Optimal Reinsurance with Regulatory Initial Capital and Default Risk

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Abstract

In a reinsurance contract, a reinsurer promises to pay the part of the loss faced by an insurer in exchange for receiving a reinsurance premium from the insurer. However, the reinsurer may fail to pay the promised amount when the promised amount exceeds the reinsurer's solvency. As a seller of a reinsurance contract, the initial capital or reserve of a reinsurer should meet some regulatory requirements. We assume that the initial capital or reserve of a reinsurer is regulated by the value-at-risk (VaR) of its promised indemnity. When the promised indemnity exceeds the total of the reinsurer's initial capital and the reinsurance premium, the reinsurer may fail to pay the promised amount or default may occur. In the presence of the regulatory initial capital and the counterparty default risk, we investigate optimal reinsurance designs from an insurer's point of view and derive optimal reinsurance strategies that maximize the expected utility of an insurer's terminal wealth or minimize the VaR of an insurer's total retained risk. It turns out that optimal reinsurance strategies in the presence of the regulatory initial capital and the counterparty default risk are different both from optimal reinsurance strategies in the absence of the counterparty default risk and from optimal reinsurance strategies in the presence of the counterparty default risk but without the regulatory initial capital.

Keywords: Optimal Reinsurance, Reinsurance Premium, Value-at-Risk, Counterparty Default Risk, Utility Function, Convex Order.

1 Introduction

Reinsurance is an important risk management tool for an insurer and has been an interesting research topic in actuarial science. In a static reinsurance model or one-period reinsurance model, one assumes that the underlying (aggregate) loss faced by an insurer in a fixed time period is a non-negative random variable X with survival function $S_X(x) = \Pr\{X > x\} = 1 - F_X(x)$. In a reinsurance contract, a reinsurer agrees to pay the part of the loss X, denoted by $I(X)$, to the insurer at the end of the contract term, while the insurer will pay a reinsurance premium, denoted by P_I, to the reinsurer when the contract is signed, where the function $I(x)$ is called ceded loss function or indemnification function. Thus, under the reinsurance contract I, the retained loss for the insurer is $R(X) = X - I(X)$, where the function $R(x) = x - I(x)$ is called retained loss function. In order to avoid any moral issue, a feasible reinsurance contract I should satisfy the following two conditions:

(1) $I : [0, \infty) \to [0, \infty)$ such that $I(0) = 0$ and I is non-decreasing;

(2) $I(y) - I(x) \leq y - x$, for any $0 \leq x \leq y$.

These two conditions imply that both $I(x)$ and $R(x)$ are continuous and non-decreasing on $[0, \infty)$. The first condition means that the larger is the incurred loss by an insurer, the larger is the covered loss by a reinsurer. The second condition implies that the growth rate of the covered loss by a reinsurer should not be faster than the growth rate of the underlying loss faced by an insurer.

Throughout this paper, we denote the set of all feasible reinsurance contracts satisfying conditions (1) and (2) by $\mathcal{I}$ and define $(a)^+ = \max\{a, 0\}$, $a \wedge b = \min\{a, b\}$, and $a \vee b = \max\{a, b\}$. In addition, we interpret the term "increasing" to mean "non-decreasing", while "decreasing" means "non-increasing".

The purpose of optimal reinsurance design is to find ceded loss functions I^*, which are optimal under certain optimization criteria. Optimal reinsurance from an insurer's point of view has been studied extensively in the literature. Two commonly used optimization criteria are maximizing the expected utility of an insurer's terminal wealth and minimizing the risk measure of an insurer's total retained risk. Some recent references on optimal reinsurance under different risk measures include Balbás et al. (2009), Asimit, Badescu and Verdoncj (2013), Cai et al. (2008), Cheung (2010), Chi (2012), Chi and Tan (2011), and so on. In addition, the one-period reinsurance model with one loss variable has been extended to models with more insurance lines of business or to models that discuss the interests of both insurers and reinsurers. Recent references on these issues can be found in Cai and Wei (2012), Cai et al. (2013), Cheung et al. (2013), Hürlimann (2011), and references therein.

In most studies on optimal reinsurance, one assumes that a reinsurer will pay the promised loss $I(X)$ regardless of its solvency or equivalently, one ignores the potential default by a reinsurer. Indeed, default risk can be reduced if a reinsurer has a sufficiently large initial capital or reserve. However, default might occur even if the initial capital of a reinsurer is very large. In a reinsurance contract I, a reinsurer may fail to pay the promised amount $I(X)$ or a reinsurer may default due to different reasons. One of the main reasons could be that the promised amount $I(X)$ exceeds the reinsurer's solvency. The larger is the initial reserve of a reinsurer, the smaller is the likelihood that default will occur. This is why the initial capital of a seller (reinsurer) of a reinsurance contract should meet some requirements by regulations to reduce default risk.

Recently, counterparty default risks in reinsurance designs or other related studies have been discussed in Asimit, Badescu, and Cheung (2013), Bernard and Ludkovski (2012), Burren (2013), Cummins et al. (2002), Dana and Scarsini (2007), Menegatti (2009), and references therein. Several models with default risks have been proposed in these references. However, in the references for reinsurance designs with default risks such as Asimit, Badescu, and Cheung (2013), Bernard and Ludkovski (2012), and so on, they assume a constant initial capital or reserve for a reinsurer regardless of how large a reinsurer's promised amount $I(X)$ is, or they do not consider the influence of a reinsurer's initial reserve on optimal reinsurance strategies. Indeed, a reasonable requirement on a reinsurer could be that the larger is the promised indemnity of a reinsurer, the larger the initial reserve of a reinsurer should be.

In this paper, we propose a reinsurance model with regulatory initial capital and default risk. We assume that the initial capital or reserve of a seller (reinsurer) of a reinsurance contract I is determined through regulation by the value-at-risk (VaR) of its promised indemnity $I(X)$, and denote the initial capital of the reinsurer by $\omega_I = \mathrm{VaR}_\alpha(I(X))$, where $\mathrm{VaR}_\alpha(Z) = \inf\{z : \Pr\{Z > z\} \le \alpha\}$ is the VaR of a random variable Z and $0 < \alpha < 1$ is called the risk level. Usually, α is a small value such as $\alpha = 0.01$ or 0.05. We assume that the reinsurer charges a reinsurance premium P_I based on the promised indemnity $I(X)$. The insurer is aware of the potential default by the reinsurer but the worst case for the insurer is that the reinsurer only pays $\omega_I + P_I$ if $I(X) > \omega_I + P_I$. Thus, when the insurer is seeking for optimal reinsurance strategies and taking account of the potential default by the reinsurer, the insurer assumes the worst indemnity $I(X) \wedge (\omega_I + P_I)$ from the reinsurer. Indeed, when $\omega_I = \mathrm{VaR}_\alpha(I(X))$, we know $\Pr\{I(X) > \omega_I + P_I\} \le \alpha$ or the probability of default by the reinsurer is not greater than the value α, which could be an acceptable risk level for the insurer. Hence, under the proposed reinsurance model, the total retained risk or cost of the insurer is $X - I(X) \wedge (\omega_I + P_I) + P_I$ and the insurer's terminal wealth is

$w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$, where w_0 is the initial capital of the insurer.

We point out that in the above proposed model, the minimum or guaranteed available capital of the reinsurer at the end of the contract is the (regulatory) initial reserve plus the reinsurance premium. However, the actual available capital of the reinsurer at the end of the contract may be different from the initial reserve plus the reinsurance premium. For example, the actual available capital of the reinsurer may be higher than the initial reserve plus the reinsurance premium if the reinsurer can use the capitals or reserves from its other portfolios or if the reinsurer has investment profits on the initial reserve and/or the reinsurance premium or if the reinsurer has other assets. On the other hand, the actual available capital of the reinsurer may be lower than the initial reserve plus the reinsurance premium if the reinsurer spends some of the initial reserve and/or the reinsurance premium or if the reinsurer has investment losses on the initial reserve and/or the reinsurance premium. Each of these scenarios may result in different reinsurance models. Indeed, our proposed model is just one of many possible mathematical models for reinsurance designs. In our proposed model, we emphasize that the initial reserve of the reinsurer is determined by the VaR of the reinsurer's promised indemnity due to regulatory requirements, the insurer believes that the guaranteed or minimum available capital of the reinsurer at the end of the contract is the initial reserve plus the reinsurance premium, and the probability of default by the reinsurer is not greater than the risk level of the VaR.

In the first part of the paper, we assume that the insurer wants to determine an optimal reinsurance strategy I^* that maximizes the expected utility of its terminal wealth of $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$ under an increasing concave utility function v. That is, we study the following optimization problem:

$$\begin{aligned} &\max_{I \in \mathcal{I}} \mathbb{E}\left[v\left(w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I\right)\right] \\ &\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p, \end{aligned} \tag{1.1}$$

where $0 < p \leq (1+\theta)\mathbb{E}(X)$ is a given reinsurance premium budget for the insurer. This optimal reinsurance problem can be viewed as the extension of the classical optimal reinsurance problem without default risk, which was first studied by Arrow (1963) and Borch (1960). As illustrated later in the paper, as $\alpha \to 0$, Problem (1.1) is reduced to the classical optimal reinsurance problem without default risk studied by Arrow (1963) and Borch (1960). We can also recover the solutions of Arrow (1963) and Borch (1960) from our solution to Problem (1.1).

In the second part of the paper, we assume that the insurer wants to use VaR at a risk level $0 < \beta < 1$ to control its total retained risk of $X - I(X) \wedge (\omega_I + P_I) + P_I$ and then seeks an optimal reinsurance strategy I^* that minimizes this VaR. That is, we consider the following optimization problem:

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\beta \left(X - I(X) \wedge (\omega_I + P_I) + P_I \right). \tag{1.2}$$

This problem is an extension of recent studies on optimal reinsurance under risk measures without default risk such as Balbás et al. (2009), Asimit, Badescu and Verdoncj (2013), Cai et al. (2008), Cheung (2010), Chi (2012), Chi and Tan (2011), and references therein. In particular, and as will be shown later, when $\alpha \leq \beta$, Problem (1.2) reduces to the problem without default risk, which was studied by Cheung et al. (2014).

As illustrated in the paper, the solutions to Problems (1.1) and (1.2) are more complicated than those without default risk. Furthermore, the optimal reinsurance strategies in the presence of regulatory initial capital and the counterparty default risk are different both from the optimal reinsurance strategies in the absence of the counterparty default risk and from the optimal reinsurance strategies in the presence of the counterparty default risk but without the regulatory initial capital.

To avoid tedious discussions and arguments, in this paper, we simply assume that the survival function $S_X(x)$ of the underlying loss random variable X is continuous and strictly decreasing on $(0, \infty)$ with $0 < S_X(0) \leq 1$ or equivalently $0 \leq F_X(0) < 1$. The assumption that the distribution function $F_X(x)$ has a possible jump at zero means it is possible that there are no claims from the insurer. Furthermore, we assume that $P_I = (1+\theta)\mathbb{E}[I(X)]$, i.e., the reinsurance premium is determined by the expected value principle, where $\theta > 0$.

The rest of the paper is organized as follows. In Section 2, we study Problem (1.1) and derive the optimal reinsurance strategy that maximizes the expected utility of an insurer's terminal wealth. In Section 3, we consider Problem (1.2) and present the optimal reinsurance strategy that minimizes the VaR of an insurer's total retained risk. In Section 4, we illustrate the results derived in Sections 2 and 3 by numerical examples and discuss the influence of the risk level α and the distribution of the underlying loss on the optimal strategies. Concluding remarks are given in Section 5. The proofs of all the results derived in this paper are given in the appendix.

2 Optimal reinsurance maximizing the expected utility of an insurer's terminal wealth

In this section, we study Problem (1.1). First, we point out that by taking $u(x) = -v(w_0 - p - x)$, Problem (1.1) is equivalent to the following minimization problem:

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] \\ &\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p, \end{aligned} \tag{2.1}$$

where u is an increasing convex function. Throughout this section, we assume $\mathbb{E}|u^{(k)}(X)| < \infty$ for $k = 0, 1, 2$ and denote the density function of $F_X(x)$ on $(0, \infty)$ by $f_X(x)$ or $f_X(x) = F'_X(x) = -S'_X(x)$ for $x \in (0, \infty)$.

Second, we notice that for any $I \in \mathcal{I}$, the function $I(x)$ is continuous on $[0, \infty)$. In addition, for any $0 \le x < y$, if $I(y) = I(x) + y - x$, then $I(t) = t - (x - I(x))$ on the interval $[x, y]$.

For any fixed premium budget $0 < p \le (1+\theta)\mathbb{E}[X]$, we denote the set of all feasible contracts with the given reinsurance premium p by

$$\mathcal{I}_p = \{I \in \mathcal{I} : P_I = (1+\theta)\mathbb{E}[I(X)] = p\}.$$

Note that if $p = (1+\theta)\mathbb{E}[X]$, then $\mathcal{I}_p = \{I(x) \equiv x\}$, which contains only one reinsurance contract $I(x) \equiv x$, and thus Problem (2.1) reduces to the trivial case. Hence, throughout this section, we assume $p \in \big(0,\ (1+\theta)\mathbb{E}[X]\big)$. Then Problem (2.1) can be written as

$$\min_{I \in \mathcal{I}_p} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] = \min_{I \in \mathcal{I}_p} H(I), \tag{2.2}$$

where

$$H(I) = \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big].$$

To solve the infinite-dimensional optimization Problem (2.2), we first show that for any given reinsurance contract $I \in \mathcal{I}_p$, there exists a contract $k_I \in \mathcal{I}_p$ such that $H(k_I) \le H(I)$ and k_I is determined by four variables. Thus, we can reduce the infinite-dimensional optimization Problem (2.2) to a finite-dimensional optimization problem. To do so, we recall the definition of convex order.

Definition 2.1. A random variable X is said to be smaller than random variable Y in convex order, denoted as $X \preccurlyeq_{cx} Y$, if $\mathbb{E}[g(X)] \le \mathbb{E}[g(Y)]$ for any convex function g such that the expectations exist.

The following lemma was given by Ohlin (1969) and it provides a useful criterion for the convex order.

Lemma 2.1. *Let X be a random variable, h_1 and h_2 be increasing functions such that $\mathbb{E}[h_1(X)] \le \mathbb{E}[h_2(X)]$. If there exists $x_0 \in \mathbb{R} \cup \{+\infty\}$ such that $h_1(x) \ge h_2(x)$ for all $x < x_0$ and $h_1(x) \le h_2(x)$ for all $x > x_0$, then $h_1(X) \preccurlyeq_{cx} h_2(X)$.*

The following theorem shows that for any given reinsurance contract $I \in \mathcal{I}_p$, there exists a contract $k_I \in \mathcal{I}_p$ such that $H(k_I) \le H(I)$. The proof of the theorem is given in the appendix.

Theorem 2.1. *Denote $a = \mathrm{VaR}_\alpha(X)$. For any $I \in \mathcal{I}_p$, there exists $k_I \in \mathcal{I}_p$ such that $H(k_I) \le H(I)$ and k_I is defined as*

$$\begin{aligned} k_I(x) &= (x-d_1)^+ - (x-(d_1+I(a)))^+ + (x-d_2)^+ - (x-(d_2+p))^+ + (x-d_3)^+ \\ &= \begin{cases} 0, & \text{for} \quad 0 \le x < d_1, \\ x-d_1, & \text{for} \quad d_1 \le x < d_1+I(a), \\ I(a), & \text{for} \quad d_1+I(a) \le x < d_2, \\ I(a)+x-d_2, & \text{for} \quad d_2 \le x < d_2+p, \\ I(a)+p, & \text{for} \quad d_2+p \le x < d_3, \\ I(a)+p+x-d_3, & \text{for} \quad d_3 \le x, \end{cases} \end{aligned} \tag{2.3}$$

for some (d_1, d_2, d_3) satisfies $0 \le d_1 \le d_1 + I(a) \le a \le d_2 < d_2 + p \le d_3 \le \infty$.

Remark 2.1. We point out that for any $I \in \mathcal{I}$, I is continuous and non-decreasing. Thus, $\omega_I = \mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X)) = I(a)$. As $\alpha \to 0$ or $a \to \infty$, we have $I(a) \to I(\infty)$ and hence $I \wedge (\omega_I + P_I) = I$. In other words, as $\alpha \to 0$ or $a \to \infty$, Problem (2.1) is reduced to the following classical problem without the default risk:

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathbb{E}\left[u\left(X - I(X)\right)\right] \\ &\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p. \end{aligned} \tag{2.4}$$

The optimal solution to Problem 2.4, given in Arrow (1963) or Borch (1960), is a stop-loss reinsurance $I_0^*(x) = (x-d^*)^+$, where d^* is uniquely determined by the premium condition. This classical result can also be recovered from our Theorem 2.1. Indeed, as $a \to \infty$, the ceded loss function k_I in (2.3) is reduced to the form

$$k_I(x) = (x-d_1)^+ - (x - d_1 - I(\infty))^+,$$

for some $0 \le d_1 \le \infty$ determined by the premium condition $(1+\theta)\mathbb{E}[k_I(X)] = p$. If $I(\infty) = \infty$, then $k_I(x) = (x-d^*)^+$. If $I(\infty) < \infty$, it is easy to see that $d_1 < d^*$ and

$x-(x-d^*)^+$ crosses $x-k_I(x)$ at most once from above on $[0,\infty)$. Thus, by Lemma 2.1, we have $X-(X-d^*)^+ \preccurlyeq_{cx} X-k_I(X)$. Thus, $\mathbb{E}\left[u\left(X-(X-d^*)^+\right)\right] \le \mathbb{E}\left[u\left(X-k_I(X)\right)\right]$ for any k_I with $I(\infty)<\infty$. Therefore, in either $I(\infty)=\infty$ or $I(\infty)<\infty$, $X-(X-d^*)^+ \preccurlyeq_{cx} X-k_I(X) \preccurlyeq_{cx} X-I(X)$. Thus, $(x-d^*)^+$ is the optimal solution to Problem (2.4).

In the rest of this paper, we assume $0<\alpha<S_X(0)$ and thus $0<a<\infty$. Otherwise, all the VaRs considered in the paper at the risk level $\alpha \in [S_X(0),1)$ are equal to zero, which are trivial cases. Theorem 2.1 reduces the infinite-dimensional optimization Problem (2.2) to a finite-dimensional optimization problem. To see that, we denote

$$\mathcal{I}_{p,0}=\left\{I\in\mathcal{I}_p \text{ and } I \text{ has the expression (2.3)}\right\}.$$

Then, thanks to Theorem 2.1, we see that Problem (2.2) is equivalent to the following minimization problem

$$\min_{I\in\mathcal{I}_{p,0}} \mathbb{E}\Big[u\Big(X-I(X)\wedge(\omega_I+P_I)\Big)\Big] = \min_{I\in\mathcal{I}_{p,0}} H(I). \tag{2.5}$$

It is still not easy to solve Problem (2.5) since it involves four variables of d_1, d_2, d_3 and $I(a)$. To solve Problem (2.5), we first need to discuss the properties of the set $\mathcal{I}_{p,0}$. We say that two contracts I_1 and I_2 in $\mathcal{I}_{p,0}$ are the same if they are equal almost everywhere with respect to Lebesgue's measure. Suppose $I_1(a)\neq I_2(a)$, since both contracts are continuous at point a, there exists $\delta\in(0,a)$ such that $I_1(x)\neq I_2(x)$ on the open interval $(a-\delta,a+\delta)$ and thus I_1 and I_2 are not the same. Therefore, for $\xi_1\neq\xi_2$,

$$\big\{I\in\mathcal{I}_{p,0}:\ I(a)=\xi_1\big\}\bigcap\big\{I\in\mathcal{I}_{p,0}:\ I(a)=\xi_2\big\}=\emptyset.$$

For any given $\xi\in[0,a]$, define contract $I_{0,\xi}\in\mathcal{I}$ as

$$I_{0,\xi}(x) = (x-a+\xi)^+-(x-a)^+ = \begin{cases} 0, & \text{for } \ 0\le x<a-\xi,\\ x-a+\xi, & \text{for } \ a-\xi\le x<a,\\ \xi, & \text{for } \ a\le x<\infty, \end{cases} \tag{2.6}$$

and denote the reinsurance premium based on $I_{0,\xi}$ by $p_{0,\xi}$ or

$$p_{0,\xi}=(1+\theta)\mathbb{E}\left[I_{0,\xi}(X)\right]=(1+\theta)\int_{a-\xi}^{a} S_X(x)\mathrm{d}x. \tag{2.7}$$

Furthermore, define contract $I_{M,\xi}\in\mathcal{I}$ as

$$I_{M,\xi}(x) = x-(x-\xi)^+ +(x-a)^+ = \begin{cases} x, & \text{for } \ 0\le x<\xi,\\ \xi, & \text{for } \ \xi\le x<a,\\ \xi+x-a, & \text{for } \ a\le x<a+p, \end{cases} \tag{2.8}$$

and denote the reinsurance premium based on $I_{M,\xi}$ by $p_{M,\xi}$ or

$$p_{M,\xi} = (1+\theta)\mathbb{E}\left[I_{M,\xi}(X)\right] = (1+\theta)\left(\int_0^{\xi} + \int_a^{\infty}\right) S_X(x)\mathrm{d}x. \tag{2.9}$$

It is easy to check that $I_{0,\xi}$, $I_{M,\xi} \in \mathcal{I}$ and that for any $I \in \mathcal{I}_{p,0}$ satisfying $I(a) = \xi$, the following inequalities hold: $I_{0,\xi}(x) \le I(x) \le I_{M,\xi}(x)$ for all $x \ge 0$. Thus, when $p_{0,\xi} \le p \le p_{M,\xi}$, the set $\{I \in \mathcal{I}_{p,0} : I(a) = \xi\}$ is not empty. In order to identify all valid ξ so that $\{I \in \mathcal{I}_{p,0} : I(a) = \xi\} \ne \emptyset$, throughout this paper, we denote

$$\xi_0 = \inf\left\{\xi \in [0, a] \text{ such that } p_{M,\xi} \ge p\right\} \tag{2.10}$$

and

$$\xi_M = \sup\big\{\xi \in [0, a] \text{ such that } p_{0,\xi} \le p\big\}. \tag{2.11}$$

It is easy to see that $0 \le \xi_0 < a$, $0 < \xi_M \le a$, $0 \le \xi_0 \le \xi_M \le a$ and that the set $\{I \in \mathcal{I}_{p,0} :\, I(a) = \xi\} \ne \emptyset$ if and only if $\xi \in [\xi_0,\, \xi_M]$. Hence, we can write $\mathcal{I}_{p,0}$ as the union of disjoint non-empty sets, namely

$$\mathcal{I}_{p,0} = \bigcup_{\xi_0 \le \xi \le \xi_M} \left\{I \in \mathcal{I}_{p,0} : I(a) = \xi\right\}.$$

It will be proved in Theorem 2.2 that Problem (2.5) is equivalent to the following two-step minimization problem:

$$\min_{0 \le \xi \le a} \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I) = \min_{\xi_0 \le \xi \le \xi_M} \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I) = \min_{\xi_0 \le \xi \le \xi_M} H(I_\xi^*), \tag{2.12}$$

where $H(I_\xi^*) = \min\limits_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I)$ and I_ξ^* is the minimizer that solves the inner minimization problem $\min\limits_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I)$ for any given $\xi \in [\xi_0,\, \xi_M]$.

To derive the expression of the minimizer I_ξ^* of (2.12), we define contract $I_{1,\xi}(x) \in \mathcal{I}$ as

$$I_{1,\xi}(x) = (x-a+\xi)^+ - (x-a-p)^+ = \begin{cases} 0, & \text{for} \quad 0 \le x < a-\xi, \\ x-a+\xi, & \text{for} \quad a-\xi \le x < a+p, \\ \xi+p, & \text{for} \quad a+p \le x < \infty, \end{cases} \tag{2.13}$$

and denote the reinsurance premium based on $I_{1,\xi}$ by $p_{1,\xi}$ or

$$p_{1,\xi} = (1+\theta)\mathbb{E}\left[I_{1,\xi}(X)\right] = (1+\theta)\int_{a-\xi}^{a+p} S_X(x)\mathrm{d}x. \tag{2.14}$$

Furthermore, we define contract $I_{2,\xi}(x) \in \mathcal{I}$ as

$$\begin{aligned} I_{2,\xi}(x) &= x-(x-\xi)^+ + (x-a)^+ - (x-a-p)^+ \\ &= \begin{cases} x, & \text{for} \quad 0 \le x < \xi, \\ \xi, & \text{for} \quad \xi \le x < a, \\ \xi + x - a, & \text{for} \quad a \le x < a+p, \\ \xi + p, & \text{for} \quad a+p \le x < \infty, \end{cases} \end{aligned} \tag{2.15}$$

and denote the reinsurance premium based on $I_{2,\xi}$ by $p_{2,\xi}$ or

$$p_{2,\xi} = (1+\theta)\mathbb{E}\left[I_{2,\xi}(X)\right] = (1+\theta)\left(\int_0^{\xi} + \int_a^{a+p}\right) S_X(x)\mathrm{d}x. \tag{2.16}$$

In addition, throughout this paper, we denote

$$\xi_1 = \begin{cases} \sup\left\{\xi \in [\xi_0, \xi_M] : \, p_{2,\xi} < p\right\}, & \text{if} \quad p_{2,\xi_0} < p, \\ \xi_0, & \text{if} \quad p_{2,\xi_0} \ge p, \end{cases} \tag{2.17}$$

and

$$\xi_2 = \begin{cases} \inf\left\{\xi \in [\xi_0, \xi_M] : \, p_{1,\xi} > p\right\}, & \text{if} \quad p_{1,\xi_M} > p, \\ \xi_M, & \text{if} \quad p_{1,\xi_M} \le p. \end{cases} \tag{2.18}$$

It is not hard to check by the definitions of ξ_1 and ξ_2 that $\xi_0 \le \xi_1 \le \xi_2 \le \xi_M$ and that at least one of the three inequalities is strict. Furthermore, we can verify that (1) $\xi_0 = \xi_1$ implies $\xi_0 = \xi_1 = \xi_2 = 0 < \xi_M$; (2) $\xi_2 = \xi_M$ implies $\xi_0 < \xi_1 = \xi_2 = \xi_M = a$; and (3) $\xi_1 = \xi_2$ implies $\xi_0 = \xi_1$ or $\xi_2 = \xi_M$. Therefore, the set $[\xi_0, \, \xi_M]$ has only the following three possible partitions: (1) $[\xi_0, \, \xi_M] = [0, \, \xi_M]$ if $\xi_0 = \xi_1$; (2) $[\xi_0, \, \xi_M] = [\xi_0, \, a]$ if $\xi_2 = \xi_M$; and (3) $[\xi_0, \, \xi_M] = [\xi_0, \, \xi_1] \cup [\xi_1, \, \xi_2] \cup [\xi_2, \, \xi_M]$ if $\xi_0 < \xi_1 < \xi_2 < \xi_M$.

Now, in the following lemma, for any given $\xi \in [\xi_0, \, \xi_M]$, we solve the inner minimization problem $\min\limits_{I \in \mathcal{I}_{p,0}, \; I(a)=\xi} H(I)$ of (2.12). The proof of the lemma is given in the appendix.

Lemma 2.2. *For a given $\xi \in [\xi_0, \xi_M]$, let I_ξ^* be the optimal solution to the minimization problem $\min\limits_{I \in \mathcal{I}_{p,0}, \; I(a)=\xi} H(I)$. Then, I_ξ^* can be summarized as follows.*

(1) If $\xi_0 \le \xi \le \xi_1$ and $\xi_0 < \xi_1$, then

$$\begin{aligned} I_\xi^*(x) &= x-(x-\xi)^+ + (x-a)^+ - (x-a-p)^+ + (x-d_{3,\xi})^+ \\ &= \begin{cases} x, & \textit{for} \quad 0 \le x < \xi, \\ \xi, & \textit{for} \quad \xi \le x < a, \\ \xi + x - a, & \textit{for} \quad a \le x < a+p, \\ \xi + p, & \textit{for} \quad a+p \le x < d_{3,\xi}, \\ \xi + p + x - d_{3,\xi}, & \textit{for} \quad d_{3,\xi} \le x, \end{cases} \end{aligned}$$

where $d_{3,\xi}$ *is determined by* $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

(2) If $\xi_1 \le \xi \le \xi_2$ *and* $\xi_1 < \xi_2$, *then*

$$\begin{aligned} I_\xi^*(x) &= (x-d_{1,\xi})^+ - (x-d_{1,\xi}-\xi)^+ + (x-a)^+ - (x-a-p)^+ \\ &= \begin{cases} 0, & \text{for} \quad 0 \le x < d_{1,\xi}, \\ x-d_{1,\xi}, & \text{for} \quad d_{1,\xi} \le x < d_{1,\xi}+\xi, \\ \xi, & \text{for} \quad d_{1,\xi}+\xi \le x < a, \\ \xi+x-a, & \text{for} \quad a \le x < a+p, \\ \xi+p, & \text{for} \quad a+p \le x < \infty, \end{cases} \end{aligned}$$

where $d_{1,\xi}$ *is determined by* $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

(3) If $\xi_2 \le \xi \le \xi_M$ *and* $\xi_2 < \xi_M$, *then*

$$\begin{aligned} I_\xi^*(x) &= (x-a+\xi)^+ - (x-a)^+ + (x-d_{2,\xi})^+ - (x-d_{2,\xi}-p)^+ \\ &= \begin{cases} 0, & \text{for} \quad 0 \le x < a-\xi, \\ x-a+\xi, & \text{for} \quad a-\xi \le x < a, \\ \xi, & \text{for} \quad a \le x < d_{2,\xi}, \\ \xi+x-d_{2,\xi}, & \text{for} \quad d_{2,\xi} \le x < d_{2,\xi}+p, \\ \xi+p, & \text{for} \quad d_{2,\xi}+p \le x < \infty, \end{cases} \end{aligned}$$

where $d_{2,\xi}$ *is determined by* $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

For any given $\xi \in [\xi_0, \xi_M]$ and the corresponding optimal ceded loss function I_ξ^* given in Lemma 2.2, we define the function h of ξ as

$$h(\xi) = H(I_\xi^*). \tag{2.19}$$

Thus, Lemma 2.2 implies that $\min\limits_{I \in \mathcal{I}_{p,0},\ I(a)=\xi} H(I) = H(I_\xi^*) = h(\xi)$.

Now, using Lemma 2.2, we obtain the optimal solution to Problem (2.2) in the following theorem. The proof of the theorem is given in the appendix.

Theorem 2.2. *Assume* $0 < p < (1+\theta)\mathbb{E}[X]$. *Then Problem (2.2) is equivalent to Problem (2.12) and the optimal solution to Problem (2.2), denoted by* I^*, *is summarized as follows.*

(1) If $\xi_1 = a$, *the optimal solution is*

$$I^*(x) = x - (x-a-p)^+ + (x-d_{3,a})^+ = \begin{cases} x, & \text{for} \quad 0 \le x < a+p, \\ a+p, & \text{for} \quad a+p \le x < d_{3,a}, \\ a+p+x-d_{3,a}, & \text{for} \quad d_{3,a} \le x, \end{cases}$$

where $d_{3,a}$ *is determined by* $(1+\theta)\mathbb{E}\left[I^*(X)\right] = p$.

(2) If $\xi_1 < a$ and $h'(\xi_M) \le 0$, then $\xi_M = a$ and the optimal solution is

$$I^*(x) = x - (x-a)^+ + (x - d_{2,a})^+ - (x - d_{2,a} - p)^+ = \begin{cases} x, & \text{if } 0 \le x < a, \\ a, & \text{if } a \le x < d_{2,a}, \\ a + x - d_{2,a}, & \text{if } d_{2,a} \le x < d_{2,a} + p, \\ a + p, & \text{if } d_{2,a} + p \le x < \infty, \end{cases}$$

where $d_{2,a}$ is determined by $(1+\theta)\mathbb{E}[I^*(X)] = p$.

(3) If $\xi_1 < a$ and $h'(\xi_M) > 0$, then there exists $\xi^* \in [\xi_2, \xi_M]$ such that $h'(\xi^*) = 0$ and the optimal solution is

$$\begin{aligned} I^*(x) &= (x - a + \xi^*)^+ - (x-a)^+ + (x - d_{2,\xi^*})^+ - (x - d_{2,\xi^*} - p)^+ \\ &= \begin{cases} 0, & \text{for } 0 \le x < a - \xi^*, \\ x - a + \xi^*, & \text{for } a - \xi^* \le x < a, \\ \xi^*, & \text{for } a \le x < d_{2,\xi^*}, \\ \xi^* + x - d_{2,\xi^*}, & \text{for } d_{2,\xi^*} \le x < d_{2,\xi^*} + p, \\ \xi^* + p, & \text{for } d_{2,\xi^*} + p \le x < \infty, \end{cases} \end{aligned}$$

where d_{2,ξ^*} is determined by $(1+\theta)\mathbb{E}[I^*(X)] = p$.

Remark 2.2. It is easy to see that the optimal solution I^* in all three cases of Theorem 2.2 can be expressed using a unified formula as

$$\begin{aligned} I^*(x) &= (x - d_1^*) - (x-a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+ \\ &= \begin{cases} 0, & \text{for } 0 \le x < d_1^*, \\ x - d_1^*, & \text{for } d_1^* \le x < a, \\ a - d_1^*, & \text{for } a \le x < d_2^*, \\ a - d_1^* + x - d_2^*, & \text{for } d_2^* \le x < d_2^* + p, \\ a - d_1^* + p, & \text{for } d_2^* + p \le x < d_3^*, \\ a - d_1^* + p + x - d_3^*, & \text{for } d_3^* \le x, \end{cases} \end{aligned} \tag{2.20}$$

where

$$(d_1^*, d_2^*, d_3^*) = \begin{cases} (0, a, d_{3,a}), & \text{if } \xi_1 = a, \\ (0, d_{2,a}, +\infty), & \text{if } \xi_1 \le a \text{ and } h'(\xi_M) \le 0, \\ (a - \xi^*, d_{2,\xi^*}, +\infty), & \text{if } \xi_1 < a \text{ and } h'(\xi_M) > 0. \end{cases}$$

Remark 2.3. We point out that for a feasible contract $I \in \mathcal{I}_p$, if $I(x) \le \omega_I + P_I = I(a) + p$ for all $x \ge 0$, then the contract is a default risk-free contact, i.e., the insurer will not face default risk with this contract.

In (1) of Theorem 2.2, which corresponds to the case where $\xi_1 = a$, if $p_{2,a} = p$, where $p_{2,\xi}$ is defined in (2.16), then $d_{3,a} = \infty$ and the optimal contract I^* is reduced to $I^*(x) = x - (x - a - p)^+ = I^*(x) \wedge (I^*(a) + p) \le I^*(a) + p$, namely the optimal contract is a default risk-free contract. However, if $p_{2,a} < p$, then there does not exist a default risk-free contract in $\mathcal{I}_p$. Indeed, suppose that $I \in \mathcal{I}_p$ is a default risk-free contract, then $I(x) \le I_{2,\xi}(x)$ for all $x \ge 0$, where $I_{2,\xi}$ is defined by (2.15) and $\xi = I(a)$. Since $\xi_1 = a$, by the definition of ξ_1 given in (2.17), we have $P_I \le p_{2,\xi} < p$. Thus, $I \notin \mathcal{I}_p$.

In (2) and (3) of Theorem 2.2, which correspond to the case where $\xi_1 < a$, it is obvious that the optimal solution I^* in both cases satisfy $I^*(x) \le I^*(a) + p$, namely the insurer will not face default risk with the two optimal contracts.

In summary, Theorem 2.2 suggests that, in order to lower default risk, an insurer should choose a contract without default risk as long as this kind of contract is available. This leads to limits for indemnities on the tails of the optimal contracts.

In addition, it has been mentioned that $I_0^* = (x - d^*)^+$ is the optimal solution to the classical Problem (2.4) in the absence of default risk. Note that $I_0^* \in \mathcal{I}_p$. It is easy to check that in all three cases of Theorem 2.2, the optimal contract I^* of Theorem 2.2 satisfies $\omega_{I^*} = I^*(a) > \omega_{I_0^*} = I_0^*(a) = (a - d^*)^+$, which means that the reinsurer will set up a higher initial reserve if the insurer chooses the optimal contract I^* of Theorem 2.2 than if the insurer chooses I_0^*. In this way, the insurer can reduce the default risk.

3 Optimal reinsurance minimizing the VaR of an insurer's total retained risk

In this section, we study Problem (1.2). To do so, for any $I \in \mathcal{I}$, we denote

$$V(I) = \mathrm{VaR}_\beta\Big(X - I(X) \wedge (\omega_I + P_I) + P_I\Big),$$

where $0 < \beta < S_X(0)$ is assumed throughout this section .

Thus Problem (1.2) is reformulated as

$$\min_{I \in \mathcal{I}} V(I). \tag{3.1}$$

Throughout this section, we denote $a = \mathrm{VaR}_\alpha(X)$ and $b = \mathrm{VaR}_\beta(X)$. For any $I \in \mathcal{I}$, we see that the function $x - I(x) \wedge (\omega_I + P_I)$ is continuous and non-decreasing on $[0, \infty)$. Thus, due to the translation invariance and preservation of VaR under continuous and non-decreasing functions, the objective function $V(I)$ can be expressed as

$$V(I) = b - I(b) \wedge (I(a) + P_I) + P_I.$$

Again, we can reduce the infinite-dimensional optimization Problem (3.1) to a finite-dimensional optimization problem. In doing so, we first give the following lemma. The proof of the lemma is given in the appendix.

Lemma 3.1. *For any $I_1, I_2 \in \mathcal{I}$, if $I_1(a) = I_2(a)$, $I_1(b) = I_2(b)$, and $P_{I_1} \leq P_{I_2}$, then $V(I_1) \leq V(I_2)$.*

Using this lemma, we can show in the following theorem that for any $I \in \mathcal{I}$, there exists $m_I \in \mathcal{I}$ such that $V(m_I) \leq V(I)$. The proof of the theorem is given in the appendix.

Theorem 3.1. *For any $I \in \mathcal{I}$, there exists $m_I \in \mathcal{I}$ satisfying $V(m_I) \leq V(I)$ and m_I is defined as*

$$\begin{aligned} m_I(x) &= (x-d_1)^+ - (x - a\wedge b)^+ + (x-d_2)^+ - (x - a\vee b)^+ \\ &= \begin{cases} 0, & \textit{for} \quad 0 \leq x < d_1, \\ x - d_1, & \textit{for} \quad d_1 \leq x < a\wedge b, \\ a\wedge b - d_1, & \textit{for} \quad a \wedge b \leq x < d_2, \\ a\wedge b - d_1 + x - d_2, & \textit{for} \quad d_2 \leq x < a\vee b, \\ a + b - d_1 - d_2, & \textit{for} \quad a\vee b \leq x < \infty, \end{cases} \end{aligned} \tag{3.2}$$

where $d_1 = a\wedge b - I(a\wedge b)$ and $d_2 = a\vee b - (I(a\vee b) - I(a\wedge b))$ satisfying $0 \leq d_1 \leq a\wedge b \leq d_2 \leq a \vee b$.

For any $(d_1, d_2) \in [0,\ a\wedge b] \times [a\wedge b,\ a\vee b]$, we define

$$\mathcal{I}_{d_1,d_2} = \{I \in \mathcal{I} :\ I \text{ has the expression (3.2)}\}.$$

Then, for $I \in \mathcal{I}_{d_1,d_2}$, define the function

$$v(d_1, d_2) = V(I) = b - I(b) \wedge (I(a) + P_I) + P_I.$$

Thus, by Theorem 3.1, we see that Problem (3.1) is equivalent to the following optimization problem:

$$\min_{I \in \mathcal{I}_{d_1,\, d_2}} V(I) = \min_{(d_1,\, d_2) \in [0,\, a\wedge b] \times [a\wedge b,\, a\vee b]} v(d_1,\, d_2), \tag{3.3}$$

which is a finite-dimensional optimization problem.

By solving Problem (3.3), we obtain the optimal solution to Problem (3.1) in the following theorem. The proof of the theorem is given in the appendix.

Theorem 3.2. *Let I^* be the optimal solution to Problem (3.1).*

(1) If $\alpha \le \beta$, then

$$\begin{aligned} I^*(x) &= (x - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ - (x-b)^+ \\ &= \begin{cases} 0, & \textit{for} \quad 0 \le x < b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X), \\ x - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X), & \textit{for} \quad b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \le x < b, \\ b - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X), & \textit{for} \quad b \le x < \infty. \end{cases} \end{aligned}$$

(2) If $\alpha > \beta$ and $\alpha \le \frac{1}{1+\theta}$, there exists $d_0 \in \mathbb{R}$ satisfying

$$(1+\theta)\int_{d_0}^{b} S_X(x)\mathrm{d}x = b - a. \tag{3.4}$$

Then

$$\begin{aligned} I^*(x) &= (x - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\})^+ - (x-b)^+ \\ &= \begin{cases} 0, & \textit{for} \quad 0 \le x < \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, \\ x - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, & \textit{for} \quad \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\} \le x < b, \\ b - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, & \textit{for} \quad b \le x < \infty, \end{cases} \end{aligned}$$

Remark 3.1. The optimal solution I^* in the two cases of Theorem 3.2 can be expressed in a unified formula as

$$I^*(x) = (x - d^*)^+ - (x-b)^+,$$

where $d^* = b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$ if $\alpha \le \beta$ and $d^* = \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}$ if $\alpha > \beta$ and $\alpha \le \frac{1}{1+\theta}$.

When $\alpha \le \beta$, which means the reinsurer is more conservative than the insurer, we have $I(b) \le I(a)$, thus Problem (1.2) is reduced to the following model without default risk:

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\beta(X - I(X) + P_I), \tag{3.5}$$

which was studied by Cheung et al. (2014).

Since the insurer measures its risk, based on VaR, at a higher risk level $\beta \ge \alpha$, the initial reserve $\omega_I = \mathrm{VaR}_\alpha(I(X))$ set by the reinsurer at a lower risk level α is high enough to ensure that default will not occur.

On the other hand, if the insurer is more conservative than the reinsurer or $\alpha > \beta$, in order to reduce the default risk, the insurer should require a lower deductible or ask the reinsurer to cover more loss, to force the reinsurer to set up a higher initial reserve. For the case where $\alpha > 1/(1+\theta)$, the optimal solution I^* has no closed form and the case is not interesting since in practice, α is a small value and usually $\alpha < 1/(1+\theta)$ holds.

4 Numerical Examples

In this section, we use numerical examples to illustrate the optimal solutions derived in Sections 2 and 3. We assume the underlying loss faced by the insurer has an exponential distribution or a Pareto distribution. Thus, in one case, the loss is light-tailed, and in the other case, the loss is heavy-tailed. We will calculate the optimal forms under the two loss distributions and consider the influence of the distribution, the risk level α, and the reinsurance premium budget p on the optimal reinsurance contracts.

Suppose random variables X and Y have exponential and Pareto distributions, respectively. Assume their survival functions are $S_X(x) = e^{-x/\mu}$ and $S_Y(x) = \left(\frac{\lambda}{x+\lambda}\right)^\gamma$ for any $x \geq 0$, respectively. All numerical results are given under the setting of $\theta = 0.1$, $\mu = 100$, $\lambda = 200$ and $\gamma = 3$. Under this setting, X and Y have the same mean 100, and the fixed premium budget $p \in (0, 110)$.

Example 4.1. *(Numerical results for the optimal reinsurance maximizing the expected utility)* In this case, we know that the optimal contract can be expressed as the unified formula given by (2.20) as

$$I^*(x) = (x - d_1^*) - (x - a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+.$$

We take the convex function $u(x) = x^2$, which means that the insurer would like to minimize the variance of its retained loss. We first consider the exponential underlying loss X with survival distribution $S_X(x) = e^{-x/100}$ for any $x \geq 0$ for the insurer. In this case, $a = \mathrm{VaR}_\alpha(X) = -\frac{\ln(1-\alpha)}{100}$. We obtain the optimal contract under $\alpha = 0.01$, $\alpha = 0.05$, and different premium budget values of p. The numerical results are given in Tables 1 and 2.

Table 1: Exponential Risk X with $\alpha = 0.01$

p	d_1^*	d_2^*	d_3^*
80.000	31.225	461.168	∞
99.200	9.921	460.940	∞
105.880	3.456	460.806	∞
108.100	1.396	460.809	∞
109.631	0	460.811	∞
109.800	0	460.517	649.089

Next, we consider the Pareto underlying loss Y with survival distribution $S_Y(y) = \left(\frac{200}{y+200}\right)^3$ for $y \geq 0$ for the insurer. In this case, $\mathrm{VaR}_\alpha(Y) = 200(\alpha^{-1/3} - 1)$. The nu-

Table 2: Exponential Risk X with $\alpha = 0.05$

p	d_1^*	d_2^*	d_3^*
80.000	28.691	302.681	∞
99.200	8.229	301.653	∞
105.880	1.971	301.407	∞
108.100	0	301.332	∞
109.631	0	299.573	431.620
109.800	0	299.573	420.917

merical results for (d_1^*, d_2^*, d_3^*) when $\alpha = 0.01$, $\alpha = 0.05$, and different premium budget values of p are summarized in Tables 3 and 4.

Table 3: Pareto Risk Y with $\alpha = 0.01$

p	d_1^*	d_2^*	d_3^*
80.000	28.405	734.196	∞
99.200	6.305	732.488	∞
105.880	0	732.107	∞
108.100	0	728.318	1215.400
109.631	0	728.300	888.275
109.800	0	728.300	864.518

Table 4: Pareto Risk Y with $\alpha = 0.05$

p	d_1^*	d_2^*	d_3^*
80.000	19.200	356.748	∞
99.200	0	352.764	∞
105.880	0	342.900	633.469
108.100	0	342.884	520.208
109.631	0	342.884	464.486
109.800	0	342.884	459.096

It is easy to see from Tables 1-4 that when the risk level α is fixed, the optimal ceded loss function $I^*(x)$ associated with a higher reinsurance premium is larger than the optimal ceded loss function $I^*(x)$ with a lower reinsurance premium, which means that the larger is the premium charged by the reinsurer, the larger is the loss that the reinsurer should cover. Furthermore, Tables 1-4 suggest that when the reinsurance premium p is fixed, the higher is the risk level α, the lower are the deductible levels d_i^* for $i = 1, 2, 3$, which means that the smaller is the reinsurer's initial reserve, the larger is the loss that the insurer should cover. Moreover, Tables 1-4 imply that when both the reinsurance premium p and risk level α are fixed, the optimal ceded loss function $I^*(x)$ for Pareto loss Y is larger than the optimal ceded loss function $I^*(x)$ for exponential loss X, which means that the insurer should cede more loss to the reinsurer for a heavy tailed loss or a riskier loss. □

Example 4.2. *(Numerical results for the optimal reinsurance minimizing the VaR of its retained risk)* In this case, denote the optimal reinsurance contract for X and Y by I_X^* and I_Y^*, respectively. Denote $d_{0,X}$ and $d_{0,Y}$ be the solutions to the equation (3.4) for X and Y, respectively. According to Theorem 3.2, we have

$$I_X^*(x) = (x - d_X^*)^+ - (x - \mathrm{VaR}_\beta(X))^+ \text{ and } I_Y^*(y) = (y - d_Y^*)^+ - (y - \mathrm{VaR}_\beta(Y))^+,$$

where $d_X^* = \mathrm{VaR}_\beta(X) \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$ if $\alpha \le \beta$ and $d_X^* = \max\{0, d_{0,X} \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}$ if $\alpha > \beta$ and $\alpha \le \frac{1}{1+\theta}$, and d_Y^* has a similar expression as d_X^*.

Using Theorem 3.2, we obtain the optimal values of d_X^* and d_Y^* for different risk levels of (α, β) in Table 5. Table 5 suggests that a Pareto loss Y will result in lower deductible levels than an exponential loss X. These numerical results are consistent with those in Example 4.1 □

Table 5: Deductible Values

(α, β)	d_X^*	$VaR_\beta(X)$	d_Y^*	$VaR_\beta(Y)$
(0.0100, 0.0500)	9.531	299.573	6.456	342.884
(0.0100, 0.0280)	9.531	357.555	6.456	458.634
(0.0185, 0.0150)	9.531	419.971	6.456	610.960
(0.0500, 0.0100)	0	460.517	0	728.318
(0.0280, 0.0100)	5.549	460.517	0	728.318
(0.0280, 0.0185)	9.531	398.999	4.448	556.205
(0.0150, 0.0185)	9.531	398.999	6.456	556.205

5 Concluding remarks

In this paper, we propose a reinsurance risk model that incorporates the regulatory requirements on the initial reserve of a reinsurance contract seller (reinsurer) and the possible default by the seller. Mathematically, the proposed model can be reduced to existing reinsurance risk models that do not consider the possible default by a reinsurer. Practically, the proposed model allows more realistic settings. We derive the optimal reinsurance strategy from the insurer's point of view under the proposed model. The results show that the regulatory initial reserve and the default risk have a significant impact on the optimal reinsurance strategy. The optimal reinsurance strategies under the proposed model are more complicated than those in the existing default risk-free reinsurance risk models. The model proposed in the paper can be further explored in different ways as well, something we plan to do in future work.

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Appendix

Throughout the appendix, for any $I \in \mathcal{I}$, we denote $\tilde{I}(x) = I(x) \wedge (\omega_I + P_I) = I(x) \wedge (I(a) + P_I)$. Thus, for any $I \in \mathcal{I}_p$, $\tilde{I}(x) = I(x) \wedge (I(a) + p)$.

Proof of Theorem 2.1. For any $I \in \mathcal{I}_p$, we denote $x_I = \sup\big\{x \geq 0 : I(x) < I(a) + p\big\}$. Then $x_I \in (a, \infty]$. Since I is non-decreasing and continuous on $[0, \infty)$, one has $\omega_I = \mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X)) = I(a)$. Therefore, for any $I \in \mathcal{I}_p$, $\tilde{I}(x) = I(x) \wedge (I(a) + p)$ for $x \geq 0$. Note that for any $I \in \mathcal{I}_p$,

$$\begin{aligned} H(I) &= \mathbb{E}\left[u\left(X - \tilde{I}(X)\right)\right] = u(0) + \mathbb{E}\left[u\left(X - I(X)\right) \mid 0 < X < a\right]\mathbb{P}(0 < X < a) \\ &\quad + \mathbb{E}\left[u\left(X - I(X)\right) \mid a \leq X < x_I\right]\mathbb{P}(a \leq X < x_I) + \mathbb{E}\left[u\left(X - I(a) - p\right)\right]\mathbb{P}(X \geq x_I). \end{aligned}$$

Firstly, we construct $k_I \in I_p$ on the interval $[0, a)$, $[a, x_I)$, and $[x_I, \infty)$, respectively. Here, we assume $x_I < \infty$. If $x_I = \infty$, we only need to consider the intervals of $[0, a)$ and $[a, x_I)$.

(1) For $0 \leq x < a$, we define, with respect to each $d \in [0,\, a - I(a)]$,

$$k_d^1(x) = \max\left\{(x-d)^+,\, I(a)\right\} = \begin{cases} 0, & \text{for} \quad 0 \leq x < d, \\ x - d, & \text{for} \quad d \leq x < d + I(a), \\ I(a), & \text{for} \quad d + I(a) \leq x < a. \end{cases}$$

When $d = 0$, we have $k_0^1(x) = x \wedge I(a)$ and thus $k_0^1(x) \geq I(x)$ on the interval $[0, a)$ and $\mathbb{E}\left[k_0^1(X) \mid 0 < X < a\right] \geq \mathbb{E}\left[I(X) \mid 0 < X < a\right]$. When $d = a - I(a)$, we have $k^1_{a-I(a)}(x) =$

$(x-a+I(a))^+$ and thus $k^1_{a-I(a)}(x) \le I(x)$ on $[0,a)$ and $\mathbb{E}\big[k^1_{a-I(a)}(X)\,|\,0<X<a\big] \le \mathbb{E}\big[I(X)\,|\,0<X<a\big]$. It is obvious that $\mathbb{E}\big[k^1_d(X)\,|\,0<X<a\big]$ is continuous in d. Therefore, there exists $d_1 \in [0,\, a-I(a)]$ such that $\mathbb{E}\left[k^1_{d_1}(X)\middle|\, 0<X<a\right] = \mathbb{E}\left[I(X)\middle|\, 0<X<a\right]$. Define $k_I(x) = k^1_{d_1}(x)$ on the interval $[0,a)$.

(2) For $a \le x < x_I$, define for each $d \in [a, x_I]$,

$$k^2_d(x) = \max\left\{I(a) + (x-d)^+,\, I(a)+p\right\} = \begin{cases} I(a), & \text{for} \quad a \le x < d, \\ I(a) + x - d, & \text{for} \quad d \le x < d+p, \\ I(a)+p, & \text{for} \quad d+p \le x < x_I. \end{cases}$$

Using similar arguments as in (1), there exists $d_2 \in [a, x_I]$ such that $\mathbb{E}\left[I(X)\middle|\, a \le X < x_I\right] = \mathbb{E}\left[k^2_{d_2}(X)\middle|\, a \le X < x_I\right]$. Define $k_I(x) = k^2_{d_2}(x)$ on $[a, x_I)$.

(3) For $x \ge x_I$, define for each $d \in [x_I, \infty]$,

$$k^3_d(x) = I(a) + p + (x-d)^+ = \begin{cases} I(a)+p, & \text{for} \quad x_I \le x < d, \\ I(a)+p+x-d, & \text{for} \quad d \le x. \end{cases}$$

Using similar arguments as in (1), there exists $d_3 \in [x_I, \infty]$ such that $\mathbb{E}\left[I(X)\middle|\, X \ge x_I\right] = \mathbb{E}\left[k^3_{d_3}(X)\middle|\, X \ge x_I\right]$. Define $k_I(x) = k^3_{d_3}(x)$ for $x \ge x_I$.

Thus, for any $x \ge 0$,

$$k_I(x) = \begin{cases} k^1_{d_1}(x), & \text{for } 0 \le x < a, \\ k^2_{d_2}(x), & \text{for } a \le x < x_I, \\ k^3_{d_3}(x), & \text{for } x_I \le x < \infty. \end{cases}$$

It is easy to see by the construction of k_I that

$$\begin{aligned} \mathbb{E}\big[k_I(X)\big] &= \mathbb{E}\big[k_I(X)\,|\,0 \le X < a\big]\mathbb{P}(0 \le X < a) + \mathbb{E}\left[k_I(X)\middle|\, a \le X < x_I\right]\mathbb{P}(a \le X < x_I) \\ &\quad + \mathbb{E}\left[k_I(X)\right]\mathbb{P}(X \ge x_I) = \mathbb{E}\left[I(X)\right] = p, \end{aligned}$$

and thus $k_I \in \mathcal{I}_p$.

It remains to show that $H(k_I) \le H(I)$. Define a random variable X_1 with distribution function $F_1(x) = \frac{F_X(x)-F_X(0)}{F_X(a)-F_X(0)}$ for $0 < x < a$. Then for any Borel measurable function $b(\cdot)$, one has

$$\mathbb{E}\left[b(X)\middle|\, 0<X<a\right] = \int_0^a \frac{b(x)}{F_X(a)-F_X(0)}\,\mathrm{d}F_X(x) = \int_0^a b(x)\mathrm{d}F_1(x) = \mathbb{E}\left[b(X_1)\right].$$

Note that, on the interval $(0,a)$, $k_I(x) = \max\left\{(x-d_1)^+,\, I(a)\right\}$. If $d_1 = 0$ or $d_1 = a - I(a)$, $k_I(x) = I(x)$ on $[0,a]$; otherwise, $I(x)$ crosses $k_I(x)$ at most once from above on $[0,a]$, say crossing at $c_1 \in [0,a)$. Thus, $x - k_I(x) \ge x - I(x)$ for $x < c_1$ and $x - k_I(x) \le x - I(x)$

for $x \geq c_1$. Together with $\mathbb{E}[I(X_1)] = \mathbb{E}[k_I(X_1)]$ and $x - k_I(x)$ and $x - I(x)$ are both continuous and non-decreasing functions, Lemma 2.1 implies $X_1 - k_I(X_1) \preccurlyeq_{cx} X_1 - I(X_1)$. Therefore, for any convex function u, $\mathbb{E}\big[u(X - I(X)) \,|\, 0 \leq X < a\big] = \mathbb{E}\big[u(X_1 - I(X_1))\big] \geq \mathbb{E}\big[u(X_1 - k_I(X_1))\big] = \mathbb{E}\big[u(X - k_I(X)) \,|\, 0 < X < a\big]$.

Using the same arguments, we see that

$$\mathbb{E}\left[u\left(X - I(X)\right)\right| a \leq X < x_I] \geq \mathbb{E}\left[u\left(X - k_I(X)\right)\right| a \leq X < x_I]$$

and $\mathbb{E}\left[u\left(X - I(X)\right)\right| X \geq x_I] \geq \mathbb{E}\left[u\left(X - k_I(X)\right)\right| X \geq x_I]$. Therefore, we can conclude that $H(I) \geq H(k_I)$. $\square$

Proof of Lemma 2.2. For any $I \in \mathcal{I}_{p,0}$ with $I(a) = \xi$ for some $\xi \in [\xi_0, \xi_M]$, or for any I that has expression (2.3) for some (d_1, d_2, d_3, ξ), we have

$$\begin{aligned}
\tilde{I}(x) &= I(x) \wedge (\xi + p) = (x - d_1)^+ - (x - (d_1 + \xi))^+ + (x - d_2)^+ - (x - (d_2 + p))^+ \\
&= \begin{cases} 0, & \text{for} \quad 0 \leq x < d_1, \\ x - d_1, & \text{for} \quad d_1 \leq x < d_1 + \xi, \\ \xi, & \text{for} \quad d_1 + \xi \leq x < d_2, \\ \xi + x - d_2, & \text{for} \quad d_2 \leq x < d_2 + p, \\ \xi + p, & \text{for} \quad d_2 + p \leq x. \end{cases}
\end{aligned}$$

Note that $\tilde{I}(x) = I(x) \wedge (\xi + p) = I(x)$ for $0 \leq x \leq d_2 + p$. Thus, by integration by parts, we have

$$\begin{aligned}
H(I) &= \mathbb{E}\left[u\left(X - \tilde{I}(X)\right)\right] = \int_0^\infty u(x - \tilde{I}(x)) dF_X(x) \\
&= u(0) + \left(\int_0^{d_1} + \int_{d_1+\xi}^{d_2}\right) S_X(x) u'\left(x - I(x)\right) \mathrm{d}x + \int_{d_2+p}^\infty S_X(x) u'\left(x - \xi - p\right) \mathrm{d}x, \quad \text{(A.1)}
\end{aligned}$$

where $u(0)$ is a constant. Note that for any $I \in \mathcal{I}_{p,0}$, the value of $H(I)$ does not depend on d_3. Indeed, the variable d_3 is only used to adjust the expectation of I so that

$$(1 + \theta)\mathbb{E}[I(X)] = (1 + \theta)\left(\int_{d_1}^{d_1+\xi} + \int_{d_2}^{d_2+p} + \int_{d_3}^\infty\right) S_X(x) \mathrm{d}x = p. \tag{A.2}$$

Now, we are going to minimize $H(I)$ using the expression (A.1) and the restriction (A.2) and to find the minimizers of d_i, denoted by $d_{i,\xi}$, for $i = 1, 2, 3$.

(1) Suppose $\xi_0 \leq \xi \leq \xi_1$ (if $\xi_0 < \xi_1$) or equivalently $p_{2,\xi} \leq p \leq p_{M,\xi}$. Then, there exists

$a \leq d_{3,\xi} \leq \infty$ such that

$$
\begin{aligned}
I_\xi^*(x) &= x - (x-a+\xi)^+ + (x-a)^+ - (x-a-p)^+ + (x-d_{3,\xi})^+ \\
&= \begin{cases} x, & \text{for} \quad 0 \leq x < a-\xi, \\ \xi, & \text{for} \quad a-\xi \leq x < a, \\ \xi + x - a, & \text{for} \quad a \leq x < a+p, \\ \xi + p, & \text{for} \quad a+p \leq x < d_{3,\xi}, \\ \xi + p + x - d_{3,\xi} & \text{for} \quad d_{3,\xi} \leq x. \end{cases}
\end{aligned}
$$

Here $d_{3,\xi}$ is determined by the premium condition $(1+\theta)\mathbb{E}[I_\xi^*(X)] = p$. Thus, I_ξ^* has form (2.3) with $(d_1, d_2, d_3) = (0, a, d_{3,\xi})$. Moreover, I_ξ^* is the optimal solution because for any $I \in \mathcal{I}_{p,0}$ satisfying $I(a) = \xi$, it is easy to check that for all $x \geq 0$, we have

$$
\begin{aligned}
\tilde{I}_\xi^*(x) - \tilde{I}(x) &= I_\xi^*(x) \wedge (I^*(a)+p) - I_\xi(x) \wedge (I(a)+p) \\
&= \left[x - (x-d_1)^+\right] + \left[(x-d_1-\xi)^+ - (x-a+\xi)^+\right] \\
&\quad + \left[(x-a)^+ - (x-d_2)^+\right] + \left[(x-d_2-p)^+ - (x-a-p)^+\right] \\
&\geq 0,
\end{aligned}
$$

where $d_1 + \xi \leq a \leq d_2$.

(2) Suppose $\xi_1 \leq \xi \leq \xi_M$ (if $\xi_1 < \xi_M$) or equivalently $p_{0,\xi} \leq p \leq p_{2,\xi}$. In this case, there exist d_1 and d_2 such that

$$
\begin{aligned}
I(x) &= (x-d_1)^+ - (x-d_1-\xi)^+ + (x-d_2)^+ - (x-d_2-p)^+ \\
&= \begin{cases} 0, & \text{for} \quad 0 \leq x < d_1, \\ \xi, & \text{for} \quad d_1+\xi \leq x < d_2, \\ \xi + x - d_2, & \text{for} \quad d_2 \leq x < d_2+p, \\ \xi + p, & \text{for} \quad d_2+p \leq x, \end{cases}
\end{aligned}
\tag{A.3}
$$

which satisfies the premium condition $(1+\theta)\mathbb{E}\left[I(X)\right] = p$. Note that expression (2.3) is reduced to expression (A.3) when $d_3 = \infty$. We claim that the minimizer I_ξ^* should have the form (A.3). This claim can be proved using the following arguments.

For any $I \in \mathcal{I}_{p,0}$ with $d_3 < \infty$, if there exists $\tilde{d}_2 \geq a$ such that $I_1(x) = (x-d_1)^+ - (x-d_1-\xi)^+ + (x-\tilde{d}_2)^+ - (x-\tilde{d}_2-p)^+ \in \mathcal{I}_{p,0}$, then the following equation

$$
0 = \mathbb{E}[I_1(X)] - \mathbb{E}[I(X)] = \int_{d_2}^{d_2+p} S_X(x+\tilde{d}_2-d_2) - S_X(x)\mathrm{d}x - \int_{d_3}^{\infty} S_X(x)\mathrm{d}x,
$$

implies that $\tilde{d}_2 < d_2$ and thus $\tilde{I}_1(x) - \tilde{I}(x) = I_1(x) \wedge (I_1(a)+p) - I(x) \wedge (I(a)+p) \geq 0$ for all $x \geq 0$. If such $\tilde{d}_2$ does not exist, then there exists $0 \leq \tilde{d}_1 \leq d_1$ such that $I_1(x) =$

$(x-\tilde{d}_1)^+ - (x-\tilde{d}_1-\xi)^+ + (x-a)^+ - (x-a-p)^+ \in \mathcal{I}_{p,0}$ and thus $\tilde{I}_1(x) \geq \tilde{I}(x)$ for all $x \geq 0$. In short, we can find another contract I_1 of the form (A.3) in $\mathcal{I}_{p,0}$ such that $\tilde{I}_1(x) \geq \tilde{I}(x)$ for all $x \geq 0$ and thus $H(I_1) \leq H(I)$. Therefore, the insurer should choose the reinsurance contract satisfying the form (A.3) or $I = \tilde{I}$.

For any contract $I \in \mathcal{I}$ of the form (A.3), the premium condition

$$(1+\theta)\left(\int_{d_1}^{d_1+\xi} + \int_{d_2}^{d_2+p}\right) S_X(x)\mathrm{d}x = p$$

implies that d_2 can be written as an implicit function of d_1 or $d_2 = d_2(d_1)$. It is not hard to see that $d_2(d_1)$ is a non-increasing function of d_1 and its derivative satisfies

$$S_X(d_1+\xi) - S_X(d_1) + \left(S_X(d_2+p) - S_X(d_2)\right) d_2'(d_1) = 0.$$

The objective function $H(I)$ given by expression (A.1) now depends on d_1 only and thus denote it as a one-variable function $H_\xi(d_1)$. Taking the derivative with respect to d_1, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}d_1} H_\xi(d_1) = \left(u'(d_1) - u'(d_2-\xi)\right)\left(S_X(d_1) - S_X(d_1+\xi)\right) \leq 0.$$

Therefore, one should choose d_1 as large as possible to have the smallest expectation of the utility.

(2.1) When $\xi_1 \leq \xi \leq \xi_2$ (if $\xi_1 < \xi_2$) or equivalently $p_{1,\xi} \leq p \leq p_{2,\xi}$, the largest possible value for d_1 is $d_{1,\xi}$ which satisfies $d_2(d_{1,\xi}) = a$ and the corresponding optimal solution is

$$I_\xi^*(x) \quad = \quad (x-d_{1,\xi})^+ - (x-d_{1,\xi}-\xi)^+ + (x-a)^+ - (x-a-p)^+.$$

Then I_ξ^* is of the form (2.3) with $(d_1, d_2, d_3) = (d_{1,\xi}, a, \infty)$ where $d_{1,\xi}$ is determined by the expectation condition $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

(2.1) When $\xi_2 \leq \xi \leq \xi_M$ (if $\xi_2 < \xi_M$) or equivalently $p_{0,\xi} \leq p \leq p_{1,\xi}$, the largest possible value for d_1 is $a-\xi$ and the corresponding optimal solution is

$$I_\xi^*(x) \quad = \quad (x-a+\xi)^+ - (x-a)^+ + (x-d_{2,\xi})^+ - (x-d_{2,\xi}-p)^+,$$

where $d_{2,\xi} = d_2(a-\xi)$, namely I_ξ^* has form (2.3) with $(d_1, d_2, d_3) = (a-\xi,\, d_{2,\xi},\, \infty)$. □

Proof of Theorem 2.2. Define the function $h(\xi)$ on $[\xi_0, \xi_M]$ as follows:

$$h(\xi) = \min_{I \in \mathcal{I}_p,\, I(a)=\xi} \mathbb{E}[u(X - I(X) \wedge (\xi+p))] = \mathbb{E}[u(X - I_\xi^*(X) \wedge (\xi+p))].$$

From the results of Lemma 2.2, we discuss the following cases.

(1) When $\xi \in [\xi_0, \xi_1]$ (if $\xi_0 < \xi_1$), we see that $d_{3,\xi}$ is an increasing function of ξ and

$$h(\xi) = u(0) + \int_{\xi}^{a} S_X(x)u'(x-\xi)\mathrm{d}x + \int_{a+p}^{\infty} S_X(x)u'(x-\xi-p)\mathrm{d}x.$$

Clearly,

$$h'(\xi) = -\int_{\xi}^{a} S_X(x)u''(x-\xi)\mathrm{d}x - S_X(\xi)u'(0) - \int_{a+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x \le 0,$$

because $u(x)$ is an increasing convex function with $u'(x) \ge 0$ and $u''(x) \ge 0$.

(2) When $\xi \in [\xi_1, \xi_2]$ (if $\xi_1 < \xi_2$), we have

$$h(\xi) = u(0) + \int_{0}^{d_{1,\xi}} S_X(x)u'(x)\mathrm{d}x + \int_{d_{1,\xi}+\xi}^{a} S_X(x)u'(x-\xi)\mathrm{d}x + \int_{a+p}^{\infty} S_X(x)u'(x-\xi-p)\mathrm{d}x.$$

The premium condition implies that $d_{1,\xi}$ can be written as an implicit function of ξ using the equation $(1+\theta)\left(\int_{d_{1,\xi}}^{d_{1,\xi}+\xi} + \int_{a}^{a+p}\right) S_X(x)\mathrm{d}x = p$. By taking the derivative with respect to ξ on both sides of the above equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi} d_{1,\xi} = \frac{S_X\left(d_{1,\xi}+\xi\right)}{S_X(d_{1,\xi}) - S_X(d_{1,\xi}+\xi)} \ge 0.$$

Thus, we obtain

$$h'(\xi) = -\int_{d_{1,\xi}+\xi}^{a} S_X(x)u''(x-\xi)\mathrm{d}x - \int_{a+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x \le 0.$$

(3) When $\xi \in [\xi_2, \xi_M]$ (if $\xi_2 < \xi_M$), we have

$$h(\xi) = u(0) + \int_{0}^{a-\xi} S_X(x)u'(x)\mathrm{d}x + \int_{a}^{d_{2,\xi}} S_X(x)u'(x-\xi)\mathrm{d}x + \int_{d_{2,\xi}+p}^{\infty} S_X(x)u'(x-\xi-p)\mathrm{d}x.$$

The premium condition implies that $d_{2,\xi}$ can be written as an implicit function of ξ by the equation $(1+\theta)\left(\int_{a-\xi}^{a} + \int_{d_{2,\xi}}^{d_{2,\xi}+p}\right) S_X(x)\mathrm{d}x = p$. By taking the derivative with respect to ξ on both sides of the above equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi} d_{2,\xi} = \frac{S_X\left(a-\xi\right)}{S_X(d_{2,\xi}) - S_X(d_{2,\xi}+p)} \ge 1.$$

Thus, we obtain

$$\begin{aligned} h'(\xi) &= S_X(a-\xi)\big(u'(d_{2,\xi}-\xi) - u'(a-\xi)\big) \\ &\quad - \int_{a}^{d_{2,\xi}} S_X(x)u''(x-\xi)\mathrm{d}x - \int_{d_{2,\xi}+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x. \end{aligned}$$

Using the above expression of $h'(\xi)$, it is not hard to check that the second derivative of h is

$$\begin{aligned} h''(\xi) &= f_X(a-\xi)\left(u'(d_{2,\xi}-\xi)-u'(a-\xi)\right)+u''(a-\xi)(S_X(a-\xi)-S_X(a)) \\ &\quad +u''(d_{2,\xi}-\xi)\left[S_X(a-\xi)-S_X(d_{2,\xi})+S_X(d_{2,\xi}+p)\right]\left(\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi}-1\right) \\ &\quad +\int_a^{d_{2,\xi}} u''(x-\xi)f_X(x)\mathrm{d}x+\int_{d_{2,\xi}+p}^{\infty} u''(x-\xi-p)f_X(x)\mathrm{d}x \geq 0, \end{aligned}$$

where $h''(\xi) \geq 0$ is due to the following facts: u is convex; $S_X(x)$ is non-increasing; $d_{2,\xi} \geq a \geq a-\xi$; and $\frac{\mathrm{d}}{\mathrm{d}\,\xi}d_{2,\xi}-1 \geq 0$. Therefore, h' in non-decreasing in ξ. It is easy to see from the definition of $d_{2,\xi}$ that when $\xi=\xi_2$, we have $d_{2,\xi_2}=a$ and thus $h'(\xi_2)=-\int_{a+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x<0$.

Denote

$$\xi^* = \sup\left\{\xi \in [\xi_2,\xi_M] : h'(\xi)<0\right\}.$$

If $h'(\xi_M) \leq 0$, then h' is always non-positive for any $\xi \in [\xi_2,\xi_M]$ or $\xi^*=\xi_M$; if $h'(\xi_M)>0$, then $\xi^*<\xi_M$ and $h'(\xi^*)=0$.

In summary, $h(\xi)$ is continuous on $[0,\infty)$, $h'(\xi) \leq 0$ for $\xi \in [\xi_0,\xi^*]$, and $h'(\xi) \geq 0$ for $\xi \in [\xi^*,\xi_M]$. Therefore, $h(\xi)$ achieves its minimal value at ξ^* and the reinsurance contract $I^*_{\xi^*}$ summarized in the theorem is the optimal solution of the two-step minimization problem (2.12), namely $\min\limits_{\xi_0 \leq \xi \leq \xi_M} \min\limits_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I) = H(I^*_{\xi^*})$.

Since $\xi^* \in [\xi_0,\xi_M]$, the corresponding contract $I^*_{\xi^*}$ is in $\mathcal{I}_{p,0}$. Thus, $H(I^*_{\xi^*}) \geq \min\limits_{I \in \mathcal{I}_{p,0}} H(I)$. Furthermore

$$\min_{\xi_0 \leq \xi \leq \xi_M} \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I) \geq \min_{I \in \mathcal{I}_{p,0}} H(I). \tag{A.4}$$

On the other hand, for any $k \in \mathcal{I}_{p,0}$,

$$H(k) \geq \min_{I \in \mathcal{I}_{p,0},\, I(a)=k(a)} H(I) \geq \min_{\xi_0 \leq \xi \leq \xi_M} \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I).$$

Therefore,

$$\min_{k \in \mathcal{I}_{p,0}} H(k) \geq \min_{\xi_0 \leq \xi \leq \xi_M} \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I). \tag{A.5}$$

Combining inequalities (A.4) and (A.5), we conclude that

$$\min_{I \in \mathcal{I}_{p,0}} H(I) = \min_{\xi_0 \leq \xi \leq \xi_M} \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I),$$

and $I^*_{\xi^*}$ is also the optimal solution to Problem (2.5). $\square$

Proof of Lemma 3.1. Denote $\xi_a = I_1(a) = I_2(a)$ and $\xi_b = I_1(b) = I_2(b)$. Then $V(I_i) = b - \xi_b \wedge (\xi_a + P_{I_i}) + P_{I_i}$ for $i = 1, 2$.

(1) Suppose $\alpha \le \beta$ (or equivalently $b \le a$). In this case, $\xi_b \le \xi_a \le \xi_a + P_{I_i}$ for $i = 1, 2$. Hence, $V(I_2) - V(I_1) = (b - \xi_b + P_{I_2}) - (b - \xi_b + P_{I_1}) = P_{I_2} - P_{I_1} \ge 0$.

(2) Suppose $\alpha \ge \beta$ (or equivalently $b \ge a$). In this case, we have $\xi_a \le \xi_b$ and

$$\begin{aligned} V(I_2) - V(I_1) &= (b - \xi_b \wedge (\xi_a + P_{I_2}) + P_{I_2}) - (b - \xi_b \wedge (\xi_a + P_{I_1}) + P_{I_1}) \\ &= \xi_b \wedge (\xi_a + P_{I_1}) - \xi_b \wedge (\xi_a + P_{I_2}) + P_{I_2} - P_{I_1}. \end{aligned}$$

Furthermore, if $\xi_b \le \xi_a + P_{I_2}$,

$$\begin{aligned} V(I_2) - V(I_1) = \xi_b \wedge (\xi_a + P_{I_1}) - \xi_b + P_{I_2} - P_{I_1} &= \begin{cases} P_{I_2} - P_{I_1}, & \text{if} \quad \xi_b \le \xi_a + P_{I_1}, \\ P_{I_2} + \xi_a - \xi_b, & \text{if} \quad \xi_b > \xi_a + P_{I_1}, \end{cases} \\ &\ge 0. \end{aligned}$$

If $\xi_b > \xi_a + P_{I_2}$, we have $\xi_a + P_{I_1} \le \xi_a + P_{I_2} < \xi_b$ and thus

$$V(I_2) - V(I_1) = (\xi_a + P_{I_1}) - (\xi_a + P_{I_2}) + P_{I_2} - P_{I_1} = 0.$$

In short, when $\alpha \ge \beta$, we have $V(I_2) - V(I_1) \ge 0$. This completes the proof of the lemma. $\square$

Proof of Theorem 3.1. For any $I \in \mathcal{I}$, for $d_1 = a \wedge b - I(a \wedge b)$ and $d_2 = a \vee b - (I(a \vee b) - I(a \wedge b))$ and substituting them into expression (3.2), it is not hard to verify that $m_I(a) = I(a)$, $m_I(b) = I(b)$ and $m_I(x) \le I(x)$ for all $x \ge 0$. Moreover, $P_{m_I} \le P_I$. Hence, we obtain the desired result using Lemma 3.1. $\square$

Proof of Theorem 3.2. To solve Problem (3.3), we can rewrite it as a two-step minimization problem. For any fixed $d_1 \in [0,\, a \wedge b]$, if the function $v(d_1, \cdot)$ is continuous on the closed interval $[a \wedge b,\, a \vee b]$, then there exists $d_2^*(d_1) \in [a \wedge b,\, a \vee b]$ such that

$$\min_{d_2 \in [a \wedge b,\, a \vee b]} v(d_1,\, d_2) = v(d_1,\, d_2^*(d_1)). \tag{A.6}$$

Moreover, if $v(d_1, d_2^*(d_1))$ is continuous in $d_1 \in [0,\, a \wedge b]$, then there exists $d_1^* \in [0,\, a \wedge b]$ such that

$$\min_{d_1 \in [0,\, a \wedge b]} v(d_1,\, d_2^*(d_1)) = v(d_1^*,\, d_2^*(d_1^*)). \tag{A.7}$$

For any $(d_1,\, d_2) \in [0,\, a \wedge b] \times [a \wedge b,\, a \vee b]$,

$$v(d_1,\, d_2) \ge \min_{d_2 \in [a \wedge b,\, a \vee b]} v(d_1,\, d_2) \;=\; v(d_1,\, d_2^*(d_1)) \ge \min_{d_1 \in [0,\, a \wedge b]} v(d_1,\, d_2^*(d_1)) = v(d_1^*,\, d_2^*(d_1^*)).$$

Thus,

$$\min_{(d_1,\,d_2)\in[0,\,a\wedge b]\times[a\wedge b,\,a\vee b]} v(d_1,\,d_2) \geq v(d_1^*,\,d_2^*(d_1^*)).$$

On the other hand, $(d_1^*,\,d_2^*(d_1^*)) \in [0,\,a\wedge b]\times[a\wedge b,\,a\vee b]$ implies

$$\min_{(d_1,\,d_2)\in[0,\,a\wedge b]\times[a\wedge b,\,a\vee b]} v(d_1,\,d_2) \leq v(d_1^*,\,d_2^*(d_1^*)).$$

Therefore,

$$\min_{(d_1,\,d_2)\in[0,\,a\wedge b]\times[a\wedge b,\,a\vee b]} v(d_1,\,d_2) \;=\; v(d_1^*,\,d_2^*(d_1^*)) = \min_{d_1\in[0,\,a\wedge b]}\ \min_{d_2\in[a\wedge b,\,a\vee b]} v(d_1,\,d_2),$$

which is a two-step minimization problem.

For each $d_1 \in [0,\,a\wedge b]$, define $v_{d_1}(d_2) = v(d_1, d_2)$ as the function of $d_2 \in [a\vee b,\,\infty)$.

(1) Suppose $\alpha \leq \beta$ (or equivalently $b \leq a$). In this case, for any fixed $d_1 \in [0, b]$, and for any $d_2 \in [b, a]$ and I given by (3.2), we have

$$v_{d_1}(d_2) = v(d_1, d_2) = V(I) = b - (b - d_1) + P_I = d_1 + (1+\theta)\left(\int_{d_1}^{b} + \int_{d_2}^{a}\right) S_X(x)\mathrm{d}x.$$

Clearly, the first derivative of $v_{d_1}(d_2)$ satisfies

$$v'_{d_1}(d_2) = \frac{\partial}{\partial d_2} v(d_1, d_2) = -\,(1+\theta)\,S_X(d_2) < 0.$$

Thus, for any $d_1 \in [0, b]$, $d_2^*(d_1) = a$ is the minimizer to the minimization problem $\min\limits_{d_2\in[b,a]} v(d_1, d_2)$ of (A.6). Hence,

$$\min_{d_1\in[0,b]}\ \min_{d_2\in[b,a]} v(d_1, d_2) = \min_{d_1\in[0,b]} v(d_1, d_2^*(d_1)) = \min_{d_1\in[0,b]} v(d_1, a). \tag{A.8}$$

Next, we consider the function

$$v(d_1, d_2^*(d_1)) = v(d_1, a) = d_1 + (1+\theta)\int_{d_1}^{b} S_X(x)\mathrm{d}x.$$

If $S_X(0) \leq \beta \leq 1$, then $b = 0$ and the only possible value for d_1 is zero. Assume $0 < \beta < S_X(0)$ and thus $b > 0$. Obviously, the function $v(d_1, d_2^*(d_1))$ is continuous in d_1 and its first derivative is $\frac{\mathrm{d}}{\mathrm{d}d_1} v(d_1, d_2^*(d_1)) = 1 - (1+\theta)\,S_X(d_1)$. Since

$$d_1 \leq \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \Leftrightarrow S_X(d_1) \geq \frac{1}{1+\theta},$$

thus, $d_1^* = \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b$ is the minimizer to the minimization problem $\min\limits_{d_1\in[0,b]} v(d_1, a)$ of (A.8). It follows that the optimal contract I^* has the form (3.2) with $d_1 = d_1^*$ and

$d_2 = d_2^*(d_1^*) = a$. Namely, we have

$$\begin{aligned} I^*(x) &= (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b)^+ - (x-b)^+ \\ &= \begin{cases} 0, & \text{for} \quad 0 \le x < \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b, \\ x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b, & \text{for} \quad \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b \le x < b, \\ b - \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b, & \text{for} \quad b \le x < \infty. \end{cases} \end{aligned}$$

(2) Suppose $\alpha \ge \beta$ (or equivalently $b \ge a$) and $\alpha \le \frac{1}{1+\theta}$. In this case, for any fixed $d_1 \in [0, a]$, we first define $v_{d_1}(d_2)$ and $G_{d_1}(d_2)$ as functions of $d_2 \in [a, b]$ as

$$\begin{aligned} v_{d_1}(d_2) &= v(d_1, d_2) = V(I) = b - (a - d_1 + b - d_2) \wedge (a - d_1 + P_I) + P_I \\ &= \begin{cases} -a + d_1 + d_2 + P_I, & \text{if} \quad b \le P_I + d_2, \\ -a + d_1 + b, & \text{if} \quad b > P_I + d_2, \end{cases} \end{aligned}$$

and

$$G_{d_1}(d_2) = P_I - (b - d_2) = (1+\theta)\left(\int_{d_1}^{a} + \int_{d_2}^{b}\right) S_X(x)\mathrm{d}x - b + d_2.$$

It is not hard to see $\frac{\mathrm{d}}{\mathrm{d}d_2} G_{d_1}(d_2) = 1 - (1+\theta) S_X(d_2) \ge 0$ for any $d_2 \in [a, b]$ because $\alpha \le \frac{1}{1+\theta}$ or $a \ge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$. Thus, $G_{d_1}(d_2)$ is a continuous and non-decreasing function of d_2 on $[a, b]$. If $G_{d_1}(a) \ge 0$, then $G_{d_1}(d_2) \ge 0$ on the interval $[a, b]$. If $G_{d_1}(a) \le 0$, there exists $c(d_1) \in [a, b]$ such that $G_{d_1}(d_2) \le 0$ for any $d_2 \in [a, c(d_1)]$ and $G_{d_1}(d_2) \ge 0$ for any $d_2 \in [c(d_1), b]$. Thus, to determine the optimal solution I^*, we need to consider the following three cases.

<u>Case 1</u>. Suppose $(1+\theta)\int_0^b S_X(x)\mathrm{d}x - b + a \le 0$. In this case, $G_{d_1}(a) \le 0$ for any $d_1 \in [0, a]$. Thus,

$$v_{d_1}(d_2) = \begin{cases} -a + d_1 + b, & \text{for} \quad a \le d_2 \le c(d_1), \\ -a + d_1 + d_2 + P_I, & \text{for} \quad c(d_1) \le d_2 \le b, \end{cases}$$

with non-negative first derivative

$$v'_{d_1}(d_2) = \begin{cases} 0, & \text{for} \quad a \le d_2 \le c(d_1), \\ 1 - (1+\theta) S_X(d_2), & \text{for} \quad c(d_1) \le d_2 \le b. \end{cases}$$

Thus $d_2^*(d_1) = a$ and then $v(d_1,\, d_2^*(d_1)) = v(d_1,\, a) = -a + d_1 + b$ is a continuous function of d_1. It implies that $\min\limits_{(d_1,\, d_2) \in [0,\, a] \times [a,\, b]} v(d_1,\, d_2) = \min\limits_{d_1 \in [0,a]} \min\limits_{d_2 \in [a,b]} v(d_1,\, d_2) = \min\limits_{d_1 \in [0,a]} v(d_1,\, d_2^*(d_1)) = \min\limits_{d_1 \in [0,a]} -a + d_1 + b = b - a$, namely, the optimal pair is $(d_1^*,\, d_2^*) = (0, a)$. The corresponding optimal contract is

$$I^*(x) \;=\; x - (x-b)^+ = \begin{cases} x, & \text{for} \quad 0 \le x < b; \\ b, & \text{for} \quad b \le x < \infty. \end{cases}$$

Case 2. Suppose $(1+\theta)\int_a^b S_X(x)\mathrm{d}x - b + a \geq 0$. In this case, $G_{d_1}(a) \geq 0$ for any $d_1 \in [0,a]$. Thus $v_{d_1}(d_2) = -a + d_1 + d_2 + P_I$ with non-negative first derivative $\frac{\mathrm{d}}{\mathrm{d}d_2} v_{d_1}(d_2) = 1 - (1+\theta)S_X(d_2)$. It implies that $d_2^*(d_1) = a$ for any $d_1 \in [0,a]$. Now, $v(d_1, d_2^*(d_1)) = v(d_1, a) = d_1 + P_I$ is a continuous function of d_1, then

$$\min_{(d_1,\,d_2)\in[0,\,a]\times[a,\,b]} v(d_1,\,d_2) = \min_{d_1\in[0,\,a]}\min_{d_2\in[a,\,b]} v(d_1,\,d_2) = \min_{d_1\in[0,\,a]} v(d_1,\,d_2^*(d_1)).$$

Note that $\frac{\mathrm{d}}{\mathrm{d}d_1} v(d_1,\,d_2^*(d_1)) = 1-(1+\theta)S_X(d_1) \leq 0 \Leftrightarrow d_1 \leq \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$ and that $\mathrm{VaR}_{\frac{1}{1+\theta}}(X) \leq a$ by assumption. Thus, the optimal pair is $(d_1^*,\,d_2^*) = (\mathrm{VaR}_{\frac{1}{1+\theta}}(X),\,a)$ and the corresponding optimal contract is

$$I^*(x) = (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ - (x-b)^+ = \begin{cases} 0, & \text{for} \quad 0 \leq x < \mathrm{VaR}_{\frac{1}{1+\theta}}(X), \\ x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X), & \text{for} \quad \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \leq x < b, \\ \mathrm{VaR}_{\frac{1}{1+\theta}}(X), & \text{for} \quad b \leq x < \infty. \end{cases}$$

Case 3. Suppose $(1+\theta)\int_a^b S_X(x)\mathrm{d}x - b + a \leq 0 \leq (1+\theta)\int_0^b S_X(x)\mathrm{d}x - b + a$. In this case, there exists $d_0 \in [0,a]$ satisfying $G_{d_1}(a) = (1+\theta)\int_{d_1}^b S_X(x)\mathrm{d}x - b + a \geq 0$ for $0 \leq d_1 \leq d_0$ and $G_{d_1}(a) \leq 0$ for $d_0 \leq d_1 \leq a$. For $d_1 \in [0, d_0]$ or $G_{d_1}(a) \geq 0$, one has $v_{d_1}(d_2^*(d_1)) = d_1 + P_I$ using the same arguments as in Case 2. For $d_1 \in [d_0, a]$ or $G_{d_1}(a) \leq 0$, one has $v_{d_1}(d_2^*(d_1)) = -a + d_1 + b$ using the same arguments as in Case 1. Thus, on the whole interval $[0,a]$, $v_{d_1}(d_2^*(d_1)) = d_1 + P_I \vee (b-a)$ is a continuous function of d_1 and its first derivative is

$$\frac{\mathrm{d}}{\mathrm{d}d_1} v_{d_1}(d_2^*(d_1)) = \begin{cases} 1 - (1+\theta)S_X(d_1), & \text{for} \quad 0 \leq d_1 \leq d_0, \\ 1, & \text{for} \quad d_0 \leq d_1 \leq a. \end{cases}$$

It implies that

$$\min_{(d_1,\,d_2)\in[0,\,a]\times[a,\,b]} v(d_1,\,d_2) = \min_{d_1\in[0,\,a]}\min_{d_2\in[a,\,b]} v(d_1,\,d_2) = \min_{d_1\in[0,\,a]} v(d_1,\,d_2^*(d_1)) = v(d_1^*,\,d_2^*(d_1^*)),$$

where $d_1^* = \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge d_0$. The corresponding optimal solution is

$$I^*(x) = (x - d_1^*)^+ - (x-b)^+ = \begin{cases} 0, & \text{for} \quad 0 \leq x < b_1^*, \\ x - d_1^*, & \text{for} \quad b_1^* \leq x < b, \\ d_1^*, & \text{for} \quad b \leq x < \infty. \end{cases}$$

Combining the above three cases, the optimal solution of Problem (3.1) can be summarized into a unified formula, which is a limited stop-loss reinsurance contract given as follows:

$$
\begin{aligned}
I^*(x) &= (x - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\})^+ - (x-b)^+ \\
&= \begin{cases}
0, & \text{for} \quad 0 \le x < \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, \\
x - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, & \text{for} \quad \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\} \le x < b, \\
b - \max\{0,\, d_0 \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\}, & \text{for} \quad b \le x < \infty,
\end{cases}
\end{aligned}
$$

where $d_0 \in \mathbb{R}$ is the solution of the equation $(1+\theta)\int_{d_0}^{b} S_X(x)\mathrm{d}x = b - a$. □

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