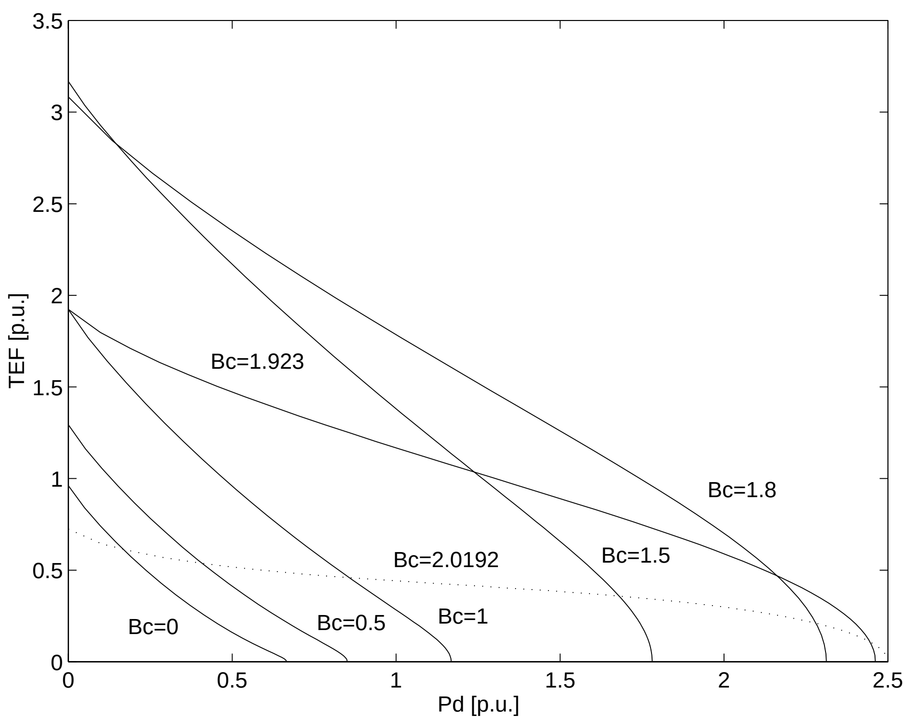


Fig. 10. Transient Energy Function profiles for the closest u.e.p. at different levels of shunt compensation B_C as $\lambda = P_d$ changes. The system undergoes a Hopf bifurcation at $B_C = 1.923$.