A risk model with varying premiums: its risk management implications

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Abstract

In this paper, we consider a risk model which allows the insurer to partially reflect the recent claim experience in the determination of the next period's premium rate. In a ruin context, similar mechanisms to the one proposed in this paper have been studied by, e.g., Tsai and Parker (2004), Afonso et al. (2009) and Loisel and Trufin (2013). In our proposed risk model, we assume the surplus process is monitored at some review times only and the premium review decision is based on the surplus increment between two successive review times. When review times are distributed as a combination of exponentials and the claim arrival dynamic is compound Poisson, we derive a matrix-form defective renewal equation for the Gerber-Shiu function, as well as an explicit expression for the discounted joint density of the surplus prior to ruin and the deficit at ruin. Finally, we numerically compute the ruin probability and some tail properties of the deficit at ruin. A comparison with their counterparts in a constant premium rate model is also presented, and some risk management conclusions are made.

Keywords: varying premiums; experience-based premium policy; risk management; Gerber-Shiu function; defective renewal equation; discounted density.

1 Introduction

In the ruin theory literature, it is typically assumed that the incoming premium rate per unit time is constant over time. This assumption is often justified at the macro level by assuming that the insurer's aggregate insurance portfolio is fairly large and stable over time. For instance, an insurance product is gradually phased out and replaced by a more trendy alternative which overall maintains a fairly stable stream of cash flows. Also, terminating customers are replaced by new ones, which is expected to keep the insurance portfolio reasonably homogeneous over time.

In this paper, we propose to instead examine an insurer's surplus process on a smaller scale. The constant premium rate assumption is thus harder to justify in this case. We consider a strategy where the incoming premium rate is no longer constant, but is allowed to vary based on the recent claim experience of a particular insurance (sub-)portfolio. This can be viewed as a mechanism to have a premium rate policy which is somehow responsive to the recent claim experience, a well known practice in industry supported by the so-called *credibility theory* in insurance mathematics (see, e.g., Klugman et al. (2012)). Our premium review policy described below can also be regarded as a different allocation of the insurer's revenues over time, which we will show has great merit from a risk management standpoint. Indeed, the premium strategy is expected to provide a more optimal matching of the cash inflows and outflows of an insurer over time, and can thus contribute

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to an enhanced set of asset and liability management techniques available to insurers to manage the solvency risk.

In the same spirit as credibility theory, the main idea of the insurer's premium policy is to generate supplementary premium income following a period of bad claim experience, while reducing the incoming premium rate when a period of good claim experience is observed. Such a strategy is often consistent with the insurer's new perception of the risk insured (even though the risk itself may have remained unchanged). As such, we consider an insurer with the following finite set of premium rate options: $\{c_i\}_{i=1}^m$ with $c_i < c_j$ for $i < j$. For instance, these premium rates can result from the application of a set of security loadings to the underlying risk. We assume that the insurer has the ability to modify the incoming premium rate at some time points based on the increment value of the surplus process between two successive review times. Also, as in Albrecher et al. (2011, 2013), ruin will be monitored at these discrete time points only. A complete description of the risk model is presented in Section 2.

To better reflect the surplus cash flows of an insurance portfolio, many researchers have considered risk models whose premium rate is allowed to take different values over time. One typical research direction is to consider risk models under a dividend strategy (see, e.g., Avanzi (2009)) of a barrier type (e.g., Gerber (1979), Lin et al. (2003) and Li and Garrido (2004)) or a threshold type (e.g., Asmussen and Albrecher (2010), Lin and Pavlova (2006) and Lin and Sendova (2008)). Another direction is to study risk models with a credibility-based premium policy. For instance, Asmussen (1999) considered a risk process where the premium rate is calculated according to past claims statistics, while Tsai and Parker (2004), Afonso (2008), Afonso et al. (2009, 2010) and Loisel and Trufin (2013) examined the dynamic experience rating premium policy where premium rate adapts itself using Bühlmann's credibility theory. Since the set of premium rates is not specified in advance, the focus is put on the numerical calculation of ruin quantities, as well as some of their asymptotic properties. Recently, Landriault et al. (2012) proposed the idea of an adaptive premium policy where premium rates are fixed ahead of time, and with the assumption of mixed Erlang claim sizes, an explicit expression for the ruin probability in the classical risk model is derived. The model considered here builds on this recent work.

The rest of the paper is structured as follows. In Section 2, we define the risk model and the Gerber-Shiu function of interest in this paper. To analyze them, we first consider the distribution of the increments between successive review times. More precisely, in Section 3, we characterize the two one-sided density of increments between successive exponentially distributed review times through their Laplace transforms. In Section 4, when review times are distributed as a combination of exponentials, a matrix-form defective renewal equation for the Gerber-Shiu function is derived, with claim arrivals following a compound Poisson process. By employing Rouche's theorem and the initial value theorem, we derive explicit expressions for some densities of ruin-related quantities. In Section 5, we illustrate the benefit of the proposed premium policy from a risk management standpoint via some numerical examples. Finally, we conclude with a brief discussion of another variant of the risk model under study in this paper.

2 Description of the risk model

We assume a claim arrival dynamic as in the classical risk model. For completeness, we recall that in the classical risk model, the surplus process is defined as

$$U_t = u + Z_t, \; t \geq 0, \tag{2.1}$$

where $u \geq 0$ is the initial surplus level, $c > 0$ is the premium rate, and $Z_t = ct - S_t$. The aggregate claim amount process $\{S_t;\ t \geq 0\}$ is defined as

$$S_t = \begin{cases} \sum_{i=1}^{N_t} J_i, & N_t > 0, \\ 0, & N_t = 0, \end{cases}$$

where $\{N_t;\ t \geq 0\}$ is a Poisson process with arrival rate $\lambda > 0$ and the claim sizes $\{J_i;\ i \geq 1\}$ are a sequence of independent and identically distributed (iid) random variables (rv's) with density p and mean μ, independent of $\{N_t;\ t \geq 0\}$. In the classical risk model, the premium rate c per unit time is assumed to be constant over time.

In this paper, we consider instead a risk model with an embedded premium policy. This risk model is such that both premium changes and ruin detection can only occur at the random times $\{X_k; k \geq 1\}$, where X_k is the k-th review time with $X_0 = 0$. Thus, to analyze the ruin event, it suffices to consider the surplus process at the review times $\{X_k; k \geq 1\}$ only. Hence, let $\mathcal{U}_k$ be the surplus process value at time X_k, and define η_k to be the effective premium rate between the successive review times X_{k-1} and X_k. It follows that

$$\mathcal{U}_k = u + \sum_{j=1}^{k} Y_j, \tag{2.2}$$

where $Y_j = \eta_j T_j - \left(S_{X_j} - S_{X_{j-1}}\right)$ and $T_j = X_j - X_{j-1}$ is the j-th inter-review time. We assume that a review period beginning with premium rate c_i has an inter-review time that is distributed according to a rv K_i with density k_i and mean κ_i, i.e., $T_j|(\eta_j = c_i) \overset{d}{=} K_i$, for $i = 1, \ldots, m$ and $j = 1, 2, \ldots$. It is also assumed that conditional on $\{\eta_k; k \geq 1\}$, the inter-review times $\{T_k; k \geq 1\}$ are mutually independent, as well as independent of the aggregate claim process $\{S_t; t \geq 0\}$. Furthermore, the premium rate process $\{\eta_k; k \geq 1\}$ is assumed to be a (possibly non-time homogeneous) discrete-time Markov process. Given that $\eta_k = c_i$,

$$\eta_{k+1} = \begin{cases} c_{\min(i+1,m)}, & \text{if } c_i T_k - \left(S_{X_k} - S_{X_{k-1}}\right) \leq 0, \\ c_{\max(i-1,1)}, & \text{if } c_i T_k - \left(S_{X_k} - S_{X_{k-1}}\right) > 0. \end{cases}$$

In other words, for an inter-review period operating at a premium rate c_i, the subsequent inter-review period will operate at a premium rate $c_{\max(i-1,1)}$ ($c_{\min(i+1,m)}$) with transition probability $\Pr\left(c_i T_k - \left(S_{X_k} - S_{X_{k-1}}\right) \leq 0\right)$ $\left(\Pr\left(c_i T_k - \left(S_{X_k} - S_{X_{k-1}}\right) > 0\right)\right)$.

It follows that the time to ruin T^* is defined as $T^* = X_{k^*}$, where

$$k^* = \inf\{k \geq 1; \mathcal{U}_k < 0\},$$

with $T^* = \infty$ if $\mathcal{U}_k \geq 0$ for $k = 0, 1, 2, \ldots$. Also, define $\mathcal{U}_{k^*-1}$ and $|\mathcal{U}_{k^*}|$ to be the surplus prior to ruin and the deficit at ruin, respectively. A Gerber-Shiu function (see, e.g., Gerber and Shiu (1998)) of interest in this context is

$$m_{i,j,\delta}(u) = E\left[e^{-\delta T^*} w\left(\mathcal{U}_{k^*-1}, |\mathcal{U}_{k^*}|\right) I_{\{\eta_{k^*}=j\}} I_{\{T^*<\infty\}} | \mathcal{U}_0 = u, \eta_1 = c_i\right], \tag{2.3}$$

for $i, j = 1, \ldots, m$, where $\delta \geq 0$ can be viewed as a discount factor or a Laplace transform (LT) argument, $w(x, y)$ is a penalty function which is assumed to satisfy mild integrability conditions, and I_A is the indicator function of the event A.

To analyze the Gerber-Shiu function defined in (2.3), it will be particularly helpful to examine the distribution of the increments of the surplus process $\{U_t; t \geq 0\}$ over an exponentially distributed time horizon, which is studied in the next section.

3 The two one-sided densities of Z_{e_α}

In this section, we assume e_α is an exponential rv with finite mean $1/\alpha$ (independent of any other rv's). Also, define Z_{e_α} to be the increment of the surplus process over the horizon e_α. The two one-sided densities of Z_{e_α}, namely g_+ and g_-, are defined respectively through their one-sided LTs as

$$E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] = \int_0^\infty e^{-sx} g_+(x)dx,$$

and

$$E\left[e^{-s(-Z_{e_\alpha})} I_{\{Z_{e_\alpha}<0\}}\right] = \int_0^\infty e^{-sx} g_-(x)dx.$$

Our objective is to identify g_+ and g_- in the classical risk model (2.1). The main results can be found in Propositions 3.1 and 3.2, respectively. We point out that Kyprianou (2006, Corollary 8.9) derived the same result in the more general class of spectrally negative Lévy processes. However, we suggest a simpler proof to this result in the context of the classical risk model, which relies on a simple set of LT arguments.

3.1 The one-sided density g_+

To determine g_+, we first define the first passage time at level 0 for the process (2.1) as $\tau_0^- = \inf\{t \geq 0 | U_t < 0\}$ (with $\tau_0^- = \infty$ if $U_t \geq 0$ for $t \geq 0$). Thus, by conditioning on whether the review time e_α or the first passage time τ_0^- occurs first, we have

$$E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] = E\left[e^{-sZ_{e_\alpha}} I_{\{e_\alpha<\tau_0^-\}}\right] + E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}} I_{\{e_\alpha\geq\tau_0^-\}}\right]. \tag{3.1}$$

To obtain an expression for the first term on the right-hand side of (3.1), let

$$\varphi_\alpha(u) = E\left[e^{-sU_{e_\alpha}} I_{\{e_\alpha<\tau_0^-\}} | U_0 = u\right],$$

for which $E\left[e^{-sZ_{e_\alpha}} I_{\{e_\alpha<\tau_0^-\}}\right] = \varphi_\alpha(0)$. Note that $\varphi_\alpha(u)$ is the LT of U_{e_α} killed if the surplus process reaches negative values before the generic time e_α. The term "killed" is used here to specify that all sample paths with $\left\{e_\alpha > \tau_0^-\right\}$ are discarded. The reader is referred to Appendix A for the proof of Lemma 3.1.

Lemma 3.1 *The LT of U_{e_α} for all sample paths with $\left\{e_\alpha > \tau_0^-\right\}$ is given by*

$$\varphi_\alpha(u) = \alpha \left\{ \frac{1}{s+\rho} v_{\alpha,c}(u) - \int_0^u e^{-sx} v_{\alpha,c}(u-x)dx \right\},$$

where $v_{\alpha,c}(u)$ is defined on $[0,\infty)$ through its LT

$$\widetilde{v}_{\alpha,c}(z) = \frac{1}{cz - \lambda(1-\widetilde{p}(z)) - \alpha},$$

and $\rho = \rho_c(\alpha)$ is the unique non-negative solution of $cz - \lambda(1-\widetilde{p}(z)) - \alpha = 0$.

Note that $v_{\alpha,c}(\cdot)$ is known as the *α-scale function* in the literature on Lévy processes. We also remark that the inversion of $\varphi_\alpha(u)$ wrt s yields

$$E\left\{\Pr(U_{e_\alpha} \in (x, x+dx), e_\alpha < \tau_0^- | U_0 = u)\right\} \cong \alpha \left\{e^{-\rho x} v_{\alpha,c}(u) - v_{\alpha,c}(u-x) I_{\{u>x\}}\right\} dx.$$

From Lemma 3.1 and the fact that $v_{\alpha,c}(0) = 1/c$, it is immediate that

$$E\left[e^{-sZ_{e_\alpha}} I_{\{e_\alpha<\tau_0^-\}}\right] \equiv \varphi_\alpha(0) = \frac{\alpha}{c}\frac{1}{s+\rho}. \tag{3.2}$$

As for the second term on the right-hand side of (3.1), we have the following lemma.

Lemma 3.2 *The LT of the one-sided density of Z_{e_α} with $\{e_\alpha \geq \tau_0^-, Z_{e_\alpha>0}\}$ is given by*

$$E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}} I_{\{e_\alpha\geq\tau_0^-\}}\right] = \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} e^{-\rho y} E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] dy, \tag{3.3}$$

where $\mathcal{T}_r p(x) = \int_x^\infty e^{-r(y-x)}p(y)dy$ is the Dickson-Hipp operator with $\mathrm{Re}(r) \geq 0$.

Proof. For $\tau_0^- \leq e_\alpha$, we shall first condition on the distribution of the deficit at ruin $|U_{\tau_0^-}|$ together with $\{\tau_0^- \leq e_\alpha\}$, which corresponds to the discounted density of the deficit at ruin

$$E\left[e^{-\alpha\tau_0^-} I_{\{|U_{\tau_0^-}|\in(y,y+dy)\}}|U_0 = 0\right] \cong \frac{\lambda}{c}\mathcal{T}_\rho p(y)dy,$$

as per a result of Gerber and Shiu (1998, Equation 3.4) with $\delta = \alpha$.

From a deficit of y, the skip-free upward surplus process must then return to level 0 before the exponential time e_α, which happens with probability $e^{-\rho y}$ (see Asmussen and Albrecher (2010, Chapter V, Lemma 3.1)). Then, using the strong Markov property, we have

$$\begin{aligned} E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}} I_{\{e_\alpha\geq\tau_0^-\}}\right] &= \int_0^\infty E\left[e^{-\alpha\tau_0^-} I_{\{|U_{\tau_0^-}|\in(y,y+dy)\}}|U_0 = 0\right] e^{-\rho y} E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] \\ &= \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} e^{-\rho y} E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] dy. \end{aligned}$$

This completes the proof of Lemma 3.2. ■

We now make use of Lemma 3.2 together with Equation (3.2) to identify the one-sided density g_+.

Proposition 3.1 *The defective density of $Z_{e_\alpha} I_{\{Z_{e_\alpha}>0\}}$ is*

$$g_+(x) = \alpha\Phi_{\alpha,c}e^{-\rho x}, \tag{3.4}$$

for $x > 0$, where

$$\Phi_{\alpha,c} = \frac{1}{c - \lambda\mathcal{T}_\rho^2 p(0)}, \tag{3.5}$$

and $\mathcal{T}_\rho^2 p(0) = \int_0^\infty e^{-\rho y}\mathcal{T}_\rho p(y)dy$.

Proof. Substituting (3.2) and (3.3) into (3.1) yields

$$E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] = \frac{\alpha}{c}\frac{1}{s+\rho} + \left\{\int_0^\infty e^{-\rho y}\frac{\lambda}{c}\mathcal{T}_\rho p(y)dy\right\} E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right],$$

which implies that

$$E\left[e^{-sZ_{e_\alpha}} I_{\{Z_{e_\alpha}>0\}}\right] = \alpha\Phi_{\alpha,c}\frac{1}{s+\rho}, \tag{3.6}$$

where $\Phi_{\alpha,c}$ is defined in (3.5). The LT inversion of (3.6) wrt s yields (3.4). ■

Also, it is easy to prove that $\Phi_{\alpha,c} > 0$, which will be used in Section 4.

3.2 The one-sided density g_-

We are now interested in the other one-sided density of Z_{e_α}. Define $\tau_b^+ = \inf\{t \ge 0 | U_t \ge b\}$ to be the first passage time of $\{U_t;\ t \ge 0\}$ at level b.

Conditioning on the first excursion below 0 (before e_α), and then on whether the review time e_α or the recovery time τ_0^+ occurs first, we have

$$E\left[e^{-s(-Z_{e_\alpha})} I_{\{Z_{e_\alpha}<0\}}\right] = \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} e^{-\rho y} E\left[e^{-s(-Z_{e_\alpha})} I_{\{Z_{e_\alpha}<0\}}\right] dy + \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} \phi_\alpha(y) dy, \tag{3.7}$$

where

$$\phi_\alpha(y) = E\left[e^{-s(-U_{e_\alpha})} I_{\{e_\alpha<\tau_0^+\}} | U_0 = -y\right].$$

An explicit expression for $\phi_\alpha(y)$ is given in Lemma 3.3 (see Appendix B for the proof).

Lemma 3.3 *The ruin quantity $\phi_\alpha(y)$ can be expressed as*

$$\phi_\alpha(y) = \alpha\left(e^{-\rho y} - e^{-sy}\right)\widetilde{v}_{\alpha,c}(s), \tag{3.8}$$

where $\widetilde{v}_{\alpha,c}(s)$ and ρ are as defined in Lemma 3.1.

We are now in a position to provide a closed-form expression for the one-sided density g_-.

Proposition 3.2 *The defective density of $-Z_{e_\alpha} I_{\{Z_{e_\alpha}<0\}}$ is given by*

$$g_-(x) = \alpha\left\{\Phi_{\alpha,c} e^{\rho x} - v_{\alpha,c}(x)\right\},\ x > 0. \tag{3.9}$$

Proof. Substituting (3.8) into the second term on the right-hand side of (3.7), it follows

$$\begin{aligned}
&\int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} E\left[e^{-s(-U_{e_\alpha})} I_{\{e_\alpha<\tau_0^+\}} | U_0 = -y\right] dy \\
&= \int_0^\infty \left\{\frac{\lambda}{c}\mathcal{T}_\rho p(y)\right\} \left\{\alpha\left(e^{-\rho y} - e^{-sy}\right)\widetilde{v}_{\alpha,c}(s)\right\} dy \\
&= \alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - \frac{\lambda}{c}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{s-\rho}\right\} \\
&= \alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - 1\right\} + \frac{\alpha}{c}\frac{1}{s-\rho},
\end{aligned} \tag{3.10}$$

where the scale function identity

$$\widetilde{v}_{\alpha,c}(s) = \frac{1}{c(s-\rho)}\frac{1}{1-\frac{\lambda}{c}\frac{\widetilde{p}(\rho)-\widetilde{p}(s)}{s-\rho}},$$

was used in the process.

Substituting (3.10) into (3.7) yields

$$E\left[e^{-s(-Z_{e_\alpha})} I_{\{Z_{e_\alpha}<0\}}\right] = \frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) E\left[e^{-s(-Z_{e_\alpha})} I_{\{Z_{e_\alpha}<0\}}\right] + \alpha\widetilde{v}_{\alpha,c}(s)\left\{\frac{\lambda}{c}\mathcal{T}_\rho^2 p(0) - 1\right\} + \frac{\alpha}{c}\frac{1}{s-\rho},$$

i.e.,

$$E\left[e^{-s(-Z_{e_\alpha})} I_{\{Z_{e_\alpha}<0\}}\right] = \alpha\Phi_{\alpha,c}\frac{1}{s-\rho} - \alpha\widetilde{v}_{\alpha,c}(s). \tag{3.11}$$

The inversion of (3.11) wrt s results in (3.9). ■

Example 3.1 *We assume that claim sizes are exponentially distributed with mean $1/\beta$. Let $\rho > 0$ and $-R < 0$ be the two solutions of*

$$s^2 + \left(\beta - \frac{\lambda+\alpha}{c}\right) s - \frac{\alpha\beta}{c} = 0.$$

It is clear that $\Phi_{\alpha,c} = \frac{1}{c}\frac{\beta+\rho}{R+\rho}$ and

$$v_{\alpha,c}(x) = \frac{1}{c}\frac{\beta+\rho}{\rho+R}e^{\rho x} - \frac{1}{c}\frac{\beta-R}{\rho+R}e^{-Rx}.$$

Using (3.4) and (3.9), it follows that, for $x > 0$,

$$g_+(x) = \frac{\alpha}{c}\frac{\beta+\rho}{R+\rho}e^{-\rho x} \text{ and } g_-(x) = \frac{\alpha}{c}\frac{\beta-R}{\rho+R}e^{-Rx}.$$

This is in agreement with Albrecher et al. (2013, Example 4.1).

Remark 3.1 (Discounted one-sided densities) *It is no more difficult to consider the two one-sided "discounted" densities of Z_{e_α}, namely g_+^δ and g_-^δ. We first consider g_+^δ defined through its LT*

$$E\left[e^{-\delta e_\alpha}e^{-sZ_{e_\alpha}}I_{\{Z_{e_\alpha}>0\}}\right] = \int_0^\infty e^{-sx}g_+^\delta(x)dx. \tag{3.12}$$

Indeed,

$$E\left[e^{-\delta e_\alpha}e^{-sZ_{e_\alpha}}I_{\{Z_{e_\alpha}>0\}}\right] = E\left[e^{-sZ_{e_\alpha}}I_{\{Z_{e_\alpha}>0\}}I_{\{e_\delta^*>e_\alpha\}}\right],$$

where e_δ^* is an exponential rv with mean $1/\delta$, independent of all other rv's. Using the fact that $\min(e_\delta^*, e_\alpha)$ is distributed as $e_{\alpha+\delta}$, $I_{\{e_\delta^*>e_\alpha\}}$ is Bernoulli distributed with mean $\frac{\alpha}{\alpha+\delta}$, and $\min(e_\delta^*, e_\alpha)$ and $I_{\{e_\delta^*>e_\alpha\}}$ are independent (see, e.g., Ross (2010)), it follows that

$$E\left[e^{-\delta e_\alpha}e^{-sZ_{e_\alpha}}I_{\{Z_{e_\alpha}>0\}}\right] = \frac{\alpha}{\alpha+\delta}E\left[e^{-sZ_{e_{\alpha+\delta}}}I_{\{Z_{e_{\alpha+\delta}}>0\}}\right].$$

Thus,

$$g_+^\delta(x) = \frac{\alpha}{\alpha+\delta}g_+(x),$$

for $x > 0$, where $g_+(x)$ is as given in (3.4) but with the parameter α substituted by $\alpha+\delta$. We can apply the same arguments for g_-^δ. Therefore, the two one-sided discounted densities of Z_{e_α} are

$$g_+^\delta(x) = \alpha\Phi_{\alpha+\delta,c}e^{-\rho x}, \tag{3.13}$$

and

$$g_-^\delta(x) = \alpha\left\{\Phi_{\alpha+\delta,c}e^{\rho x} - v_{\alpha+\delta,c}(x)\right\},$$

for $x > 0$ where $\rho = \rho_c(\alpha+\delta)$.

4 Matrix-form defective renewal equation

In this section, we assume that a review period beginning with premium rate c_i has an inter-review time distributed as a rv K_i that is a combination of exponentials, i.e., it has a density given by

$$k_i(t) = \sum_{k=1}^{n}\xi_{ik}\alpha_{ik}e^{-\alpha_{ik}t},\ t > 0, \tag{4.1}$$

and $\widetilde{k}_i(s) = \sum_{j=1}^{n}\xi_{ij}\frac{\alpha_{ij}}{s+\alpha_{ij}}$. Note that the class of combinations of exponentials is dense in the set of all continuous probability distributions on the positive axis (see, e.g., Dufresne (2007)).

Proposition 4.1 *Let $g_{i,+}^{\delta}$ and $g_{i,-}^{\delta}$ be the two one-sided discounted densities of the increments over the review time K_i with density (4.1). For $x > 0$, we have*

$$g_{i,+}^{\delta}(x) = \sum_{k=1}^{n} \xi_{ik}\alpha_{ik}\Phi_{ik}e^{-\rho_{ik}x}, \tag{4.2}$$

and

$$g_{i,-}^{\delta}(x) = \sum_{k=1}^{n} \xi_{ik}\alpha_{ik}\left\{\Phi_{ik}e^{\rho_{ik}x} - v_{ik}(x)\right\}, \tag{4.3}$$

where $\rho_{ik} = \rho_{c_i}(\alpha_{ik}+\delta)$, $\Phi_{ik} = \Phi_{\alpha_{ik}+\delta,c_i}$ and $v_{ik}(x) = v_{\alpha_{ik}+\delta,c_i}(x)$.

Proof. *By the definition of the LT of $g_{i,+}^{\delta}$, along with (3.12) and (3.13), it follows*

$$\begin{aligned}\int_0^\infty e^{-sx}g_{i,+}^{\delta}(x)dx &= E\left[e^{-\delta K_i}e^{-sZ_{K_i}}I_{\{Z_{K_i}>0\}}\right] \\ &= \int_0^\infty E\left[e^{-\delta t}e^{-sZ_t}I_{\{Z_t>0\}}\right]k_i(t)dt \\ &= \sum_{k=1}^{n}\xi_{ik}E\left[e^{-\delta e_{\alpha_{ik}}}e^{-sZ_{e_{\alpha_{ik}}}}I_{\{Z_{e_{\alpha_{ik}}}>0\}}\right] \\ &= \int_0^\infty e^{-sx}\sum_{k=1}^{n}\xi_{ik}\alpha_{ik}\Phi_{ik}e^{-\rho_{ik}x}dx,\end{aligned}$$

By the uniqueness of LT, we obtain (4.2). Similar arguments apply to (4.3). ■

4.1 Laplace transform of the Gerber-Shiu function

With the two one-sided discounted densities, we now consider ruin-related quantities in the risk model (2.2) with the embedded premium policy.

By conditioning on the increment of the surplus process at the first review time, the Gerber-Shiu function defined in (2.3) can be expressed as

$$m_{i,j,\delta}(u) = \int_0^u m_{\min(i+1,m),j,\delta}(u-y)g_{i,-}^{\delta}(y)dy + b_{ij}(u) + \int_0^\infty m_{\max(i-1,1),j,\delta}(u+y)g_{i,+}^{\delta}(y)dy, \tag{4.4}$$

where $g_{i,+}^{\delta}(\cdot)$ and $g_{i,-}^{\delta}(\cdot)$ are defined in (4.2) and (4.3), and

$$b_{ij}(u) = \left\{\int_u^\infty w(u, y-u)g_{i,-}^{\delta}(y)dy\right\} I_{\{i=j\}}. \tag{4.5}$$

Taking the LT on both sides of (4.4), one finds that

$$\widetilde{m}_{i,j,\delta}(z) = \widetilde{m}_{\min(i+1,m),j,\delta}(z)\widetilde{g}_{i,-}^{\delta}(z) + \widetilde{b}_{ij}(z) + \sum_{k=1}^{n}\xi_{ik}\alpha_{ik}\Phi_{ik}\frac{\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik}) - \widetilde{m}_{\max(i-1,1),j,\delta}(z)}{z-\rho_{ik}}. \tag{4.6}$$

In matrix form, Equation (4.6) becomes

$$(\mathbf{I} - \mathbf{A}(z))\,\widetilde{\mathbf{m}}_{\delta}(z) = \widetilde{\mathbf{B}}(z) + \sum_{k=1}^{n}\mathbf{D}_k(z)\mathbf{C}_{k,\delta}, \tag{4.7}$$

where $\widetilde{\mathbf{m}}_\delta(z) = [\widetilde{m}_{i,j,\delta}(z)]_{i,j=1}^m$, $\widetilde{\mathbf{B}}(z) = \text{diag}\left\{\widetilde{b}_{ii}(z)\right\}_{i=1}^m$, $\mathbf{D}_k(z) = \text{diag}\left\{\frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z-\rho_{ik}}\right\}_{i=1}^m$, $\mathbf{C}_{k,\delta} = \left[\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik})\right]_{i,j=1}^m$, and

$$\mathbf{A}(z) = \begin{bmatrix} \sum_{k=1}^n \frac{\xi_{1k}\alpha_{1k}\Phi_{1k}}{\rho_{1k}-z} & \widetilde{g}_{1,-}^\delta(z) & 0 & \cdots & 0 \\ \sum_{k=1}^n \frac{\xi_{2k}\alpha_{2k}\Phi_{2k}}{\rho_{2k}-z} & 0 & \widetilde{g}_{2,-}^\delta(z) & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \sum_{k=1}^n \frac{\xi_{(m-1)k}\alpha_{(m-1)k}\Phi_{(m-1)k}}{\rho_{(m-1)k}-z} & 0 & \widetilde{g}_{m-1,-}^\delta(z) \\ 0 & \cdots & 0 & \sum_{k=1}^n \frac{\xi_{mk}\alpha_{mk}\Phi_{mk}}{\rho_{mk}-z} & \widetilde{g}_{m,-}^\delta(z) \end{bmatrix}.$$

Remark 4.1 *It is not difficult to check that $\mathbf{A}(0)|_{\delta=0}$ is the transition matrix for the discrete-time Markov process $\{\eta_k; k \geq 1\}$, whose stationary probabilities $\boldsymbol{\vartheta}^T = [\vartheta_1, \ldots, \vartheta_m]$ satisfy*

$$\begin{cases} \boldsymbol{\vartheta}^T \mathbf{A}(0)|_{\delta=0} = \boldsymbol{\vartheta}^T, \\ \boldsymbol{\vartheta}^T \mathbf{1} = 1, \end{cases}$$

while the the stationary probabilities of the process $\{\mathcal{U}_k; k \geq 0\}$ are given by $\boldsymbol{\pi}^T = [\pi_1, \ldots, \pi_m]$, where

$$\pi_i = \frac{\vartheta_i \kappa_i}{\sum_{j=1}^m \vartheta_j \kappa_j}, \tag{4.8}$$

for $i = 1, \ldots, m$ (see, e.g., Ross (2010)). Given these stationary probabilities, we can write the positive security loading condition for model (2.2) as

$$\sum_{i=1}^m \pi_i c_i > \lambda\mu, \tag{4.9}$$

which is very intuitive, since it is equivalent to $\sum_{i=1}^m \vartheta_i E\left[c_i K_i - S_{K_i}\right] > 0$.

Assuming $(\mathbf{I} - \mathbf{A}(z))$ is invertible, it follows that

$$\widetilde{\mathbf{m}}_\delta(z) = \frac{\text{adj}\,(\mathbf{I} - \mathbf{A}(z))\left(\widetilde{\mathbf{B}}(z) + \sum_{k=1}^n \mathbf{D}_k(z)\mathbf{C}_{k,\delta}\right)}{\det(\mathbf{I} - \mathbf{A}(z))}, \tag{4.10}$$

where adj $(\mathbf{I} - \mathbf{A}(z))$ is the adjoint matrix of $(\mathbf{I} - \mathbf{A}(z))$. Note that the matrices $\{\mathbf{C}_{k,\delta}\}_{k=1}^n$ in (4.10) contain $m \times m \times n$ unknown constants, namely $\widetilde{m}_{1,j,\delta}(\rho_{1k})$ and $\widetilde{m}_{(i-1),j,\delta}(\rho_{ik})$ for $i = 2, \ldots, m$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Thus, our objective is to identify these constants in order to fully characterize the closed-form expression for $\widetilde{\mathbf{m}}_\delta(z)$ given in (4.10), enabling its use in the numerical implementation of Section 5. An application of a matrix generalization of Rouche's theorem (see Dshalalow (1995)) will be useful in this context.

Lemma 4.1 *For $\delta > 0$, there are $m \times n$ non-negative solutions, namely $\gamma_1, \ldots, \gamma_{mn}$, to* $\det(\mathbf{I} - \mathbf{A}(z)) = 0$.

Proof. Define the contour $D = \lim_{r\to\infty}(D_r \cup D_0)$, where $D_r = \{z : \ |z| = r \text{ and } \text{Re}(z) \geq 0\}$ and $D_0 = \{z : \ |z| < r \ \ \text{Re}(z) = 0\}$. It can be shown that $\sum_{j=1}^m |a_{ij}(z)| < 1$ on D, where $a_{ij}(z)$ is the (i,j) entry of $\mathbf{A}(z)$.

Given that $g^{\delta}_{i,-}(y) \geq 0$, for $y > 0$, it follows that

$$\left|\widetilde{g}^{\delta}_{i,-}(z)\right| \leq \widetilde{g}^{\delta}_{i,-}(0) = \sum_{k=1}^{n} \xi_{ik}\alpha_{ik} \left\{ \frac{\Phi_{ik}}{-\rho_{ik}} + \frac{1}{\alpha_{ik}+\delta} \right\}$$

for $i = 1, \ldots, m$ and $\mathrm{Re}(z) \geq 0$.

Let us now assume that

$$\left| \sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z - \rho_{ik}} \right| \leq \widetilde{g}^{\delta}_{i,+}(0), \tag{4.11}$$

holds for all z in $D_r \cup D_0$. It follows that

$$\sum_{j=1}^{m} |a_{ij}(z)| = \left| \sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z - \rho_{ik}} \right| + \left|\widetilde{g}^{\delta}_{i,-}(z)\right| \leq \widetilde{g}^{\delta}_{i,+}(0) + \widetilde{g}^{\delta}_{i,-}(0) = \widetilde{k}_i(\delta) < 1.$$

To show that (4.11) holds on the contour $D_r \cup D_0$ for r sufficiently large, let us first consider the imaginary part of the contour. It is clear that for any z such that $\mathrm{Re}(z) = 0$,

$$\left| \sum_{k=1}^{n} \xi_{ik}\alpha_{ik} \frac{\Phi_{ik}}{z - \rho_{ik}} \right| = |\widetilde{g}_{i,+}(-z)| \leq \widetilde{g}^{\delta}_{i,+}(0).$$

Also, for all $z \in D_r$ such that $r > r_0 = \max_{i,k} \rho_{ik} + \frac{\sum_{k=1}^{n} |\xi_{ik}\alpha_{ik}\Phi_{ik}|}{\sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{\rho_{ik}}}$, we have

$$\left| \sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{z - \rho_{ik}} \right| \leq \sum_{k=1}^{n} \frac{|\xi_{ik}\alpha_{ik}\Phi_{ik}|}{|z| - \max_{i,k}\rho_{ik}} \leq \sum_{k=1}^{n} \frac{\xi_{ik}\alpha_{ik}\Phi_{ik}}{\rho_{ik}} = \widetilde{g}^{\delta}_{i,+}(0).$$

Therefore, $\sum_{j=1}^{m} |a_{ij}(z)| < 1$ holds on D_r for any $r > r_0$ and the imaginary axis defined by D_0. Now we can apply the matrix form of Rouche's theorem. Since $\det \mathbf{I} = 1 \neq 0$, $\det(\mathbf{I} - \mathbf{A}(z))$ satisfies $\mathcal{N}_{I-A} - \mathcal{P}_{I-A} = 0$, where $\mathcal{N}_{I-A}$ and $\mathcal{P}_{I-A}$ are the number of zeros and poles inside D of $\det(\mathbf{I} - \mathbf{A}(z))$, respectively. It is clear from the definition of $\mathbf{A}(z)$ that $\det(\mathbf{I} - \mathbf{A}(z))$ has $m \times n$ poles, namely $z = \rho_{ik}$ for $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Therefore, $\det(\mathbf{I} - \mathbf{A}(z))$ must have $m \times n$ zeros inside D. ■

Note that in the limiting case $\delta \to 0^+$, if the positive security loading condition (4.9) is satisfied, there are still $m \times n$ non-negative solutions to $\det(\mathbf{I} - \mathbf{A}(z)) = 0$ among which one solution is 0.

Henceforth, we assume that $\gamma_1, \ldots, \gamma_{mn}$ are distinct. For $i = 1, \ldots, mn$, let the non-zero row vector $\underline{\mathbf{h}}_i^T = [\underline{h}_{i1}, \ldots, \underline{h}_{im}]$ be the left eigenvector of $(\mathbf{I} - \mathbf{A}(\gamma_i))$ associated with the eigenvalue 0. Now we are ready to provide an explicit expression for $\{\mathbf{C}_{k,\delta}\}_{k=1}^{n}$.

Proposition 4.2 *If the matrix* $\mathbf{V} = \left[\underline{\mathbf{h}}_i^T \mathbf{D}_k(\gamma_i)\right]_{i=1,k=1}^{mn,n}$ *is invertible, then the matrices* $\{\mathbf{C}_{k,\delta}\}_{k=1}^{n}$ *are given by*

$$\begin{pmatrix} \mathbf{C}_{1,\delta} \\ \vdots \\ \mathbf{C}_{n,\delta} \end{pmatrix} = \mathbf{W} \begin{pmatrix} \underline{\mathbf{h}}_1^T \widetilde{\mathbf{B}}(\gamma_1) \\ \vdots \\ \underline{\mathbf{h}}_{mn}^T \widetilde{\mathbf{B}}(\gamma_{mn}) \end{pmatrix}, \tag{4.12}$$

where $\mathbf{W} = -\mathbf{V}^{-1}$.

Proof. By definition, for $i = 1, \ldots, mn$,

$$\underline{\mathbf{h}}_i^T \left(\mathbf{I} - \mathbf{A}(\gamma_i)\right) = \mathbf{0}^T,$$

which implies

$$\underline{\mathbf{h}}_i^T \left(\mathbf{I} - \mathbf{A}(\gamma_i)\right) \widetilde{\mathbf{m}}_\delta(\gamma_i) = \mathbf{0}^T \widetilde{\mathbf{m}}_\delta(\gamma_i) = \mathbf{0}^T. \tag{4.13}$$

Multiplying (4.7) at $z = \gamma_i$ by the left eigenvector $\underline{\mathbf{h}}_i^T$ and using (4.13), one finds

$$\underline{\mathbf{h}}_i^T \left(\mathbf{I} - \mathbf{A}(\gamma_i)\right) \widetilde{\mathbf{m}}_\delta(\gamma_i) = \underline{\mathbf{h}}_i^T \left(\widetilde{\mathbf{B}}(\gamma_i) + \sum_{k=1}^{n} \mathbf{D}_k(\gamma_i)\mathbf{C}_{k,\delta}\right) = \mathbf{0}^T,$$

which results in the following system of linear equations,

$$\mathbf{V} \begin{pmatrix} \mathbf{C}_{1,\delta} \\ \vdots \\ \mathbf{C}_{n,\delta} \end{pmatrix} = - \begin{pmatrix} \underline{\mathbf{h}}_1^T \widetilde{\mathbf{B}}(\gamma_1) \\ \vdots \\ \underline{\mathbf{h}}_{mn}^T \widetilde{\mathbf{B}}(\gamma_{mn}) \end{pmatrix}.$$

For $\mathbf{V}$ an invertible matrix, the result easily follows. ■

We point out that the matrix $\mathbf{V}$ is a generalized Cauchy matrix (see, e.g., Heinig (1995)) of the form

$$\left[\frac{\mathbf{z}_i^T \mathbf{y}_j}{c_i - d_j}\right]_{i,j=1}^{mn},$$

where $c_i = \gamma_i$, $d_j = \rho_{st}$, $\mathbf{z}_i^T = \underline{\mathbf{h}}_i^T$, $\mathbf{y}_j = \xi_{st}\alpha_{st}\Phi_{st}\mathbf{e}_s$ with $\mathbf{e}_s$ the canonical vectors, $s = j - \lfloor \frac{j}{m} \rfloor \times m$, and $t = \lceil \frac{j}{m} \rceil$. Conditions under which such a matrix is invertible have been widely analyzed in the literature (see, e.g., Heinig (1995, 1998) for more details). For instance, when $\mathbf{V}$ is invertible, an application of Theorem 2.2 in Heinig (1995) leads to an expression for $\mathbf{W} = -\mathbf{V}^{-1}$. Let $\mathbf{Z} = \mathrm{col}(\mathbf{z}_i^T)_{i=1}^{mn}$ and $\mathbf{Y} = \mathrm{col}(\mathbf{y}_j^T)_{j=1}^{mn}$, then

$$\mathbf{W} = [w_{i,j}]_{i,j=1}^{mn} = \left[\frac{\mathbf{x}_i^T \mathbf{p}_j}{d_i - c_j}\right]_{i,j=1}^{mn},$$

where $\mathbf{X} = \mathrm{col}(\mathbf{x}_i^T)_{i=1}^{mn}$ and $\mathbf{P} = \mathrm{col}(\mathbf{p}_j^T)_{j=1}^{mn}$ are the solutions to $\mathbf{VX} = \mathbf{Z}$ and $\mathbf{P}^T\mathbf{V} = \mathbf{Y}^T$.

Therefore, using (4.10) and (4.12), we have an explicit expression for $\widetilde{\mathbf{m}}_\delta(z)$, whose inversion results in an expression for $m_{ij,\delta}(u)$ in terms of the solutions $\gamma_1, \ldots, \gamma_{mn}$.

4.2 Matrix-form defective renewal equation and discounted joint densities

Intuitively, we expect the Gerber-Shiu function to satisfy a matrix-form defective renewal equation, also known as Markov renewal equation in the ruin theory literature (see, e.g., Cheung and Feng (2013)). Interest in such a representation comes from the fact that its solution is known to have some particularly nice properties (see, e.g., Miyazawa (2002) and Li and Luo (2005)).

Let $h_{1,ij}^{*\delta}(y|u)$ and $h_{2,ij}^{*\delta}(x, y|u)$ be the discounted density of the deficit at ruin $|\mathcal{U}_{k^*}|$ for ruin occurring at time X_1 and the discounted joint density of $(\mathcal{U}_{k^*-1}, |\mathcal{U}_{k^*}|)$ for ruin occurring after X_1, respectively. By conditioning on the first drop in surplus at a review time, the Gerber-Shiu function $m_{i,j,\delta}(u)$ can be represented as

$$m_{i,j,\delta}(u) = \sum_{l=1}^{m} \int_0^u m_{l,j,\delta}(u - y) h_{il}^{*\delta}(y|0) dy + f_{ij}(u), \tag{4.14}$$

where

$$h_{ij}^{*\delta}(y|u) = h_{1,ij}^{*\delta}(y|u) + \int_0^\infty h_{2,ij}^{*\delta}(x,y|u)dx,$$

and

$$f_{ij}(u) = \int_u^\infty w(u, y-u) h_{1,ij}^{*\delta}(y|0)dy + \int_u^\infty \int_0^\infty w(x+u, y-u) h_{2,ij}^{*\delta}(x,y|0)dxdy.$$

In a matrix form, we have

$$\mathbf{m}_\delta(u) = \mathbf{H} * \mathbf{m}_\delta(u) + \mathbf{F}(u), \tag{4.15}$$

where $\mathbf{m}_\delta(u) = [m_{i,j,\delta}(u)]_{i,j=1}^m$, $\mathbf{H}(y) = \left[h_{ij}^{*\delta}(y|0)\right]_{i,j=1}^m$, $\mathbf{F}(u) = [f_{ij}(u)]_{i,j=1}^m$ and the convolution of two matrices is defined as

$$[\mathbf{H} * \mathbf{m}_\delta(u)]_{ij} = \sum_{l=1}^m \int_0^u h_{il}^{*\delta}(y|0) m_{l,j,\delta}(u-y)dy.$$

Given that $\sum_{l=1}^m \int_0^\infty h_{il}^{*\delta}(y|0)dy = E\left[e^{-\delta T^*} I_{\{T^*<\infty\}} | \mathcal{U}_0 = 0, \eta_1 = c_i\right] < 1$ under $\delta > 0$ (or the positive security loading if $\delta = 0$), (4.15) is a matrix-form defective renewal equation with a unique solution.

In the following, we complete the characterization of (4.15) by identifying the discounted densities $h_{1,ij}^{*\delta}(y|0)$ and $h_{2,ij}^{*\delta}(x,y|0)$.

Proposition 4.3 *The discounted densities $h_{1,ij}^{*\delta}(y|0)$ and $h_{2,ij}^{*\delta}(x,y|0)$ are given by*

$$h_{1,ij}^{*\delta}(y|0) = g_{i,-}^\delta(y) I_{\{i=j\}}, \tag{4.16}$$

and

$$h_{2,ij}^{*\delta}(x,y|0) = \sum_{k=1}^n \xi_{ik}\alpha_{ik}\Phi_{ik} \sum_{l=1}^{mn} w_{(mk-m+i),l} \underline{h}_{lj} e^{-\gamma_l x} g_{j,-}^\delta(y+x). \tag{4.17}$$

Proof. Taking $u = 0$ and a penalty function of the form $w(x,y) = e^{-s_1 x - s_2 y}$ in (4.14), we have

$$m_{i,j,\delta}(0) = \int_0^\infty e^{-s_2 y} h_{1,ij}^{*\delta}(y|0)dy + \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y} h_{2,ij}^{*\delta}(x,y|0)dxdy. \tag{4.18}$$

Also, an application of the initial value theorem (see, e.g., Spiegel (1965)) to the transform $\widetilde{\mathbf{m}}_\delta(z)$ given in (4.10) leads to

$$\mathbf{m}_\delta(0) = \mathbf{B}(0) + \sum_{k=1}^n \mathbf{D}_k \mathbf{C}_{k,\delta},$$

where $\mathbf{D}_k = \text{diag}\{\xi_{ik}\alpha_{ik}\Phi_{ik}\}_{i=1}^m$. Alternatively, for $i,j = 1,\ldots,m$,

$$m_{i,j,\delta}(0) = b_{ij}(0) + \sum_{k=1}^n \xi_{ik}\alpha_{ik}\Phi_{ik} \widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik}). \tag{4.19}$$

Now, making use of (4.5) and (4.12), it is immediate that

$$b_{ij}(0) = \int_0^\infty w(0,y) g_{i,-}^\delta(y)dy I_{\{i=j\}} = \int_0^\infty e^{-s_2 y} g_{i,-}^\delta(y) I_{\{i=j\}}dy \tag{4.20}$$

and

$$
\begin{aligned}
\widetilde{m}_{\max(i-1,1),j,\delta}(\rho_{ik}) &= \sum_{l=1}^{mn} w_{(mk-m+i),l}\left(\sum_{s=1}^{m} \underline{h}_{ls}\widetilde{b}_{sj}(\gamma_l)\right) \\
&= \sum_{l=1}^{mn}\sum_{s=1}^{m} w_{(mk-m+i),l}\underline{h}_{ls}\int_0^\infty e^{-\gamma_l u}\int_u^\infty e^{-s_1 u - s_2(y-u)} g_{s,-}^{\delta}(y)dydu I_{\{s=j\}} \\
&= \int_0^\infty\int_0^\infty \sum_{l=1}^{mn}\sum_{s=1}^{m} w_{(mk-m+i),l}\underline{h}_{ls} e^{-\gamma_l x} e^{-s_1 x - s_2 y} g_{s,-}^{\delta}(y+x) I_{\{s=j\}} dydx. \quad (4.21)
\end{aligned}
$$

Substituting (4.20) and (4.21) into (4.19), one obtains

$$
\begin{aligned}
m_{i,j,\delta}(0) &= \int_0^\infty e^{-s_2 y} g_{i,-}^{\delta}(y) I_{\{i=j\}} dy \\
&+ \int_0^\infty\int_0^\infty \sum_{k=1}^{n} \xi_{ik}\alpha_{ik}\Phi_{ik}\sum_{l=1}^{mn}\sum_{s=1}^{m} w_{(mk-m+i),l}\underline{h}_{ls} e^{-\gamma_l x} e^{-s_1 x - s_2 y} g_{s,-}^{\delta}(y+x) I_{\{s=j\}} dydx.
\end{aligned} \tag{4.22}
$$

A comparison of (4.18) and (4.22) immediately leads to (4.16) and (4.17). ■

5 Numerical examples

In this section, we numerically implement the theoretical results obtained in Section 4. We show that the proposed embedded premium policy mitigates the risk of an insurer's insolvency. Our conclusions are consistent with similar work performed on the impact of experience-rating system on ruin-related quantities (see, e.g., Loisel and Trufin (2013) and Tsai and Parker (2004)).

First, we introduce a constant premium model with randomized reviews (CPMRR) (see, e.g., Albrecher et al. (2011, 2013)), where the classical risk model (2.1) can only be observed at random review times with density k. In other words, the CPMRR is the same as the model defined in (2.2) but with premium rates $c_i \equiv \bar{c}$ and review time distributions $k_i(t) \equiv k(t)$ for $i = 1, \ldots, m$. In the following, we propose to compare ruin quantities in the risk model (2.2) with the embedded premium policy to their counterparts in the CPMRR. As a basis for comparison, we assume that $\bar{c}$ is fixed at the stationary level of the embedded premium policy, i.e.,

$$
\bar{c} = \sum_{j=1}^{m} \pi_j c_j,
$$

where $\boldsymbol{\pi}^T = [\pi_1, \ldots, \pi_m]$ are the stationary probabilities defined in (4.8). Also, given that for our embedded premium policy, ruin quantities are defined conditional on the initial premium rate, we remove this dependence on η_1 by mixing the ruin quantities over the stationary probabilities, i.e., we consider

$$
m_{st,\delta}(u) = \sum_{i=1}^{m} \pi_i m_{i,\delta}(u), \tag{5.1}
$$

where $m_{i,\delta}(u) = \sum_{j=1}^{m} m_{i,j,\delta}(u)$ is the Gerber-Shiu function with an initial premium rate c_i. In addition, we assume the review time's density does not depend on the premium rate in effect, i.e., $k_i(t) = k(t)$ for $i = 1, \ldots, m$.

5.1 Ruin probability

We begin our analysis with the ruin probability. Let $\psi_{st}(u)$ be the stationary ruin probability resulting from Equation (5.1) with $\delta = 0$ and $w(x, y) = 1$. Also, define $\psi_{\bar{c}}(u)$ to be the ruin probability of the CPMRR.

Example 5.1 *We consider an example with two premium rates $c_1 = 11$ and $c_2 = 14$. Claim sizes are assumed to be exponentially distributed with mean 10, while the inter-review times are also exponentially distributed with $k(t) = \alpha e^{-\alpha t}$. Finally, we set the claim arrival rate to $\lambda = 1$. Results for $\psi_{st}(u)$ and $\psi_{\bar{c}}(u)$ are provided in Table 1, for different u and α.*

	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$
α	u=0		u=25		u=50		u=100	
0.1	0.5410	0.5158	0.3418	0.3458	0.2143	0.2318	0.0831	0.1042
0.5	0.7104	0.7053	0.4583	0.4772	0.2946	0.3229	0.1213	0.1478
1	0.7688	0.7666	0.5122	0.5289	0.3407	0.3650	0.1505	0.1737
10	0.8808	0.8807	0.6660	0.6700	0.5036	0.5098	0.2880	0.2950
∞	0.9091	0.9091	0.7243	0.7243	0.5770	0.5770	0.3663	0.3663

Table 1: Ruin probability with different values of u and α

From Table 1, we observe that:

1. As expected, the ruin probability is a decreasing function of the initial surplus u.
2. The ruin probability is an increasing function of α. As the rate α increases, the frequency of solvency checks increases, making it more likely to identify a ruin event. Also, given that c_1 and c_2 have positive security loadings, a larger α implies that the premium review will be conducted more often to reduce the premium rate to c_1, and thus making the surplus process riskier.

   As expected, when α goes to ∞, all ruin probabilities converge to the ruin probability in the (continuous time) classical risk model (2.1) with a constant premium rate of c_1.
3. For relatively large surplus values, the ruin probabilities ψ_{st} are smaller than $\psi_{\bar{c}}$, which implies that our embedded premium policy reduces the risk of insolvency in the long run. However, the opposite conclusion is reached for small initial surplus values, an observation also made by Tsai and Parker (2004) and Loisel and Trufin (2013) in a similar context.

In the following example, our goal is to investigate the effect of the distribution of the inter-review times on the ruin probability.

Example 5.2 *We reconsider Example 5.1 under two following alternative distributional assumptions for the inter-review times:*

M1: $k(t) = (\alpha_1 e^{-\alpha_1 t} + \alpha_2 e^{-\alpha_2 t})/2$ *with* (α_1, α_2) *such that the mean is* $1/\alpha$ *and the variance is* $1.5/\alpha^2 > 1/\alpha^2$.

M2: $k(t) = \left(3\alpha_1 e^{-\alpha_1 t} - \alpha_2 e^{-\alpha_2 t}\right)/2$ *with* (α_1, α_2) *such that the mean is* $1/\alpha$ *and the variance is* $0.5/\alpha^2 < 1/\alpha^2$.

	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$
α	u=0		u=25		u=50		u=100	
0.1	0.5486	0.5198	0.3444	0.3436	0.2164	0.2285	0.0858	0.1020
0.5	0.7126	0.7057	0.4612	0.4768	0.2992	0.3237	0.1261	0.1498
1	0.7696	0.7664	0.5160	0.5304	0.3465	0.3682	0.1564	0.1778
10	0.8799	0.8798	0.6668	0.6706	0.5054	0.5112	0.2903	0.2971
∞	0.9091	0.9091	0.7243	0.7243	0.5770	0.5770	0.3663	0.3663

Table 2: Ruin probability under M1

	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$	$\psi_{st}(u)$	$\psi_{\bar{c}}(u)$
α	u=0		u=25		u=50		u=100	
0.1	0.5359	0.5123	0.3397	0.3458	0.2130	0.2330	0.0820	0.1055
0.5	0.7086	0.7047	0.4570	0.4781	0.2928	0.3236	0.1191	0.1478
1	0.7680	0.7665	0.5108	0.5291	0.3384	0.3644	0.1479	0.1725
10	0.8814	0.8814	0.6656	0.6698	0.5026	0.5088	0.2863	0.2936
∞	0.9091	0.9091	0.7243	0.7243	0.5770	0.5770	0.3663	0.3663

Table 3: Ruin probability under M2

Tables 2 and 3 contain the values of the resulting ruin probabilities for Example 5.2. Similar conclusions as those provided for Example 5.1 are also valid here. As far as the distributional assumptions of inter-review times are concerned, we remark a tendency for the ruin probability to increase as the variance of the inter-review time distribution increases. However, this conclusion is not general, as when $u = 0$ and $\alpha = 10$, the opposite ordering is observed.

5.2 Deficit at ruin

In this subsection, we shift our attention to the deficit at ruin, more precisely to its tail properties. By taking $\delta = 0$ and $w(x, y) = e^{-sy}$ in Equation (5.1), one finds that

$$\sum_{i=1}^{m} \pi_i E\left[e^{-s|\mathcal{U}_{k^*}|} I_{\{T^*<\infty\}} | \mathcal{U}_0 = u, \eta_1 = c_i\right] = \int_0^\infty e^{-sy} \Pr\left(L_{st} \in dy\right),$$

where L_{st} corresponds to the deficit at ruin in the stationary risk model of (2.2). Clearly L_{st} is a defective rv, and we alternatively consider the proper rv $L^*_{st} = L_{st}|T^* < \infty$. In what follows, we focus on the Value at Risk (VaR) of the mixing deficit at ruin L_{st} (and L^*_{st}), which is defined as

$$\mathrm{VaR}^{(*)}_{st,q} = \inf\left\{y \geq 0 : \Pr\left(L^{(*)}_{st} > y\right) \leq 1 - q\right\}. \tag{5.2}$$

In the CPMRR, the counterparts to (5.2) are denoted by $\mathrm{VaR}_{\bar{c},q}$ and $\mathrm{VaR}^*_{\bar{c},q}$, respectively.

Example 5.3 *We reconsider Example 5.1 under assumption M1 in Example 5.2. Table 4 contains the VaR value when $\alpha = 0.5$, in which case we have $\alpha_1 = 1/3$ and $\alpha_2 = 1$.*

Table 4 leads to similar conclusions as those for the ruin probabilities of Table 1, even though the impact is less noticeable. Indeed, with the exception of small surplus levels, the values of VaR of the deficit at ruin (both defective and proper) in the proposed premium policy risk model are smaller than their counterparts in the CPMRR. This is another numerical evidence of the merit of the embedded premium policy proposed in this paper from a risk management standpoint.

q	u=0		u=25		u=50		u=100		u=200	
	$\mathrm{VaR}_{st,q}$	$\mathrm{VaR}_{\bar{c},q}$	$\mathrm{VaR}_{st,q}$	$\mathrm{VaR}_{\bar{c},q}$	$\mathrm{VaR}_{st,q}$	$\mathrm{VaR}_{\bar{c},q}$	$\mathrm{VaR}_{st,q}$	$\mathrm{VaR}_{\bar{c},q}$	$\mathrm{VaR}_{st,q}$	$\mathrm{VaR}_{\bar{c},q}$
0.95	51.37	50.59	43.02	43.71	34.54	36.29	17.70	21.22	0	0
0.98	69.82	68.74	61.37	62.00	52.75	54.56	35.59	39.30	2.15	9.13
0.99	83.99	82.65	75.46	76.00	66.73	68.55	49.34	53.16	15.41	22.61
0.995	98.32	96.70	89.70	90.13	80.88	82.67	63.28	67.17	28.87	36.28
0.9995	146.86	144.14	137.89	137.76	128.77	130.27	110.60	114.50	74.92	82.77
	$\mathrm{VaR}^*_{st,q}$	$\mathrm{VaR}^*_{\bar{c},q}$	$\mathrm{VaR}^*_{st,q}$	$\mathrm{VaR}^*_{\bar{c},q}$	$\mathrm{VaR}^*_{st,q}$	$\mathrm{VaR}^*_{\bar{c},q}$	$\mathrm{VaR}^*_{st,q}$	$\mathrm{VaR}^*_{\bar{c},q}$	$\mathrm{VaR}^*_{st,q}$	$\mathrm{VaR}^*_{\bar{c},q}$
0.95	58.15	57.46	58.50	58.47	58.59	58.81	58.60	58.98	58.58	59.00
0.98	76.72	75.72	77.11	76.97	77.20	77.40	77.19	77.60	77.17	77.62
0.99	90.97	89.70	91.36	91.11	91.44	91.59	91.42	91.81	91.39	91.84
0.995	105.38	103.82	105.76	105.35	105.82	105.88	105.78	106.12	105.74	106.15
0.9995	154.09	151.40	154.31	153.25	154.28	153.86	154.17	154.15	154.11	154.19

Table 4: VaR of the defective and proper deficit at ruin

6 Conclusion

To conclude, we point out that another variant of the proposed risk model can be analyzed using the methodology developed in this paper. This risk model consists of replacing the natural performance level 0 for the increment between successive review times by a random threshold (which may or may not depend on the premium rate effective at the beginning of the period). A matrix-form defective renewal equation for the Gerber-Shiu function as well as discounted joint densities of interest can also be obtained in this context.

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Appendix

A Proof for Lemma 3.1

By conditioning on the first occurrence between a claim instant and a review time, we get

$$\begin{aligned}\varphi_\alpha(u) &= \int_0^\infty \lambda e^{-(\lambda+\alpha)t}\left\{\int_0^{u+ct}\varphi_\alpha(u+ct-y)p(y)dy\right\}dt + \int_0^\infty \alpha e^{-(\lambda+\alpha)t}e^{-s(u+ct)}dt \\ &= \frac{\lambda}{c}\int_u^\infty e^{-\frac{\lambda+\alpha}{c}(x-u)}\int_0^x \varphi_\alpha(x-y)p(y)dydx + \frac{\alpha}{\lambda+\alpha+cs}e^{-su} \\ &= \frac{\lambda}{c}\mathcal{T}_{\frac{\lambda+\alpha}{c}}r_\alpha(u) + \frac{\alpha}{\lambda+\alpha+cs}e^{-su}, \end{aligned} \tag{A.1}$$

where

$$r_\alpha(x) = \int_0^x \varphi_\alpha(x-y)p(y)dy.$$

Taking the LT on both sides of (A.1), one obtains

$$\begin{aligned}
\widetilde{\varphi}_\alpha(z) &= \frac{\lambda}{c}\frac{\widetilde{r}_\alpha\left(\frac{\lambda+\alpha}{c}\right)-\widetilde{r}_\alpha(z)}{z-\frac{\lambda+\alpha}{c}}+\frac{\alpha}{\lambda+\alpha+cs}\frac{1}{z+s}\\
&= \frac{\lambda}{c}\frac{\widetilde{\varphi}_\alpha\left(\frac{\lambda+\alpha}{c}\right)\widetilde{p}\left(\frac{\lambda+\alpha}{c}\right)-\widetilde{\varphi}_\alpha(z)\widetilde{p}(z)}{z-\frac{\lambda+\alpha}{c}}+\frac{\alpha}{\lambda+\alpha+cs}\frac{1}{z+s}.
\end{aligned} \tag{A.2}$$

A simple rearrangement of (A.2) yields

$$\{cz-\lambda\left(1-\widetilde{p}(z)\right)-\alpha\}\,\widetilde{\varphi}_\alpha(z)=\left\{\lambda\widetilde{\varphi}_\alpha\left(\frac{\lambda+\alpha}{c}\right)\widetilde{p}\left(\frac{\lambda+\alpha}{c}\right)+\frac{\alpha c}{\lambda+\alpha+cs}\right\}-\frac{\alpha}{z+s}. \tag{A.3}$$

The first term on the right-hand side of (A.3) does not depend on z, and thus by taking $z=\rho$, we can express it as

$$\lambda\widetilde{\varphi}_\alpha\left(\frac{\lambda+\alpha}{c}\right)\widetilde{p}\left(\frac{\lambda+\alpha}{c}\right)+\frac{\alpha c}{\lambda+\alpha+cs}=\frac{\alpha}{\rho+s}. \tag{A.4}$$

Substituting (A.4) into (A.3), we have

$$\{cz-\lambda\left(1-\widetilde{p}(z)\right)-\alpha\}\,\widetilde{\varphi}_\alpha(z)=\alpha\left(\frac{1}{\rho+s}-\frac{1}{s+z}\right),$$

i.e.,

$$\widetilde{\varphi}_\alpha(z)=\alpha\left(\frac{1}{s+\rho}-\frac{1}{s+z}\right)\widetilde{v}_{\alpha,c}(z).$$

By taking the LT inversion wrt z, we complete the proof.

B Proof for Lemma 3.3

By reflection, we get

$$\phi_\alpha(y)=E\left[e^{-sR_{e_\alpha}}I_{\{e_\alpha<\tau_0^{*-}\}}|R_0=y\right],$$

where $R_t=u-Z_t$ is the dual risk model and $\tau_0^{*-}=\inf\{t\geq 0|R_t\leq 0\}$ is the first passage time of $\{R_t;\ t\geq 0\}$ at level 0. Thus, $\phi_\alpha(y)$ is the LT of R_{e_α} given that the review time e_α occurs before ruin. Intuitively, it is clear that

$$\phi_\alpha(y)=e^{-s\epsilon}\phi_\alpha(y-\epsilon)+e^{-\rho(y-\epsilon)}\phi_\alpha(\epsilon),$$

for all $\epsilon\in[0,u]$. Integrating over ϵ from 0 to y, it follows that

$$y\phi_\alpha(y)=\int_0^y e^{-s\epsilon}\phi_\alpha(y-\epsilon)d\epsilon+\int_0^y e^{-\rho(y-\epsilon)}\phi_\alpha(\epsilon)d\epsilon, \tag{B.1}$$

Taking the LT on both sides of (B.1), we obtain

$$\int_0^\infty e^{-zy}y\phi_\alpha(y)dy=\left(\frac{1}{s+z}+\frac{1}{\rho+z}\right)\widetilde{\phi}_\alpha(z).$$

Note that $\int_0^\infty e^{-zy}y\phi_\alpha(y)dy=\frac{d}{dz}\widetilde{\phi}_\alpha(z)$. Thus, solving this ordinary differential equation yields

$$\phi_\alpha(y)=c(s)\left(e^{-sy}-e^{-\rho y}\right), \tag{B.2}$$

where $c(s)$ is a constant involving s.

To identify $c(s)$, we condition on the time and amount of the first jump, i.e.,

$$\begin{aligned}
\phi_\alpha(y) &= \int_0^{y/c} \lambda e^{-(\lambda+\alpha)t} \left\{ \int_0^\infty \phi_\alpha(y-ct+x)p(x)dx \right\} dt + \int_0^{y/c} \alpha e^{-(\lambda+\alpha)t} e^{-s(y-ct)} dt \\
&= c(s) \left\{ \frac{\lambda}{\lambda+\alpha-cs} \widetilde{p}(s) \left(e^{-sy} - e^{-(\lambda+\alpha)y/c} \right) - \frac{\lambda}{\lambda+\alpha-c\rho} \widetilde{p}(s) \left(e^{-\rho y} - e^{-(\lambda+\alpha)y/c} \right) \right\} \\
&+ \frac{\alpha}{\lambda+\alpha-cs} \left(e^{-sy} - e^{-(\lambda+\alpha)y/c} \right) \\
&= \left\{ c(s) \frac{\lambda}{\lambda+\alpha-cs} \widetilde{p}(s) + \frac{\alpha}{\lambda+\alpha-cs} \right\} e^{-sy} - c(s) e^{-\rho y} \\
&- \left\{ c(s) \frac{\lambda}{\lambda+\alpha-cs} \widetilde{p}(s) - c(s) + \frac{\alpha}{\lambda+\alpha-cs} \right\} e^{-(\lambda+\alpha)y/c}.
\end{aligned}$$

Matching the coefficients of e^{-sy}, we get

$$c(s) = \frac{\alpha}{\alpha+\lambda-cs-\lambda\widetilde{p}(s)} = -\alpha \widetilde{v}_{\alpha,c}(s). \tag{B.3}$$

Thus, substituting (B.3) into (B.2) yields

$$\phi_\alpha(y) = \alpha \left(e^{-\rho y} - e^{-sy} \right) \widetilde{v}_{\alpha,c}(s).$$

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