Improvements on the star discrepancy of (t, s)-sequences

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Abstract

The term low-discrepancy sequences is widely used to refer to s-dimensional sequences X for which the bound $D^*(N, X) \le c_s(\log N)^s + O((\log N)^{s-1})$ is satisfied, where D^* denotes the usual star discrepancy. In this article, we are concerned with (t, s)-sequences in base b, one of the most famous families of low-discrepancy sequences along with Halton sequences. The constants c_s for (t, s)-sequences were first computed by Sobol' and Faure in special cases and then achieved in general form by Niederreiter in the eighties. Then, quite recently, Kritzer improved these constants for $s \ge 2$ by a factor $1/2$ for an odd base and $b/(2(b+1))$ for an even base $b \ge 4$. Our aim is to further improve this result in the case of an even base by a factor $2[(b+1)/b][(b-1)/b]^{s-1}$ $(s \ge 2)$, hence obtaining the ratio $3/2^{s-1}$ for $b = 2$. Combining this estimate with best known t-values from the database MinT, we obtain new values of c_s for Niederreiter-Xing sequences where base 2 recovers supremacy on base 3 (in Kritzer's

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paper). Our proof relies on a method of Atanassov to bound the discrepancy of Halton sequences. We also investigate $(t, 1)$-sequences for which the approach of Kritzer does not work.

1 Introduction

In the number theoretic setting, low-discrepancy sequences are usually compared against each other by looking at the asymptotic behavior of bounds on their discrepancy. More precisely, the term low-discrepancy sequences refers to sequences X satisfying the bound $D^*(N, X) \leq c_s(\log N)^s + O((\log N)^{s-1})$ (see Section 1.1). A better understanding of the properties of such bounds provides useful insights on the corresponding sequences in applications to numerical analysis and to (quasi-) Monte Carlo methods. Among many publications on both these areas, the reader can consult the recent monographs [3, 12, 18] and [14] for a complete overview.

Two well-known families of multi-dimensional low-discrepancy sequences are the Halton sequences [9] and the so-called (t, s)-sequences introduced by Sobol' [21], Faure [5] and generalized by Niederreiter [13]. Recently, there has been a renewed interest in Halton sequences due to an important improvement discovered by Atanassov [1]. His results provided a drastic change in our understanding of how bounds on the discrepancy vary with the dimension s for these sequences (for readers who do not have access to the paper [1], we recommend [3, Theorem 3.36] where a detailed proof of [1, Theorem 2.1] is given in a slightly less general form and [23] for a complete investigation of the proofs of Theorem 2.1 and Theorem 2.2 from [1]). For (t, s)-sequences in dimension $s \geq 2$, another recent improvement was provided by Kritzer in [11], who was able to reduce by a factor of about two the leading constant c_s obtained in [13]. His results are presented in Section 1.3 below. In the same paper, Kritzer also states a conjecture for (t, s)-sequences in even bases that would substantially improve his own bounds. A careful comparison of these improved bounds — through the constant c_s — for both Halton sequences and (t, s)-sequences can be found in [7, Section 2.3].

One of the objectives of our study is to closely approach the conjecture of Kritzer for even bases (within a factor of about 1/2). To this end, we deepen the process from [8] which consists in using Atanassov's method for Halton sequences to obtain discrepancy bounds for (t, s)-sequences. We also provide a new result for the discrepancy of $(t, 1)$-sequences and extend to a larger class of (t, s)-sequences the bounds presented in [8, 11].

The paper is organized as follows. The rest of this first section is used to cover background material on discrepancy bounds for (t,s)-sequences together with previously known results on c_s. Section 2 is devoted to the one-dimensional case, where we provide a new result for the discrepancy bound of $(t,1)$-sequences. Section 3 describes how to use Atanassov's approach to study the discrepancy of (t,s)-sequences (in the "broad sense", a terminology to be defined shortly), closely following the recent study [8]. The important improvement mentioned above for even bases b is given in Section 4, along with some numerical comparisons showing how this result improves upon the previously established best bounds from [11].

1.1 Discrepancy

We start with a review of the notion of discrepancy, which will be used throughout the paper. Various types exist but here, for short, we only consider the so-called *extreme discrepancy*, which corresponds to the worst case error in the domain of complexity of multivariate problems. Assume we have a point set $\mathcal{P}_N = \{X_1, \dots, X_N\} \subseteq I^s := [0,1]^s$ and denote $\mathcal{J}$ (resp $\mathcal{J}^*$) the set of intervals J of I^s of the form $J = \prod_{j=1}^{s}[y_j, z_j)$, where $0 \le y_j < z_j \le 1$ (resp. $J = \prod_{j=1}^{s}[0, z_j)$). Then the *discrepancy function* of $\mathcal{P}_N$ on such an interval J is the difference

$$E(J;N) = A(J;\mathcal{P}_N) - NV(J),$$

where $A(J;\mathcal{P}_N) = \#\{n; 1 \le n \le N, X_n \in J\}$ is the number of points in $\mathcal{P}_N$ that fall in the subinterval J, and $V(J) = \prod_{j=1}^{s}(z_j - y_j)$ is the volume of J.

Then, the *star (extreme) discrepancy* D^* and the *(extreme) discrepancy* D of $\mathcal{P}_N$ are defined by

$$D^*(\mathcal{P}_N) = \sup_{J \in \mathcal{J}^*} |E(J;N)| \quad \text{and} \quad D(\mathcal{P}_N) = \sup_{J \in \mathcal{J}} |E(J;N)|.$$

It is well known that $D^*(\mathcal{P}_N) \le D(\mathcal{P}_N) \le 2^s D^*(\mathcal{P}_N)$. For an infinite sequence X, we denote by $D(N,X)$ and $D^*(N,X)$ the discrepancies of its first N points. Note that several authors have a $1/N$ factor when defining the above quantities.

A sequence satisfying $D^*(N,X) \in O((\log N)^s)$ is typically considered to be a *low-discrepancy sequence*. But the constant hidden in the O notation needs to be made explicit to make comparisons possible across sequences. This is achieved in many papers with an inequality of the form

$$D^*(N,X) \le c_s (\log N)^s + O((\log N)^{s-1}). \tag{1}$$

The constant c_s in this inequality is the main object of our study, as in many other papers dealing with theoretical comparisons of low-discrepancy sequences.

1.2 Review of (t, s)-sequences

The concept of (t, s)-sequences has been introduced by Niederreiter [13] to give a general framework for various constructions including Sobol' sequences [21], Faure sequences [5], and later a more general class of constructions referred to as Niederreiter-Xing sequences [17].

Definition 1. Given an integer $b \geq 2$, an *elementary interval* in I^s is an interval of the form $\prod_{i=1}^{s}[a_i b^{-d_i}, (a_i + 1)b^{-d_i})$ where a_i, d_i are nonnegative integers with $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$.
Given integers t, m with $0 \leq t \leq m$, a (t, m, s)*-net in base* b is an s-dimensional set with b^m points such that any elementary interval in base b with volume b^{t-m} contains exactly b^t points of the set.
An s-dimensional sequence $(X_n)_{n\geq1}$ in I^s is a (t, s)*-sequence* if the subset $\{X_n : kb^m < n \leq (k+1)b^m\}$ is a (t, m, s)-net in base b for all integers $k \geq 0$ and $m \geq t$.

In order to give sense to new important constructions, Tezuka [22] and then Niederreiter and Xing [16, 17] introduced a new definition using the so-called truncation operator that we now define.

Truncation : Let $x = \sum_{i=1}^{\infty} x_i b^{-i}$ be a b-adic expansion of $x \in [0, 1]$, with the possibility that $x_i = b - 1$ for all but finitely many i. For every integer $m \geq 1$, we define the m-truncation of x by $[x]_{b,m} = \sum_{i=1}^{m} x_i b^{-i}$ (depending on x via its expansion). In the case where $X \in I^s$, the notation $[X]_{b,m}$ means an m-truncation is applied to each coordinate of X.

Definition 2. An s-dimensional sequence $(X_n)_{n\geq1}$, with prescribed $b-$adic expansions for each coordinates, is a (t, s)*-sequence (in the broad sense)* if the subset $\{[X_n]_{b,m}; kb^m < n \leq (k + 1)b^m\}$ is a (t, m, s)-net in base b for all integers $k \geq 0$ and $m \geq t$.

The former (t, s)-sequences are now called (t, s)-sequences *in the narrow sense* and the others just (t, s)-sequences (Niederreiter-Xing [17], Definition 2 and Remark 1); in this paper, we will sometimes use intentionally the expression *in the broad sense* to emphasize the difference.

1.3 Review of bounds for the discrepancy of (t, s)-sequences

Bounds for the discrepancy of (t, s)-sequences in the narrow sense have been established by Niederreiter in [13] with constant c_s in (1):

$$c_s^{Ni} = \frac{b^t}{s!} \frac{b-1}{2\lfloor \frac{b}{2} \rfloor} \left(\frac{\lfloor \frac{b}{2} \rfloor}{\log b} \right)^s . \tag{2}$$

In the same paper an improved bound with $c_s^{Ni} = \frac{b^t}{s} \left(\frac{b-1}{2 \log b} \right)^s$, is obtained for $s = 2$, for $s = 3$ and $b = 2$, and for $s = 4$ and $b = 2$.

More recently, Kritzer [11], still for (t, s)-sequences in the narrow sense and in dimension $s \geq 2$, obtained constants

$$c_s^{Kr} = \frac{b^t}{s!} \frac{b-1}{2(b+1)} \left(\frac{b}{2 \log b} \right)^s \text{ when } b \text{ is an even base and} \tag{3}$$

$$c_s^{Kr} = \frac{b^t}{s!} \frac{1}{2} \left(\frac{b-1}{2 \log b} \right)^s \text{ when } b \text{ is an odd base,} \tag{4}$$

hence improving the constants from [13] by a multiplicative factor 1/2 for an odd b and $\frac{b}{2(b+1)}$ for an even b (apart from the special cases where $b = 2$ and b even, $b \geq 4$ when $s = 2$).

Further, in the same paper, Kritzer states the conjecture that, when the base b is even, the constant c_s for $s \geq 2$ should be

$$c_s^{conj} = \frac{b^t}{s!} \frac{b^2}{2(b^2-1)} \left(\frac{b-1}{2 \log b} \right)^s . \tag{5}$$

As for (t, s)-sequences (in the broad sense), Niederreiter and Xing [16] showed that constant c_s^{Ni} in (2) is still valid, but Kritzer in [11] did not take into account this generalization.

2 The one-dimensional case

In his paper [11], Kritzer obtains improved bounds for dimensions $s \geq 2$. His method is based on a result for $(t, m, 2)$-nets (shown in [2]), and thus it cannot be used to get corresponding improved bounds in one dimension. Hence, this section is devoted to the special case $s = 1$, for which the corresponding bounds are obtained in Corollary 1 (note the slight subtlety in the case of an even base). Furthermore, we believe that Theorem 1 and Corollary 1 below are likely to be the starting point for proving Kritzer's conjecture. But until now we could not manage to insert our better bounds in one dimension in our proof using the method of Atanassov. See the remarks at the end of Sections 2 and 4 for further details. Before presenting our results (Section 2.2), we must review known facts about the discrepancy of one-dimensional sequences (Section 2.1).

2.1 Review of van der Corput sequences

Generalized van der Corput sequences have been introduced in [4] to improve the behavior of the original van der Corput sequence in base 2.

Definition 3. Let $b \geq 2$ be an integer. For integers n and N with $n \geq 1$ and $1 \leq N \leq b^n$, write $N-1 = \sum_{r=0}^{\infty} a_r(N)\ b^r$ in the b-adic system (so that $a_r(N) = 0$ if $r \geq n$) and let $\Sigma = (\sigma_r)_{r\geq 0}$ be a sequence of permutations of $\{0, 1, \ldots, b-1\}$. Then the *generalized van der Corput sequence in base b associated with* Σ is defined by

$$S_b^{\Sigma}(N) = \sum_{r=0}^{\infty} \frac{\sigma_r\big(a_r(N)\big)}{b^{r+1}}.$$

If $\sigma_r = \sigma$ for all $r \geq 0$, we write $S_b^{\Sigma} = S_b^{\sigma}$. The original van der Corput sequence in base b, S_b^{id}, is obtained with the identical permutation id.

The study of van der Corput sequences involves two more precise notions of discrepancy defined as follows:

$$D^+(N, X) = \sup_{0\leq\alpha\leq 1} E([0,\alpha); N; X) \qquad \text{and}$$
$$D^-(N, X) = \sup_{0\leq\alpha\leq 1} (-E([0,\alpha); N; X)).$$

These discrepancies D^+ and D^- are linked to the preceding ones by the relations $D = D^+ + D^-$ and $D^* = \max(D^+, D^-)$.

Reminders on asymptotic results for S_b^{id}. We only recall here the results about van der Corput sequences useful for the statement of Corollary 1.

For any permutation σ, there exists an effectively computable constant α_b^{σ} such that

$$\limsup_{N\to\infty} \frac{D(N, S_b^{\sigma})}{\log N} = \frac{\alpha_b^{\sigma}}{\log b} \text{ and } D(N, S_b^{\sigma}) \leq \frac{\alpha_b^{\sigma}}{\log b}\log N + \alpha_b^{\sigma} + 2 \text{ for all } N \geq 1.$$

Moreover, for any sequence of permutations Σ, we have

$$D(N, S_b^{\Sigma}) \leq D(N, S_b^{\mathrm{id}}) = D^*(N, S_b^{\mathrm{id}}) \text{ for all } N \geq 1,$$

i.e., the original van der Corput sequence S_b^{id} is the worst distributed among S_b^{Σ} sequences with respect to both the discrepancy D and the star discrepancy D^*. Since

$$\frac{\alpha_b^{\mathrm{id}}}{\log b} = \frac{b-1}{4\log b} \text{ if } b \text{ is odd and } \frac{\alpha_b^{\mathrm{id}}}{\log b} = \frac{b^2}{4(b+1)\log b} \text{ if } b \text{ is even,}$$

these constants are the worst possible leading constants for the discrepancies D and D^* of S_b^{Σ} sequences. All these results come from [4].

Reminders on van der Corput sequences towards $(0, 1)$-sequences. These results come from [6]. First, recall the relation between these two families:

Generalized van der Corput sequences S_b^{Σ} are $(0, 1)$-sequences (in the broad sense). Here, the truncation is required for sequences S_b^{Σ} such that $\sigma_r(0) = b - 1$ for all sufficiently large r [6, Proposition 3.1].

Next, a result about the star discrepancy: The original van der Corput sequences S_b^{id} are the worst distributed with respect to the star discrepancy among all $(0, 1)$-sequences X_b (in the broad sense) and in arbitrary base $b \geq 2$, that is [6, Theorem 5.1]

$$D^*(N, X_b) \leq D^*(N, S_b^{\mathrm{id}}) = D(N, S_b^{\mathrm{id}}) \text{ for all } N \geq 1.$$

This theorem generalizes the same result from Kritzer [10] which was restricted to $(0, 1)$-sequences (in the narrow sense).

Moreover, in the same paper examples are given showing that van der Corput sequences S_b^{id} are *not* the worst distributed with respect to the discrepancy D among all $(0, 1)$-sequences X_b (in the broad sense) for any base b and among all digital $(0, 1)$-sequences X_2 (in the broad sense) for base 2, see [6, Theorems 5.3 and 5.2] for more details.

2.2 On the star discrepancy of $(t, 1)$-sequences

We extend the above theorem to $(t, 1)$-sequences in base b, hence getting the worst possible leading constants c_1 for $(t, 1)$-sequences.

Theorem 1. *For any base b, the original van der Corput sequences are the worst distributed with respect to the star discrepancy among all $(t, 1)$-sequences X_b^t (in the broad sense) that is, for all $N \geq 1$*

$$D^*(b^t N, X_b^t) \leq b^t D^*(N, S_b^{\mathrm{id}}) = b^t D(N, S_b^{\mathrm{id}}).$$

Proof. The proof closely follows the proof of [6, Theorem 5.1] with the sequence $S_b^{\mathrm{id},t}$ consisting of b^t copies of S_b^{id} instead of the sequence S_b^{id}. For the sake of completeness, we give the main steps in the following.
Let be given an integer $m \geq 0$ and the collection of elementary intervals of length b^{-m}. By construction (see [4] Property 3.1.2 with $\sigma_{n-1} = \mathrm{id}$), the van der Corput sequence S_b^{id} is the unique $(0,1)$-sequence whose points are always placed, in increasing order, as left as possible within the given collection of elementary intervals and satisfy the elementary interval property.

Let τ be the permutation defined by $\tau(k) = b-1-k$ for $0 \leq k \leq b-1$. Then, by construction (see [4] Property 3.1.2 with $\sigma_{n-1} = \tau$), S_b^{τ} is the unique $(0,1)$-sequence whose points are always placed, in increasing order, as right as possible within the given collection of elementary intervals and satisfy the elementary interval property.

Therefore, by the elementary interval property for X_b^t, for any real $\alpha \in [0,1]$ and any integer $N \geq 1$ we have

$$
\begin{aligned}
A([0,\alpha); b^t N; X_b^t) \leq & A([0,\alpha); b^t N; S_b^{\mathrm{id},t}) = b^t A([0,\alpha); N; S_b^{\mathrm{id}}) \text{ and} \\
A([0,\alpha); b^t N; X_b^t) \geq & A([0,\alpha); b^t N; S_b^{\tau,t}) = b^t A([0,\alpha); N; S_b^{\tau}),
\end{aligned}
$$

where $S_b^{\tau,t}$ consists of b^t copies of S_b^{τ}.

Now, from the definitions of D^+ and D^-, we get

$$
D^+(b^t N, X_b^t) \leq b^t D^+(N, S_b^{\mathrm{id}}) \quad \text{and} \quad D^-(b^t N, X_b^t) \leq b^t D^-(N, S_b^{\tau}).
$$

Finally, we have $D^-(N, S_b^{\tau}) = D^+(N, S_b^{\mathrm{id}})$ (see [6, proof of Theorem 5.1] for details), so that

$$
\begin{aligned}
D^*(b^t N, X_b^t) = & \max(D^+(b^t N, X_b^t), D^-(b^t N, X_b^t)) \\
\leq & b^t D^+(N, S_b^{\mathrm{id}}) = b^t D^*(N, S_b^{\mathrm{id}}).
\end{aligned}
$$

(Notice, from [4], that $D^-(N, S_b^{\mathrm{id}}) = 0$, $D^+(N, S_b^{\tau}) = 0$ and therefore $D^*(N, S_b^{\mathrm{id}}) = D(N, S_b^{\mathrm{id}}) = D^*(N, S_b^{\tau}) = D(N, S_b^{\tau})$). □

Remark 1. The main idea of the proof is in the first sentence; it was already used by Dick and Kritzer [2] in the context of Hammersley two-dimensional point sets and by Kritzer [10]. But in this paper, Kritzer only considers $(0,1)$-sequences in the narrow sense and his proof is longer. Here, the use of S_b^{τ} together with the good control of discrepancy by means of so-called φ-functions permits a shorter proof and a more general result.

Corollary 1. *For all $(t, 1)$-sequences X_b^t (in the broad sense) and for all $N \geq 1$ we have*

$$D^*(b^t N, X_b^t) \leq b^t \left(\frac{b-1}{4 \log b} \log N + \frac{b-1}{4} + 2 \right) \text{ if } b \text{ is odd and}$$
$$D^*(b^t N, X_b^t) \leq b^t \left(\frac{b^2}{4(b+1) \log b} \log N + \frac{b^2}{4(b+1)} + 2 \right) \text{ if } b \text{ is even,}$$

Hence, we find that constants c_s^{Kr} in (4) and c_s^{conj} in (5) are valid for $s = 1$, but not c_s^{Kr} in (3). Note that for $s = 2$, $c_s^{conj} = c_s^{Kr}$ in (3).

Remark 2. 1. Corollary 1 shows that the conjecture of Kritzer is true for $s = 1$ whereas the bound (3), valid for $s \geq 2$, is false for $s = 1$. Indeed this bound is equal to $\frac{b^t(b^2-b)}{4(b+1)\log b} < \frac{b^{t+2}}{4(b+1)\log b}$ and (5) is reached with $S_b^{\mathrm{id},t}$, a contradiction. But for $s = 2$ the conjecture is true and until now this is the only dimension for which it is known to hold.

2. As for one-dimensional sequences having the smallest star discrepancy that are currently known, recent progress has been made by Ostromoukhov [19] who improves the preceding ones from the first author [4] cited in [13, Section 8] and [14, Chapter 3.1] (in the same paper, Ostromoukhov also improves the discrepancy D).

3. The results mentioned at the end of Section 2.1 for $(0, 1)$-sequences, [6, Theorems 5.3 and 5.2], should have analogs for $(t, 1)$-sequences, $t > 0$.

3 Discrepancy bound for (t, s)-sequences ($s \geq 2$)

In this section, we extend to (t, s)-sequences (in the broad sense) the discrepancy bound obtained in [8] for (t, s)-sequences in the narrow sense by using Atanassov's method first initiated for Halton sequences [1]. We also give more details toward the end of the proof than we did for Theorem 1 in [8], where they were omitted due to lack of space. In that way, apart from these slight improvements, we prepare the proof of Section 4 concerning even bases. In what follows, $\mathcal{P}_N(X)$ denotes the set containing the first N points of a sequence X and until the end, we set $n := \left\lfloor \frac{\log N}{\log b} \right\rfloor$. Also, several results in this section apply to the *truncated version* of the (t, s)-sequence under consideration, a concept that we now define.

Definition 4. Let X be a (t, s)-sequence in base b, with its kth term defined as $X_k = (X_k^{(1)}, \ldots, X_k^{(s)})$, for $k \geq 1$. Let

$$[\mathcal{P}_N(X)] = \{([X_k^{(1)}]_{b,\Delta}, \ldots, [X_k^{(s)}]_{b,\Delta}), 1 \leq k \leq N\}, \text{ where } \Delta = n + 1.$$

We refer to $[\mathcal{P}_N(X)]$ as the *first N points of a truncated version of the sequence X*.

The next result would be trivial without the truncation operator.

Lemma 1. Let X be a (t,s)-sequence in base b and $J = \prod_{i=1}^s [b_i b^{-d_i}, c_i b^{-d_i})$ with integers b_i, c_i satisfying $0 \le b_i < c_i \le b^{d_i}$. Then for $N \ge b^{d_1} \cdots b^{d_s}$, $A(J; [\mathcal{P}_N(X)])$ is a nondecreasing function of N.

Proof. If $N \ge b^{d_1} \cdots b^{d_s}$, then $\Delta \ge d_i$ for all i. Therefore as N increases, there cannot be fewer points (from the truncated sequence) inside a particular interval J. The reason why we have to make sure $\Delta \ge d_i$ for all i is that otherwise, as N increases, some points could leave the interval J as more precision is added to their digital expansion, but once the precision Δ is not less than the precision d_i used to define the interval, then this can no longer happen. □

The next lemma directly follows from the definition of (t,s)-sequences, but it requires some adaptation due to the truncation operator.

Lemma 2. Let X be a (t,s)-sequence. Let $J = \prod_{i=1}^s [b_i b^{-d_i}, c_i b^{-d_i})$ where b_i, c_i are integers satisfying $0 \le b_i < c_i \le b^{d_i}$. Then, for any integers $N \ge 1$ and $u \ge 0$ we have

$$A(J; [\mathcal{P}_N(X)]) = u b^t (c_1 - b_1) \cdots (c_s - b_s) \text{ where } N = u b^t b^{d_1} \cdots b^{d_s},$$

$$|A(J; [\mathcal{P}_N(X)]) - NV(J)| \le b^t \prod_{i=1}^s (c_i - b_i) \text{ and}$$

$$A(J; [\mathcal{P}_N(X)]) \le b^t \prod_{i=1}^s (c_i - b_i) \text{ if } N < b^t b^{d_1} \cdots b^{d_s}.$$

Proof. To prove the first statement, we first observe that from the definition of a (t,s)-sequence, the sets $\{[X_k]_{b,m}; l b^m \le k < (l+1) b^m\}$ are (t,m,s)-nets for $0 \le l < u$, where $u \ge 1$. The sets $\{[X_k]_{b,M}; l b^m \le k < (l+1) b^m\}$ are also (t,m,s)-nets for $0 \le l < u$ and $M > m$, which proves the result since $\Delta > m$ in the case $N = u b^m$.

For the second statement, let $l \ge 1$ be such that $l b^t b^{d_1} \cdots b^{d_s} \le N < (l+1) b^t b^{d_1} \cdots b^{d_s}$. Then, using Lemma 1, we get

$$A(J; [\mathcal{P}_N(X)]) - NV(J) \le (l+1) b^t \prod_{i=1}^s (c_i - b_i) - l b^t \prod_{i=1}^s b^{d_i} (c_i - b_i) b^{-d_i}$$

$$=b^t \prod_{i=1}^{s} (c_i - b_i).$$

A similar argument shows that $NV(J) - A(J;[\mathcal{P}_N(X)]) \le b^t \prod_{i=1}^s (c_i - b_i)$.

For the third statement, in the case where $N < b^t b^{d_1} \cdots b^{d_s}$, we define $\tilde{d}_i = \min(\Delta, d_i)$, where $\Delta = n+1$. Let $[J]$ be defined as the smallest interval of the form $\prod_{i=1}^s [\tilde{b}_i b^{-\tilde{d}_i}, \tilde{c}_i b^{-\tilde{d}_i})$ with $0 \le \tilde{b}_i \le \tilde{c}_i \le b^{\tilde{d}_i}$ and such that $J \subseteq [J]$. We can see that $[J]$ is obtained by using $\tilde{c}_i = \lceil c_i / b^{\tilde{d}_i - d_i} \rceil$ and $\tilde{b}_i = \lfloor b_i / b^{\tilde{d}_i - d_i} \rfloor$. It is easy to see that each interval of the form $\prod_{i=1}^s [l_i b^{-\tilde{d}_i}, (l_i+1) b^{-\tilde{d}_i})$ has at most b^t points from $[\mathcal{P}_N(X)]$. Hence

$$A(J;[\mathcal{P}_N(X)]) \le A([J];[\mathcal{P}_N(X)]) \le b^t \prod_{i=1}^{s} (\tilde{c}_i - \tilde{b}_i) \le b^t \prod_{i=1}^{s} (c_i - b_i),$$

where the last inequality follows from the definition of $\tilde{b}_i$ and $\tilde{c}_i$. □

Lemma 3. Let $b \ge 2$, $N \ge 1$ and $k \ge 1$ be integers. For integers $j \ge 0$ and $1 \le i \le k$, let $c_j^{(i)} \ge 0$ be given numbers satisfying $c_0^{(i)} \le 1$ and $c_j^{(i)} \le c$ for $j \ge 1$, where c is some fixed number. Then

$$(6) \qquad \sum_{\{(j_1,\ldots,j_k)\,;\, b^{j_1}\cdots b^{j_k} \le N\}} \prod_{i=1}^{k} c_{j_i}^{(i)} \le \frac{1}{k!} \left(c \frac{\log N}{\log b} + k \right)^k .$$

Proof. For each $m \in \{0,1,\ldots,k\}$, fix a subset $L = \{i_1,\ldots,i_m\}$ of $\{1,\ldots,k\}$ and consider the contributions of all the k-tuples $\mathbf{j}$ with $j_r > 0$ for $r \in L$, and $j_r = 0$ for $r \notin L$, with $\prod_{i=1}^k b^{j_i} = \prod_{i \in L} b^{j_i} \le N$. One can verify as in [1, Lemma 3.2] (see also [3, Lemma 3.38]) that there are $\frac{1}{m!}\left(\frac{\log N}{\log b}\right)^m$ such k-tuples, each having a contribution of

$$\prod_{i=1}^{k} c_{j_i}^{(i)} = \prod_{i \in L} c_{j_i}^{(i)} \prod_{i \notin L} c_{j_i}^{(i)} \le \prod_{i \in L} c \prod_{i \notin L} 1 = c^m.$$

Expanding both sides of (6), the result now follows since $\frac{1}{m!} \le \frac{1}{k!} k^{k-m}$. □

Definition 5. Consider an interval $J \subseteq I^s$. We call a *signed splitting* of J any collection of intervals $J_1,\ldots,J_n$ and respective signs $\epsilon_1,\ldots,\epsilon_n$ equal to ± 1, such that for any (finitely) additive function ν on the intervals in I^s, we have $\nu(J) = \sum_{i=1}^n \epsilon_i \nu(J_i)$.

The following lemma, slightly reformulated, is taken directly from [1, Lemma 3.5] (see also [3, Lemma 3.40]).

Lemma 4. Let $J = \prod_{i=1}^{s}[0, z^{(i)})$ be an s-dimensional interval and, for each $1 \leq i \leq s$, let $n_i \geq 0$ be given integers. Set $z_0^{(i)} = 0$, $z_{n_i+1}^{(i)} = z^{(i)}$ and, if $n_i \geq 1$, let $z_j^{(i)} \in [0,1]$ be arbitrary given numbers for $1 \leq j \leq n_i$. Then the collection of intervals $\prod_{i=1}^{s}[\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}))$, with signs $\epsilon(j_1, \ldots, j_s) = \prod_{i=1}^{s} \operatorname{sgn}(z_{j_i+1}^{(i)} - z_{j_i}^{(i)})$, for $0 \leq j_i \leq n_i$, is a signed splitting of J.

We can now prove the result announced at the beginning of Section 3:

Theorem 2. *For any (t, s)-sequence X (in the broad sense) in any base b and for any $N \geq 1$ we have*

$$D^*(N, X) \leq \frac{b^t}{s!}\left(\left\lfloor\frac{b}{2}\right\rfloor \frac{\log N}{\log b} + s\right)^s + b^t \sum_{k=1}^{s-1} \frac{b}{k!}\left(\left\lfloor\frac{b}{2}\right\rfloor \frac{\log N}{\log b} + k\right)^k. \tag{7}$$

Proof. As in [5] and [1], we will use special numeration systems in base b — using signed digits a_j bounded by $\lfloor b/2 \rfloor$ — to expand reals in $[0,1)$. That is, we write $z \in [0,1)$ as

$$z = \sum_{j=0}^{\infty} a_j b^{-j} \begin{cases} \text{with } |a_j| \leq \frac{b-1}{2} \text{ if } b \text{ is odd} \\ \text{with } |a_j| \leq \frac{b}{2} \text{ and } |a_j| + |a_{j+1}| \leq b-1 \text{ if } b \text{ is even.} \end{cases} \tag{8}$$

The existence and unicity of such expansions are obtained by induction, see [1, p. 21–22] or [23, p. 12–13] where more details are given. For later use, it is worth pointing out that the expansion starts at b^0 and as a result, it is easy to see that a_0 is either 0 or 1.

We can now begin the proof: Let $\boldsymbol{z} = (z^{(1)}, \ldots, z^{(s)}) \in [0,1)^s$ and consider its b-adic expansion $z^{(i)} = \sum_{j=0}^{\infty} a_j^{(i)} b^{-j}$ according to our numeration systems (8) above. Recall that $n = \left\lfloor \frac{\log N}{\log b} \right\rfloor$ and define $z_0^{(i)} = 0$ and $z_{n+1}^{(i)} = z^{(i)}$. If $n \geq 1$, consider the numbers $z_k^{(i)} = \sum_{j=0}^{k-1} a_j^{(i)} b^{-j}$ for $k = 1, \ldots, n$. Applying Lemma 4 with $n_i = n$ for all $1 \leq i \leq s$, we split up $J = \prod_{i=1}^{s}[0, z^{(i)})$ using the numbers $z_j^{(i)}$, with $0 \leq j \leq n+1$, and obtain the signed splitting

$$I(\boldsymbol{j}) := \prod_{i=1}^{s}[\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)})), \qquad 0 \leq j_i \leq n, \tag{9}$$

with signs $\epsilon(\boldsymbol{j}) = \prod_{i=1}^{s} \operatorname{sgn}(z_{j_i+1}^{(i)} - z_{j_i}^{(i)})$, where $\boldsymbol{j} = (j_1, \ldots, j_s)$.

Since V and $A(\,.\,; [\mathcal{P}_N(X)])$ are both additive, $A(J; [\mathcal{P}_N(X)]) - NV(J)$ may be expanded as

(10)

$$A(J; [\mathcal{P}_N(X)]) - NV(J) = \sum_{j_1=0}^{n} \cdots \sum_{j_s=0}^{n} \epsilon(\boldsymbol{j})\left(A(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) - NV(I(\boldsymbol{j}))\right)$$

$$=: \Sigma_1 + \Sigma_2,$$

where we rearrange the terms so that in Σ_1 we put the terms $\boldsymbol{j}$ such that $b^{j_1} \cdots b^{j_s} \leq N$ (that is $j_1 + \cdots + j_s \leq n$) and in Σ_2 the rest. Notice that in Σ_1, the $j_i's$ are small, so the corresponding $I(\boldsymbol{j})$ is bigger. Hence, Σ_1 deals with the coarser part whereas Σ_2 deals with the finer part.

Thanks to Lemmas 2 and 3, it is easy to deal with Σ_1: from Lemma 2, we have that

$$(11)\quad |\, A(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) - NV(I(\boldsymbol{j}))| \leq b^t \prod_{i=1}^{s} |\, z_{j_i+1}^{(i)} - z_{j_i}^{(i)} |\, b^{j_i} = b^t \prod_{i=1}^{s} |\, a_{j_i}^{(i)}|.$$

Then, applying Lemma 3 with $k = s$, $c_j^{(i)} = |a_j^{(i)}|$ and $c = \left\lfloor \frac{b}{2} \right\rfloor$, we obtain

$$|\Sigma_1| \leq \sum_{\{\boldsymbol{j}\,;\, b^{j_1} \cdots b^{j_s} \leq N\}} |A(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) - NV(I(\boldsymbol{j}))| \leq \frac{b^t}{s!} \left(\left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + s \right)^s$$

which is the first part of the bound of Theorem 2.

The terms gathered in Σ_2 give the second part of the bound of Theorem 2, i.e., the part in $O((\log N)^{s-1})$. The idea of Atanassov for his proof of Theorem 2.1 in [1] is to divide the set of s-tuples $\boldsymbol{j}$ in Σ_2 into s disjoint sets included in larger ones for which Lemma 3 applies and gives the desired upper bound. The adaptation to (t, s)-sequences follows the same way. For the sake of completeness, we survey the proof in the following, something we could not achieve in [8] due to lack of space.

Recall that in Σ_2, we are summing over the set of all s-tuples $\boldsymbol{j}$ such that $b^{j_1} \cdots b^{j_s} > N$ with $0 \leq j_i \leq n$ for all $1 \leq i \leq n$. We split this set a priori into s disjoint sets $B_0, \ldots, B_{s-1}$, where $B_0 = \{\boldsymbol{j}; b^{j_1} > N\}$ and

$$B_k = \{\boldsymbol{j}; b^{j_1} \cdots b^{j_k} \leq N < b^{j_1} \cdots b^{j_k} b^{j_{k+1}}\} \text{ for } k \geq 1.$$

But we immediately observe that $B_0 = \emptyset$, since $b^{j_1} > N$ implies $j_1 > n$ which is in contradiction with the definition of the signed splitting defined in (9). For the same reason, the corresponding set B_0 in the proof for Halton sequences is also empty, a remark that was skipped in papers dealing with Atanassov's Theorem 2.1 [1, 23, 3].

Hence we only have to deal with $1 \leq k \leq s-1$. For such a fixed k and for a fixed k-tuple $(j_1, \ldots, j_k)$ with $b^{j_1} \cdots b^{j_k} \leq N$, define r as the largest integer such that $b^{j_1} \cdots b^{j_k} b^{r-1} \leq N$. Then the s-tuple $\boldsymbol{j} = (j_1, \ldots, j_k, j_{k+1}, \ldots, j_s) \in B_k$ if and only if $j_{k+1} \geq r$, the remaining integers $j_{k+2}, \ldots, j_s$ being arbitrary in the interval $[0, n]$. Still for fixed k and $(j_1, \ldots, j_k)$ as above, define $K_1 = \prod_{i=1}^k [\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}))$ and

$$K = K_1 \times [\min(z_r^{(k+1)}, z^{(k+1)}), \max(z_r^{(k+1)}, z^{k+1})) \times \prod_{i=k+2}^{s} [0, z^{(i)}).$$

Coming back to the sum of terms in Σ_2 that belong to B_k, we obtain by a simple case analysis

$$\begin{aligned}(12) \quad & \pm (A(K; [\mathcal{P}_N(X)]) - NV(K)) \\ & \qquad = \sum_{\{(j_{k+1},\ldots,j_s)\,;\,\boldsymbol{j}\in B_k\}} \epsilon(\boldsymbol{j})(A(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) - NV(I(\boldsymbol{j}))\end{aligned}$$

where $\pm = \operatorname{sgn}(z^{(k+1)} - z_r^{(k+1)})$. Now, since

$$|\, z^{(k+1)} - z_r^{(k+1)}| = |\sum_{j=r}^{\infty} a_j^{(k+1)} b^{-j}| \le \left\lfloor \frac{b}{2} \right\rfloor \frac{1}{b^r} \frac{b}{b-1} \le \frac{1}{b^{r-1}},$$

it follows that $[\min(z_r^{(k+1)}, z^{(k+1)}), \max(z_r^{(k+1)}, z^{(k+1)})) \subseteq [m_1 b^{-r}, m_2 b^{-r})$ for some nonnegative integers m_1, m_2 satisfying $0 \le m_2 - m_1 < b$. Hence K is contained in the interval $K' = K_1 \times [m_1 b^{-r}, m_2 b^{-r}) \times [0,1)^{s-k-1}$. Thanks to this inclusion we can bound $A(K; [\mathcal{P}_N(X)])$ using Lemma 2 (observe that $N < b^{j_1} \cdots b^{j_k} b^r \le b^t b^{j_1} \cdots b^{j_k} b^r$):

$$\begin{aligned} A(K; [\mathcal{P}_N(X)]) \le & A(K'; [\mathcal{P}_N(X)]) \le b^t (m_2 - m_1) \prod_{i=1}^{k} b^{j_i} |\, z_{j_i+1}^{(i)} - z_{j_i}^{(i)}| \\ \le & b^t \cdot b \prod_{i=1}^{k} |\, a_{j_i}^{(i)}|. \end{aligned}$$

But on the other hand,

$$NV(K) \le b^t b^{j_1} \cdots b^{j_k} b^r \cdot V(K') = b^t (m_2 - m_1) \prod_{i=1}^{k} |\, a_{j_i}^{(i)}| \le b^t \cdot b \prod_{i=1}^{k} |\, a_{j_i}^{(i)}|,$$

so that $|A(K; [\mathcal{P}_N(X)]) - NV(K)| \le b^t \cdot b \prod_{i=1}^k |\, a_{j_i}^{(i)}|$.

Finally, collecting everything in one sum, we get

$$\begin{aligned} |\Sigma_2| &= \Big| \sum_{k=1}^{s-1} \sum_{\{(j_1,\ldots,j_k)\,;\,b^{j_1}\cdots b^{j_k} \le N\}} \sum_{\{(j_{k+1},\ldots,j_s)\,;\,\boldsymbol{j}\in B_k\}} \epsilon(\boldsymbol{j})(A(I(\boldsymbol{j}); N) - NV(I(\boldsymbol{j}))\Big| \\ &= \Big| \sum_{k=1}^{s-1} \sum_{\{(j_1,\ldots,j_k)\,;\,b^{j_1}\cdots b^{j_k} \le N\}} \pm (A(K; N) - NV(K))\Big| \end{aligned}$$

$$\leq \; b^t \sum_{k=1}^{s-1} b \sum_{\{(j_1,\ldots,j_k)\,;\, b^{j_1}\cdots b^{j_k}\leq N\}} \prod_{i=1}^{k} |\, a_{j_i}^{(i)}|$$

$$\leq \; b^t \sum_{k=1}^{s-1} \frac{b}{k!} \left(\left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + k \right)^k ,$$

where we applied Lemma 3 with $c = \lfloor \frac{b}{2} \rfloor$ and $1 \leq k \leq s-1$. This is the second part of the bound of Theorem 2.

So far, we have proved (7) only for the truncated version of the sequence but, as shown in [15, Lemma 4.2], when we go from the truncated to the untruncated version of the sequence, the bound for the discrepancy remains the same. Thus if a bound of the form (1) applies to the truncated version of a (t,s)-sequence, it applies to the untruncated version as well with the same constant c_s. Notice that [15, Lemma 4.2] is a nice improvement over [16, Rem. 2], the latter one saying that going from the truncated to the untruncated version adds a term at most s to the discrepancy of the nets involved. □

From Theorem 2 we can derive the following result for the constant c_s, which for the case where b is odd is the same as the constant c_s^{Ni}, while in the case where b is even, it is larger than c_s^{Ni} by a factor $b/(b-1)$.

Corollary 2. *The discrepancy of a (t,s)-sequence X in base b satisfies (1) with*

$$c_s = \begin{cases} \frac{b^t}{s!}\left(\frac{b-1}{2\log b}\right)^s & \text{if } b \text{ is odd} \\ \frac{b^t}{s!}\left(\frac{b}{2\log b}\right)^s & \text{if } b \text{ is even.} \end{cases}$$

Remark 3. So far, we have not been able to include in our proof the improvements in one dimension from Section 2. Hence we are not able to reach the constants c_s^{Kr} obtained by Kritzer [11]. However, for even bases we are able to substantially improve upon c_s^{Kr} in (3) and get quite close to c_s^{conj} conjectured in (5). This new result is the topic of the next section.

4 Discrepancy bound for (t,s)-sequences in an even base

The technique used to get our improved bounds for (t,s)-sequences in even bases is an adaptation of the one used by Atanassov in [1] to handle the case of an even base in Halton sequences. The result is as follows:

Theorem 3. *For any (t,s)-sequence X (in the broad sense) in an even base b and for any $N \geq b^s$ we have*

$$D^*(N,X) \leq \frac{b^t}{s!}\left(\frac{b-1}{2}\frac{\log N}{\log b}+s\right)^s + sb^t\left(\frac{b}{2}\right)^s\left(\frac{\log N}{\log b}\right)^{s-1} + b^t\sum_{k=1}^{s-1}\frac{b}{k!}\left(\frac{b}{2}\frac{\log N}{\log b}+k\right)^k. \tag{13}$$

Proof. At the outset, the proof is the same as for Theorem 2 until equation (10) for the discrepancy function $A(J;[\mathcal{P}_N(X)]) - NV(J)$. The end of the proof, concerning Σ_2, is the same too and gives the last term in the bound (13). Hence, what remains to be done is to deal with Σ_1. To this end, we split up the set $\{(j_1,\dots,j_s)\,;\, b^{j_1}\cdots b^{j_s} \leq N\}$ in two parts:

$$S' = \left\{(j_1,\dots,j_s)\,;\, b^{j_1}\cdots b^{j_s} \leq \frac{N}{b^s}\right\} \text{ and}$$
$$S'' = \left\{(j_1,\dots,j_s)\,;\, \frac{N}{b^s} < b^{j_1}\cdots b^{j_s} \leq N\right\}.$$

The part S' provides the biggest contribution to the bound and hence gives the leading term in (13): using (11), we need to prove

$$\sum_{(j_1,\dots,j_s)\in S'}\prod_{i=1}^{s}|a_{j_i}^{(i)}| \leq \frac{1}{s!}\left(\frac{(b-1)\log N}{2\log b}+s\right)^s.$$

To do so, we proceed in three steps:

First, we define integers $c_j^{(i)} = |a_{2j-1}^{(i)}| + |a_{2j}^{(i)}|$ for $j \geq 1$ and $c_0^{(i)} = 1$. We have that for $j \geq 1$,

$$|a_j^{(i)}| \leq \begin{cases} |a_{j-1}^{(i)}| + |a_j^{(i)}| = c_h^{(i)} = c_{j/2}^{(i)} & \text{if } j = 2h \text{ is even,}\\ |a_j^{(i)}| + |a_{j+1}^{(i)}| = c_h^{(i)} = c_{(j+1)/2}^{(i)} & \text{if } j = 2h-1 \text{ is odd,}\end{cases}$$

and for $j = 0$, $a_j^{(i)}$ is either 0 or 1 so that $a_0^{(i)} \leq c_0^{(i)}$. Hence

$$\prod_{i=1}^{s}|a_{j_i}^{(i)}| \leq \prod_{i=1}^{s}|c_{j_i'}^{(i)}|$$

with

$$j_i' = \begin{cases} j_i/2 & \text{if } j_i \text{ is even}\\ (j_i+1)/2 & \text{if } j_i \text{ is odd.}\end{cases} \tag{14}$$

Secondly, we apply Lemma 3 to the numbers $c_j^{(i)}$ with $k = s$, $\tilde{b} = b^2$ and $c = \sqrt{\tilde{b}} - 1 = b - 1$ (observe that, according to the numeration system (8) in case of b even, we have $c_j^{(i)} = |a_{2j-1}^{(i)}| + |a_{2j}^{(i)}| \le b-1$). Hence we get (observe that $\log \tilde{b} = 2\log b$)

$$\sum_{\{(l_1,\dots,l_s)\,;\,\tilde{b}^{l_1}\cdots\tilde{b}^{l_s}\le N\}} \prod_{i=1}^{s} c_{l_i}^{(i)} \le \frac{1}{s!}\left(\frac{(b-1)\log N}{2\log b}+s\right)^s. \tag{15}$$

Finally, we show that for $(j_1,\dots,j_s)\in S'$ the s-tuple $(j_1',\dots,j_s')$ defined in (14) verifies $\tilde{b}^{j_1'}\cdots\tilde{b}^{j_s'}\le N$. Indeed, we have

$$\tilde{b}^{j_1'}\cdots\tilde{b}^{j_s'} = b^{2\left(\sum_{\{j_i;j_i=2h\}}\frac{j_i}{2}+\sum_{\{j_i;j_i=2h-1\}}\frac{j_i+1}{2}\right)} = b^{\sum j_i}.b^{\sum_{\{j_i;j_i=2h-1\}}1} \le \frac{N}{b^s}b^s = N.$$

(Recall that we are estimating the sum for $(j_1,\dots,j_s)\in S'$, i.e., such that $b^{\sum j_i}\le \frac{N}{b^s}$). Hence, the s-tuples $(j_1',\dots,j_s')$ derived from the s-tuples $(j_1,\dots,j_s)\in S'$ as in (14) belong to the set $\{(l_1,\dots,l_s);\tilde{b}^{l_1}\cdots\tilde{b}^{l_s}\le N\}$. Therefore, according to the first step and to (15), we can assert that

$$\sum_{(j_1,\dots,j_s)\in S'}\prod_{i=1}^{s}|a_{j_i}^{(i)}| \le \sum_{\{(l_1,\dots,l_s)\,;\,\tilde{b}^{l_1}\cdots\tilde{b}^{l_s}\le N\}} \prod_{i=1}^{s} c_{l_i}^{(i)} \le \frac{1}{s!}\left(\frac{(b-1)\log N}{2\log b}+s\right)^s.$$

(observe that two reasons prevail for the first inequality: one from the inequality in the first step and the other from the fact that $(j_1',\dots,j_s')\in\{(l_1,\dots,l_s);\tilde{b}^{l_1}\cdots\tilde{b}^{l_s}\le N\}$). This concludes the study of S'.

Now we deal with $S''=\left\{(j_1,\dots,j_s)\,;\,\frac{N}{b^s}<b^{j_1}\cdots b^{j_s}\le N\right\}$. Taking the logarithm, $(j_1,\dots,j_s)\in S''$ is equivalent to

$$n-s<\sum_{i=1}^{s}j_i\le n \text{ with } n=\left\lfloor\frac{\log N}{\log b}\right\rfloor.$$

Similarly to what is done in the proof of [1, Lemma 3.2] (see also [3, Lemma 3.38]), we can show that the number of such s-tuples $(j_1,\dots,j_s)$ is at most sn^{s-1} since we have s relations $\sum_{i=1}^{s} j_i = m$ with $n-s<m\le n$. On the other hand, each s-tuple $\boldsymbol{j}=(j_1,\dots,j_s)$ contributes a term that is at most $b^t\left(\frac{b}{2}\right)^s$ in the estimate of the discrepancy function $A(I(\boldsymbol{j});N)-NV(I(\boldsymbol{j}))$, as can be seen in (11). Finally, the total contribution from S'' is at most

$$b^t\left(\frac{b}{2}\right)^s s\left(\frac{\log N}{\log b}\right)^{s-1}.$$

This is the second term in (13), which ends the main part of the proof of Theorem 3 according to the outline given at the beginning. Going from the truncated to the untruncated version of the sequence results from the arguments given in [15], as explained at the end of the proof of Theorem 2. □

Corollary 3. *The discrepancy of a (t,s)-sequence X in an even base b satisfies (1) with*

$$c_s^{FL} = \frac{b^t}{s!}\left(\frac{b-1}{2\log b}\right)^s. \tag{16}$$

Remark 4. 1. Compared to the constant c_s^{Kr} of Kritzer (3), we observe a significant improvement with $\frac{c_s^{FL}}{c_s^{Kr}} = \frac{2(b+1)}{b}\left(\frac{b-1}{b}\right)^{s-1}$, the most striking one being obtained with $b=2$ where the ratio is $\frac{3}{2^{s-1}}$, hence tending to 0 as s tends to infinity. But we have to take into account the quality parameter t to compare sequences in different bases (see the numerical results below).

2. Compared to the conjecture of Kritzer (5), we still have the gap $\frac{c_s^{conj}}{c_s^{FL}} = \frac{b^2}{2(b^2-1)}$. As we already noted, this gap comes from our inability to insert in our proof — using Atanassov's method — the better bounds in one dimension deduced from our knowledge of van der Corput sequences. Until now, these better bounds — translated in terms of two-dimensional Hammersley point sets — can only be inserted in the method of Sobol' [21], used later on in [5, 13, 11], thanks to the double recursion on m and s involving (t,m,s)-nets. But on the other hand, the ideas in the proof of Theorem 3 seem difficult to transpose in the double recursion method. We view this as a challenge for future research.

Numerical results At the end of [11], Kritzer devotes a section to numerical results concerning the smallest possible constants c_s obtained in each dimension with specific sequences taking into account the best possible values of the quality parameter t. He gives two tables for dimensions between 2 and 20:

– The first one (Table 2) uses values of t from [14] and logically, for $s \geq 3$, he obtains the same constants as [14, Table 4.4] divided by 2 since all bases are odd primes, except $b=9$ for $s=8,9$. If $s=2$ then $b=2$ and the constant from Table 4.4 is multiplied by $\frac{b^2}{2(b^2-1)} = \frac{2}{3}$.

– The second one (Table 3) uses best known values of t from the database MinT [20], a remarkable software containing updated information on the best available constructions for a given set of parameters. For base $b=2$, these values already show a drastic improvement over Table 2, but the best constants c_s^{Kr} are obtained in base $b=3$ for all dimensions $3 \leq s \leq 20$, these constants being again half of the constants deduced from constant c_s^{Ni} in (2) with t-values from MinT.

In the following, we only give one table, corresponding to [11, Table 3] above, where the smallest achievable constant c_s^{FL} — over all (t,s)-sequences

s	best c_s^{FL}	t	best c_s^{Kr}	t	s	best c_s^{FL}	t	best c_s^{Kr}	t
2	2.60e-1	0	**0.173**	0	15	**1.87e-10**	15	2.04e-10	7
3	1.25e-1	1	**6.28e-2**	0	16	**8.42e-12**	15	3.48e-11	8
4	**2.26e-2**	1	4.29e-2	1	17	**2.86e-12**	18	5.59e-12	9
5	**6.51e-3**	2	7.81e-3	1	18	**2.29e-13**	19	2.83e-13	9
6	1.57e-3	3	**1.18e-3**	1	19	**8.69e-15**	19	4.06e-14	10
7	**3.23e-4**	4	4.62e-4	2	20	**1.25e-15**	21	1.66e-14	12
8	**5.82e-5**	5	1.58e-4	3	25	**3.93e-20**	31	4.41e-20	15
9	**9.33e-6**	6	1.60e-5	3	30	**1.15e-25**	39	1.30e-25	19
10	**2.69e-6**	8	4.36e-6	4	35	**1.48e-31**	47	1.52e-30	25
11	**3.53e-7**	9	3.61e-7	4	40	**4.67e-38**	54	9.77e-37	29
12	**4.24e-8**	10	8.21e-8	5	45	**1.28e-43**	65	3.37e-43	33
13	**4.71e-9**	11	1.72e-8	6	50	4.01e-49	77	**6.72e-50**	37
14	**9.71e-10**	13	3.36e-9	7					

Table 1: The best value of c_s is indicated in boldface. For c_s^{FL} the base is always $b = 2$ and for c_s^{Kr} the base is always $b = 3$ except for $s = 2$ where $b = 2$.

— for each dimension is given. The difference between our Table 1 and Table 3 from [11] is that we use c_s^{FL} from formula (16) in the case of even bases. Of course, we also use MinT to find the smallest possible value of t for each base. We see that we are able to improve upon the results of Kritzer for all dimensions listed there except $s = 2, 3, 6, 50$ and that base 2 wins over base 3 except in three cases. Note that if Kritzer's conjecture is true, base 2 also wins for $s = 6$ but not for $s = 3$ and 50. We observe too that the best sequence in dimension 2 is a $(0, 2)$-Sobol' sequence and the best one in dimension 3 is a $(0, 3)$-Faure sequence, all others being (t, s)-Niederreiter-Xing sequences with $t > 0$. Although it is not shown here, we performed calculations for all s between 20 and 50, and observed that for these 24 additional cases, there were only two cases (in dimension 24 and 49) where base 3 won over base 2.

References

[1] E. I. Atanassov, *On the discrepancy of the Halton sequences*, Math. Balkanica (N.S.) 18.1-2 (2004), 15–32.

[2] J. Dick and P. Kritzer, *A best possible upper bound on the star discrepancy of* $(t, m, 2)$*–nets*, Monte Carlo Methods Appl. 12 (1) (2006), 1–17.

[3] J. Dick and F. Pillichshammer, *Digital Nets and Sequences: Discrepancy Theory and Quasi-Monte Carlo Integration*, Cambridge University Press, UK, 2010.

[4] H. Faure, *Discrépance de suites associées à un système de numération (en dimension un)*, Bull. Soc. Math. France 109 (1981), 143–182.

[5] H. Faure, *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. 61 (1982), 337–351.

[6] H. Faure, *Van der Corput sequences towards (0,1)–sequences in base b*, J. Théor. Nombres Bordeaux 19 (2007), 125–140.

[7] H. Faure and C. Lemieux, *Generalized Halton sequences in 2008: A comparative study*, ACM Trans. Model. Comp. Sim. 19 (2009), Article 15.

[8] H. Faure, C. Lemieux, X. Wang, *Extensions of Atanassov's methods for Halton sequences*, in: Monte Carlo and Quasi-Monte Carlo Methods 2010, H. Wozniakowski (ed.), Springer, to appear.

[9] J. H. Halton, *On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals*, Numer. Math. 2 (1960), 184–190.

[10] P. Kritzer, *A new upper bound on the star discrepancy of* $(0,1)$*-sequences*, Integers 5(3) A11 (2005), 1–9.

[11] P. Kritzer, *Improved upper bounds on the star discrepancy of* (t,m,s)*-nets and* (t,s)*-sequences*, J. Complexity 22 (2006), 336–347.

[12] C. Lemieux, *Monte Carlo and Quasi-Monte Carlo Sampling*, Springer Series in Statistics, Springer, New York, 2009.

[13] H. Niederreiter, *Point sets and sequences with small discrepancy*, Monatsh. Math. 104 (1987), 273–337.

[14] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1992.

[15] H. Niederreiter and F. Özbudak, *Low-discrepancy sequences using duality and global function fields*, Acta Arith. 130 (2007), 79–97.

[16] H. Niederreiter and C. P. Xing, *Low-discrepancy sequences and global function fields with many rational places*, Finite Fields Appl. 2 (1996), 241–273.

[17] H. Niederreiter and C. P. Xing, *Quasirandom points and global function fields*, in: Finite Fields and Applications, S. Cohen and H. Niederreiter (eds), London Math. Soc. Lectures Notes Series 233 (1996), 269–296.

[18] H. Niederreiter and C. P. Xing, *Rational Points on Curves over Finite Fields: Theory and Applications*, London Math. Soc. Lectures Notes Series vol. 285, Cambridge University Press, UK, 2001.

[19] V. Ostromoukhov, *Recent Progress in Improvement of Extreme Discrepancy and Star Discrepancy of One-dimensional Sequences*, in: Monte-Carlo and Quasi-Monte Carlo Methods 2008, P. L'Ecuyer and A. B. Owen (eds.), Springer, New York, (2009), 561–572.

[20] R. Schürer and W. Schmid, *MinT: a database for optimal net parameters*, in: Monte Carlo and Quasi-Monte Carlo Methods 2004, H. Niederreiter and D.Talay (eds.), Springer, Berlin (2006), 457–469.

[21] I. M. Sobol', *The distribution of points in a cube and the approximate evaluation of integrals*, U.S.S.R. Comput. Math. and Math. Phys. 7 (1967), 86–112.

[22] S. Tezuka, *Polynomial arithmetic analogue of Halton sequences*, ACM Trans. Model. Comp. Sim. 3 (1993), 99–107.

[23] X. Wang, C. Lemieux, H. Faure, *A note on Atanassov's discrepancy bound for the Halton sequence*, Technical report, University of Waterloo, Canada (2008).