A review of discrepancy bounds for (t, s) and $(t, \mathbf{e}, s)$-sequences with numerical comparisons

Henri Faure[a], Christiane Lemieux[b]

[a]*Institut de Mathématiques de Marseille, France, henri.faure@univ-amu.fr*
[b]*University of Waterloo, Canada, clemieux@uwaterloo.ca*

Abstract

In this paper we review recent discrepancy bounds for (t, s)-sequences and other low-discrepancy sequences. We then provide numerical comparisons of these bounds for various combinations of dimension and number of points. In some cases, we compare different bounds for the same type of construction, and in other cases, we compare the best bounds across different constructions.

Keywords: Discrepancy bounds, low-discrepancy sequences.

1. Introduction

The family of (t, s)-sequences constitute a large and important family of low-discrepancy sequences, often used in the context of quasi-Monte Carlo methods for integration and approximation. The quality of these sequences is often assessed through the concept of star discrepancy, denoted by $D^*(P_N)$, where P_N is the set of points in $[0, 1)^s$ obtained by considering the first N points of the sequence under study. The exact value of the discrepancy is typically very hard to compute, therefore the behavior of these sequences is often assessed through bounds of the form

$$ND^*(P_N) \leq c_s(\log N)^s + O(\log N)^{s-1}. \tag{1}$$

There has been a lot of work on discrepancy bounds of the form (1) for (t, s)-sequences over the last few years. In some cases, the asymptotic behavior of the bound, as studied through the quantity c_s, was improved significantly [1, 4, 6, 17]. In some other cases, the improvement was instead done within the $O(\log N)^{s-1}$ term, which can still have an important effect when comparing bounds in the non-asymptotic regime, that is, for finite values of N. When sequences are compared with the goal of determining which ones should be preferred for practical purposes, we believe it is important to not only focus on the constant c_s but also consider the whole bound, whose behaviour may be more indicative of the sequence's performance when used in practice, for possibly large values of s and not so large values of N. If a construction provides an improvement in c_s at the expense of a much worse bound overall, it is unclear that its performance in practice will be satisfying. This type of behaviour was studied in [5] for Halton sequences and the so-called *modified Halton sequences* introduced by Atanassov in [1]. The latter have a smaller constant c_s that is competing with Niederreiter-Xing sequences, but with a huge complementary $O(\log N)^{s-1}$ term. As mentioned in [5], in that case the bound does not tell the whole story, in particular because it is independent of the choice of permutations used to improve the quality of the original Halton sequences for finite values of N.

The goal of this paper is first to present an overview of these recent results, highlighting the various improvements that each result provided. This discussion is presented in Section 2. Then we provide numerical comparisons to assess the impact of these different improvements. Since these bounds also allow us to compare the performance of different choices of bases for (t,s)-sequences, we also present results comparing the best bounds for different low-discrepancy sequences.

2. Review of known bounds

Before presenting the different recent bounds that will later be compared, we first provide some background information on discrepancy and (t,s)-sequences. We then give some perspective on how the different bounds were obtained and what improvement they provide.

We start with a review of the notion of discrepancy, which will be used throughout the paper. We focus on the so-called *extreme discrepancy*, which corresponds to the worst case error in the domain of complexity of multivariate problems. Assume we have a point set $\mathcal{P}_N = \{X_1, \ldots, X_N\} \subseteq I^s := [0,1]^s$ and denote $\mathcal{J}$ (resp $\mathcal{J}^*$) the set of intervals J of I^s of the form $J = \prod_{j=1}^{s} [y_j, z_j)$, where $0 \leq y_j < z_j \leq 1$ (resp. $J = \prod_{j=1}^{s} [0, z_j)$). Then the *discrepancy function* of $\mathcal{P}_N$ on such an interval J is the difference

$$E(J; N) = A(J; \mathcal{P}_N) - NV(J),$$

where $A(J;\mathcal{P}_N) = \#\{n; 1 \leq n \leq N, X_n \in J\}$ is the number of points in $\mathcal{P}_N$ that fall in the subinterval J, and $V(J) = \prod_{j=1}^{s}(z_j - y_j)$ is the volume of J.

Then, the *star (extreme) discrepancy* D^* and the *(extreme) discrepancy* D of $\mathcal{P}_N$ are defined by

$$D^*(\mathcal{P}_N) = \sup_{J \in \mathcal{J}^*} |E(J;N)| \quad \text{and} \quad D(\mathcal{P}_N) = \sup_{J \in \mathcal{J}} |E(J;N)|.$$

It is well known that $D^*(\mathcal{P}_N) \leq D(\mathcal{P}_N) \leq 2^s D^*(\mathcal{P}_N)$. For an infinite sequence X, we denote by $D(N, X)$ and $D^*(N, X)$ the discrepancies of its first N points. Note that several authors have a $1/N$ factor when defining the above quantities.

Moving on to (t, s)-sequences, this concept was introduced by Niederreiter [12] to give a general framework for various constructions using generating matrices applied to van der Corput sequences, including Sobol' sequences [15], Faure sequences [2], and later a more general class of constructions referred to as Niederreiter-Xing sequences [14]. The definition of this last class of sequences needs an extension of the original concept of (t, s)-sequences which is now widely used in the literature. In this paper, we only consider this extension which makes use of the coordinatewise m-digit truncation in base $b \geq 2$ of elements $X \in I^s$, denoted by $[X]_{b,m}$ (see [14] for details).

Definition 1. A sequence $(X_n)_{n \geq 1}$ of points $X_n \in I^s$ is a (t, s)-*sequence* in base b if the subset $\{[X_n]_{b,m}; kb^m < n \leq (k+1)b^m\}$ is a (t, m, s)-net in base b for all integers $k \geq 0$ and $m \geq t$.

Several of the bounds presented in this paper make use of the technique introduced by Atanassov in [1] to study Halton sequences. His approach is very different from the traditional ones developed by Sobol', Faure and Niederreiter for (t, s)-sequences, which make use of a double recursion on the parameter m such that $N = b^m$, and the dimension s, along with a symmetrisation technique that provides the $1/(s! \cdot 2^s)$ term. Instead, Atanassov uses a signed splitting method together with an argument from diophantine geometry to provide an expression for c_s smaller by a factor of $s!$ compared to the previous bound for Halton sequences from [2, Section 2].

Atanassov's proof method was adapted in [8] to provide bounds for (t, s)-sequences having about the same constant c_s—equal when b is odd and larger by $b/(b-1)$ when b is even—as those that were known at the time, namely from Niederreiter [13]. However, in the subsequent paper [6], we were able to use signed splittings coupled with signed b-adic expansions in the case of an even base b to get a significant improvement on c_s in that case, more precisely by a factor of $((b-1)/b)^{s-1}$ compared to[13]. The results from [6] are given next, starting with one that is valid for any (t,s)-sequence:

Theorem 1. *Let X be a (t,s)-sequence in base b. For any $N \geq 1$ we have*

$$D^*(N,X) \leq \frac{b^t}{s!}\left(\left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + s\right)^s + b^t \sum_{k=0}^{s-1} \frac{b}{k!}\left(\left\lfloor \frac{b}{2} \right\rfloor \frac{\log N}{\log b} + k\right)^k. \tag{2}$$

We also have from [6] the result that is specialized to the case of an even base b, given by:

Theorem 2. *Let X be a (t,s)-sequence in an even base b. For any $N \geq b^s$ we have*

$$\begin{aligned} D^*(N,X) \leq & \frac{b^t}{s!}\left(\frac{b-1}{2}\frac{\log N}{\log b} + s\right)^s + sb^t\left(\frac{b}{2}\right)^s\left(\frac{\log N}{\log b}\right)^{s-1} \\ & + b^t \sum_{k=0}^{s-1} \frac{b}{k!}\left(\frac{b}{2}\frac{\log N}{\log b} + k\right)^k. \end{aligned} \tag{3}$$

Note that the two results stated above (as well as Theorem 4 later on) are slightly different from the ones given in the original papers [8, 6]: the second sum starts at $k=0$ above, correcting the result given in [8, 6] that starts at $k=1$ instead. This difference is explained in more detail in [3, 7].

On the other hand, using a completely different approach that is rather a refinement of the double recursion coupled with the symmetrisation introduced by Sobol' for his construction, Kritzer [10] was able to improve the constant c_s from [13] by a multiplicative factor of $1/2$ for odd b and $b/(2(b+1))$ for even b. His approach was then extended in a joint paper with Faure [4] to provide the smallest value for c_s known so far, improving c_s from Theorem 2 by a factor of $b^2/(2(b^2-1))$ when b is even. Their approach still uses a (unique) recursion and the same symmetrisation idea, along with two clever counting lemmas. Their bounds are described in the following result.

Theorem 3. *Let $s \geq 2$ and let X be a (t,s)-sequence in base b. Then for any $N \geq \max\{b, b^t\}$, we have*

$$D^*(N,X) \leq b^t \sum_{v=0}^{s} A_{v,b}^{(s)} \left(\frac{\log N}{\log b}\right)^v, \tag{4}$$

$$\textit{with} \quad A_{0,b}^{(s)} = \frac{b+2}{2} a_{0,b}^{(s)} \quad , \quad A_{s,b}^{(s)} = \frac{b-1}{2s} a_{s-1,b}^{(s)} \quad \textit{and}$$

$$A_{v,b}^{(s)} = \left(2^v + \frac{b-1}{2}\right) a_{v,b}^{(s)} + \frac{b-1}{2v} a_{v-1,b}^{(s)} \quad \textit{for } 1 \leq v \leq s-1, \ \textit{in which}$$

$$\begin{aligned} a_{v,b}^{(s)} \ &= \ \binom{s-2}{v} \left(\frac{b+2}{2}\right)^{s-2-v} \frac{(b-1)^v}{2^v v!} (a_{0,b}^{(2)} + s^2 - 4) \\ &\quad + \binom{s-2}{v-1} \left(\frac{b+2}{2}\right)^{s-1-v} \frac{(b-1)^{v-1}}{2^{v-1} v!} a_{1,b}^{(2)}, \end{aligned}$$

$$\textit{where} \quad a_{0,b}^{(2)} = \begin{cases} \frac{b+8}{4} & \textit{if } b \textit{ is even,} \\ \frac{b+4}{2} & \textit{if } b \textit{ is odd,} \end{cases} \quad \textit{and} \quad a_{1,b}^{(2)} = \begin{cases} \frac{b^2}{4(b+1)} & \textit{if } b \textit{ is even,} \\ \frac{b-1}{4} & \textit{if } b \textit{ is odd.} \end{cases}$$

Next and going back to Atanassov's method, in [7] we were able to get a different discrepancy bound by making use of a careful worst-case analysis when studying signed splittings, together with a tighter counting argument. The constant c_s in this new bound is the same as in [6]—and therefore not as small as the one from [4]—but overall the bound appears to be better than both (3) and (4) in the non-asymptotic regime. This bound is recalled in the following theorem [7, Theorem 1].

Theorem 4. *For any (t,s)-sequence X in any base b and for any $N \geq 1$ we have (where $\gamma_b = 2 - b \pmod 2$))*

$$D^*(N,X) \leq \frac{b^t}{s!} \left(\frac{b-1}{2}\right)^s \prod_{k=1}^{s} \left(\frac{\log N}{\log b} + \gamma_b k\right) \qquad (5)$$
$$+ b^t \sum_{k=0}^{s-1} \frac{b}{k!} \left(\frac{b-1}{2}\right)^k \prod_{l=1}^{k} \left(\frac{\log N}{\log b} + \gamma_b l\right).$$

We now define a new family of low-discrepancy sequences recently introduced by Tezuka [17], namely the family of $(t, \mathbf{e}, s)$-sequences.

Definition 2. Given integers t, m with $0 \leq t \leq m$ and an s-tuple of positive integers $\mathbf{e} = (e_1, \ldots, e_s)$, a $(t, m, \mathbf{e}, s)$*-net in base* b is an s-dimensional point set with b^m points such that any elementary interval $E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$ with $0 \leq a_i \leq b^{d_i}$ and $e_i | d_i$ for $1 \leq i \leq s$, and $V(E) = b^{t-m}$, contains exactly b^t points of the set. (Notice that these conditions imply that $m - t$ is of the form $j_1 e_1 + \cdots + j_s e_s$). The definition of $(t, \mathbf{e}, s)$-sequences is the same as for (t, s)-sequences, with $(t, m, \mathbf{e}, s)$-nets in place of the usual (t, m, s)-nets.

From the definition of $(t,m,\mathbf{e},s)$-nets, it is easy to see that (t,s)-sequences are $(t,\mathbf{e},s)$-sequences with $\mathbf{e}=(1,\dots,1)$. A stronger version of Definition 2 has been further introduced by Hofer and Niederreiter in [9].

It is quite remarkable that Atanassov's technique developed for Halton sequences in [1] also applies to $(t,\mathbf{e},s)$-sequences: using an adaptation where powers b^{e_i} take place of coprimes p_i, Tezuka was able to get the following bound for the discrepancy of an arbitrary $(t,\mathbf{e},s)$-sequence X in base b. (see his Theorem 2 in [17] for the version with $N > b^t$ instead of $N \geq 1$):

Theorem 5. *Let $b \geq 2$ be an arbitrary integer. The star discrepancy of the first $N \geq 1$ points of a $(t,\mathbf{e},s)$-sequence X in base b satisfies*

$$D^*(N,X) \leq \frac{b^t}{s!}\prod_{i=1}^{s}\left(\frac{\lfloor b^{e_i}/2\rfloor}{e_i}\left(\frac{\log N}{\log b}\right)+s\right)+ b^t\sum_{k=0}^{s-1}\frac{b^{e_{k+1}}}{k!}\prod_{i=1}^{k}\left(\frac{\lfloor b^{e_i}/2\rfloor}{e_i}\left(\frac{\log N}{\log b}\right)+k\right). \quad (6)$$

For (t,s)-sequences, Theorem 5 and Theorem 1 give the same bound.

But in the same paper, Tezuka was also able to obtain new discrepancy bounds for generalized Niederreiter sequences (as defined in [16, Section 3]) by characterizing them as $(0,\mathbf{e},s)$-sequences, where e_i is the degree of the irreducible polynomial (so-called base polynomial) in base b used in the definition of the ith generating matrix of the sequence [17, Theorem 1], hence getting the following corollary to Theorem 5:

Corollary 1. *The discrepancy of a generalized Niederreiter sequence in prime power base b, with base polynomial degrees e_i, satisfies the bound (1) with*

$$c_s^{Tez} = \frac{1}{s!}\prod_{i=1}^{s}\frac{\lfloor b^{e_i}/2\rfloor}{e_i\log b}.$$

Next, applying our variant of Atanassov's method for (t,s)-sequences, we have been able to improve this result as follows [7, Theorem 2]:

Theorem 6. *Let $b \geq 2$ be an arbitrary integer. The star discrepancy of the* *first $N \geq 1$ points of a $(t, \mathbf{e}, s)$-sequence X in base b satisfies*

$$D^*(N,X) \leq \frac{b^t}{s!} \prod_{i=1}^{s} \left(\frac{b^{e_i}-1}{2e_i} \left(\frac{\log N}{\log b} + \sum_{i=1}^{s} e_i \right) + s \right) + b^t \sum_{k=0}^{s-1} \frac{b^{e_{k+1}}}{k!} \prod_{i=1}^{k} \left(\frac{b^{e_i}-1}{2e_i} \left(\frac{\log N}{\log b} + \sum_{i=1}^{k} e_i \right) + k \right). \quad (7)$$

Hence we obtain a new constant $c_s^{FL} = \frac{1}{s!} \prod_{i=1}^{s} \frac{b^{e_i}-1}{2e_i \log b}$ for generalized Niederreiter sequences such as those considered in Corollary 1. If b is odd the two constants are equal, but if b is even c_s^{FL} is lower than c_s^{Tez}. For example, we can see in [7, Table 1] that in base 2, when the integers e_i are the degrees of the irreducible polynomials over $\mathbb{F}_2$ sorted in non-decreasing order, c_s^{FL} is smaller by a factor of about 10, at least for small dimensions (less than 10).

3. Numerical comparisons of bounds

We first note that all the bounds for $D^*(N,X)$ have been divided by N in our calculations. In Table A.1 below, we compare the above bounds in the case where $b = 2$. For each pair (N, s), we give on the first line the value of the general bound (2), on the second line the value of the bound (3) specialized for even bases, on the third line the value of the bound (4), and on the fourth line, the value of the bound (5). In all cases, the term b^t in the bound is set equal to 1 since it appears as a multiplying constant in each bound. Note that the bound is only given when N is larger than the minimum number of points for which the bound holds. We see that starting in dimension $s = 3$, the bound (5) is always smaller than the other ones.

We performed the same kind of calculation for base 3, and the results are in Table A.2, where for each pair (N, s), we give on the first line the value of the bound (2), on the second line the value of the bound (4), and on the third line, the value of the bound (5). As in the case where the base is 2, here we also omit the term b^t as it appears in all bounds as a multiplying constant. Results for base 4 and base 5 are shown in Tables A.3 and A.4 below. The results for base 3 are similar to the ones for base 2, while for bases 4 and 5, we see that the bound (2) is often more competitive, except for base 5 and dimension 20, where we see it being outperformed by (5).

Next, we compare the bounds from (6) and (7) on generalized Niederreiter sequences in base 2 and 3. More precisely, we compare these two bounds on

$(0, \mathbf{e}, s)$-sequences for different values of s and N, where e_i is the degree of the ith irreducible polynomial over $\mathbb{F}_b$, for $b = 2, 3$. For comparison purposes, we also provide the value of the bound (5) using $t = \sum_{i=1}^{s}(e_i - 1)$. The results are in Tables A.5 and A.6. In base 2, for small dimensions ($s \leq 5$), the bound (5) gives the smallest values, and for $s > 5$ (6) is the best. The results are similar for base 3, except that (6) starts to be the best as soon as $s \geq 4$, and it also always give a smaller value than (7). This makes sense since the main advantage of (7) over (6) holds for even bases only.

Overall, we see through these results that a smaller asymptotic constant c_s does not necessarily lead to a smaller discrepancy bound, when considering moderate values of N.

4. Comparisons across bases

The previous section focused on comparing recent discrepancy bounds for a fixed base b. In this section, we want to compare different constructions through their discrepancy bound. More specifically, we compare the Sobol' sequence (base 2), with Faure sequences in base $b \geq s$ (we choose b as the smallest prime larger or equal to s), and also Halton sequences and generalized Niederreiter sequences in base 2, i.e., the $(0, \mathbf{e}, s)$-sequences in base 2 studied at the end of the previous section. We also indicate which of the above bounds was used to get the result in the table.

Finally, for comparison purposes we also include the bound obtained for Halton sequences proved by Atanassov in [1], which holds for $N \geq 1$ and is given by the following inequality, where $u \neq 0$ only if one b_i is even, in which case $u = \frac{b_i}{2(s-1)!} \prod_{1 \leq j \leq s, j \neq i} \left(\frac{(b_j - 1) \log N}{2 \log b_j} + s - 1 \right)$

$$D^*(N, H) \leq \frac{1}{s!} \prod_{i=1}^{s} \left(\frac{(b_i - 1) \log N}{2 \log b_i} + s \right) + \sum_{k=0}^{s-1} \frac{b_{k+1}}{k!} \prod_{i=1}^{k} \left(\left\lfloor \frac{b_i}{2} \right\rfloor \frac{\log N}{\log b_i} + k \right) + u.$$

As we can see in Table A.7, in dimension 5 and 10 and despite its successful use in many practical applications, the Sobol' sequence leads to the largest bound, while the Faure sequence provides the smallest bound. It is worth noting that even when $s = 10$, which is still relatively small, the bounds are already useless in some sense, since they are all above 1. We would need to make comparisons for values of N much larger than 10^7 in order to get reasonable values for the bound, but such values of N are much larger than what can be "afforded" by a typical computational budget. Nevertheless, these numerical results give some information on the behavior of the sequences for very large values of N and suggest which ones might be the best in that setting. For this reason, we chose to show the value of the computed bound rather than the trivial bound of 1.

5. Conclusion

We have reported on recent important improvements for the behavior of the star discrepancy of (t, s) and $(t, \mathbf{e}, s)$-sequences from a number theoretic point of view, with leading asymptotic constants c_s divided by factors as large as $(b/(b-1))^{s-2}$ in the case of an even base since 2006 [10]. These improvements have been initiated thanks to an adaptation of Atanassov's method for Halton sequences [8, 6] and led indirectly to the new concept of $(t, \mathbf{e}, s)$-sequences introduced in [17]. However, from the point of view of effective bounds in the non-asymptotic regime, it clearly appears through our numerical comparisons that these improvements are obtained at the expense of complementary terms in Equation (1) which take a prominent place in usual ranges of samples for applications. Hence, regarding practitioners, we are still confronted with the same problem: which sequences can be recommended in quasi-Monte Carlo methods with a good expectation of reliable results? At the moment, we still think that the success of Sobol' sequences, as well as generalized Halton and Faure sequences is in the selection of good direction numbers for the first ones and good scramblings by permutations for the last ones (see the conclusions [5, Section 8] and [11, Section 7] for more comments).

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References

[1] E. I. Atanassov, On the discrepancy of the Halton sequences, Math. Balkanica (N.S.) 18.1-2 (2004), 15–32.

[2] H. Faure, Discrépance de suites associées à un système de numération (en dimension s), Acta Arith. 61 (1982), 337–351.

[3] H. Faure, On Atanassov's methods for discrepancy bounds of low-discrepancy sequences, in: Uniform Distribution and Quasi-Monte Carlo Methods, P. Kritzer, H. Niederreiter, F. Pillichshammer, A. Winterhof (Eds.), Radon Series on Computational and Applied Mathematics 15, De Gruyter (2014), 105–126.

[4] H. Faure and P. Kritzer, New star discrepancy bounds for (t, m, s)-nets and (t, s)-sequences, Monatsh. Math. 172 (2013), 55–75.

[5] H. Faure and C. Lemieux, Improved Halton sequences and discrepancy bounds, Monte Carlo Methods Appl. 16 (2010), 231–250.

[6] H. Faure and C. Lemieux, Improvements on the star discrepancy of (t, s)-sequences, Acta Arith. 61 (2012), 61–78.

[7] H. Faure and C. Lemieux, A variant of Atanassov's method for (t, s)-sequences and $(t, \mathbf{e}, s)$-sequences, J. Complexity (2014), in press.

[8] H. Faure, C. Lemieux, X. Wang, Extensions of Atanassov's methods for Halton sequences, in: Monte Carlo and Quasi-Monte Carlo Methods 2010, L. Plaskota and H. Woźniakowski (Eds.), Springer (2012), 345–362.

[9] R. Hofer and H. Niederreiter, A construction of (t,s)-sequences with finite-row generating matrices using global function fields, Finite Fields Appl. 21 (2013), 97–110.

[10] P. Kritzer, Improved upper bounds on the star discrepancy of (t, m, s)-nets and (t, s)-sequences, J. Complexity 22 (2006), 336–347.

[11] C. Lemieux and H. Faure, New perspectives on $(0, s)$-sequences, in: Monte Carlo and Quasi-Monte Carlo Methods 2008, P. L'Ecuyer and A. Owen (Eds.), Springer (2009), 113–130.

[12] H. Niederreiter, Point sets and sequences with small discrepancy, Monatsh. Math. 104 (1987), 273–337.

[13] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1992.

[14] H. Niederreiter and C. P. Xing, Quasirandom points and global function fields, in: Finite Fields and Applications, S. Cohen and H. Niederreiter (Eds), London Math. Soc. Lectures Notes Series 233 (1996), 269–296.

[15] I. M. Sobol', The distribution of points in a cube and the approximate evaluation of integrals, U.S.S.R. Comput. Math. and Math. Phys. 7 (1967), 86–112.

[16] S. Tezuka, Polynomial arithmetic analogue of Halton sequences, ACM Trans. Model. Comp. Sim. 3 (1993), 99–107.

[17] S. Tezuka, On the discrepancy of generalized Niederreiter sequences, J. Complexity 29 (2013), 240–247.

Appendix A. Tables

Table A.1: Results for base $b = 2$; each pair (N, s) shows successively bounds (2) from Theorem 1, (3) from Theorem 2, (4) from Theorem 3, and (5) from Theorem 4.

$N\backslash s$	2	3	4	5	8	10	20
10^4	1.47e-2	9.84e-2	5.43e-1	2.62	1.79e2	2.32e3	2.30e8
	9.45e-3	9.43e-2	1.16	1.67e1	5.86e4	1.29e7	–
	4.74e-3	5.70e-2	5.72e-1	4.94	1.68e3	5.58e4	–
	5.03e-3	1.90e-2	5.78e-2	1.52e-1	1.53	5.18	3.46e2
10^5	2.10e-3	1.64e-2	1.04e-1	5.72e-1	5.43e1	8.39e2	1.50e8
	1.24e-3	1.45e-2	2.22e-1	4.02	2.79e4	9.62e6	–
	6.26e-4	8.10e-3	8.93e-2	8.53e-1	3.96e2	1.60e4	–
	6.86e-4	2.97e-3	1.03e-2	3.07e-2	4.27e-1	1.74	2.37e2
10^6	2.84e-4	2.53e-3	1.82e-2	1.12e-1	1.43e1	2.61e2	8.25e7
	1.55e-4	2.08e-3	3.78e-2	8.27e-1	1.00e4	4.96e6	–
	7.96e-5	1.10e-3	1.31e-2	1.36e-1	8.32e1	3.99e3	–
	8.95e-5	4.39e-4	1.71e-3	5.67e-3	1.06e-1	5.10e-1	1.35e2
10^7	3.69e-5	3.70e-4	2.97e-3	2.03e-2	3.39	7.20e1	3.96e7
	1.90e-5	2.83e-4	5.95e-3	1.53e-1	2.94e3	1.99e6	1.84e20
	9.85e-6	1.44e-4	1.84e-3	2.06e-2	1.60e1	9.01e2	2.61e10
	1.13e-5	6.19e-5	2.68e-4	9.76e-4	2.37e-2	1.34e-1	6.57e1

Table A.2: Results for base $b = 3$; each pair (N, s) shows successively the bounds (2) from Theorem 1, (4) from Theorem 3, and (5) from Theorem 4.

$N \backslash s$	2	3	4	5	8	10	20
10^4	8.51e-3	4.39e-2	1.91e-1	7.45e-1	3.02e1	2.97e2	1.26e7
	6.82e-3	9.56e-2	1.14	1.18e1	7.23e3	3.66e5	–
	7.99e-3	3.62e-2	1.30e-1	3.98e-1	5.88	2.46e1	3.63e3
10^5	1.15e-3	6.79e-3	3.33e-2	1.44e-1	7.47	8.34e1	5.11e6
	8.86e-4	1.35e-2	1.77e-1	2.03	1.70e3	1.05e5	–
	1.09e-3	5.74e-3	2.38e-2	8.32e-2	1.77	9.15	3.10e3
10^6	1.50e-4	9.92e-4	5.40e-3	2.57e-2	1.68	2.11e1	1.89e6
	1.11e-4	1.82e-3	2.60e-2	3.25e-1	3.56e2	2.62e4	3.71e12
	1.43e-4	8.54e-4	4.01e-3	1.57e-2	4.61e-1	2.89	2.10e3
10^7	1.89e-5	1.39e-4	8.29e-4	4.30e-3	3.48e-1	4.93	6.41e5
	1.36e-5	2.38e-4	3.64e-3	4.91e-2	6.84e1	5.88e3	1.60e12
	1.81e-5	1.21e-4	6.34e-4	2.77e-3	1.08e-1	8.08e-1	1.19e3

Table A.3: Results for base $b = 4$; each pair (N, s) shows successively the bounds (2) from Theorem 1, (3) from Theorem 2, (4) from Theorem 3, and (5) from Theorem 4.

$N \backslash s$	2	3	4	5	8	10	20
10^4	1.78e-2	1.25e-1	7.13e-1	3.53	2.54e2	3.36e3	3.46e8
	1.86e-2	1.95e-1	2.38	3.36e1	1.17e5	2.58e7	-
	8.40e-3	1.36e-1	1.88	2.27e1	8.12e4	4.57e6	-
	1.59e-2	1.12e-1	6.68e-1	3.54	3.47e2	5.98e3	2.92e9
10^5	2.48e-3	2.02e-2	1.33e-1	7.51e-1	7.52e1	1.19e3	2.22e8
	2.45e-3	3.04e-2	4.55e-1	8.10	5.59e4	1.92e7	-
	1.11e-3	1.93e-2	2.94e-1	3.91	5.20e3	4.47e5	1.79e14
	2.08e-3	1.66e-2	1.10e-1	6.39e-1	7.97e1	1.56e3	1.20e9
10^6	3.28e-4	3.06e-3	2.28e-2	1.44e-1	1.94e1	3.63e2	1.21e8
	3.11e-4	4.40e-3	7.78e-2	1.67	2.00e4	9.93e6	-
	1.41e-4	2.63e-3	4.32e-2	6.25e-1	1.09e3	1.11e5	9.09e13
	2.64e-4	2.33e-3	1.69e-2	1.08e-1	1.68e1	3.71e2	4.42e8
10^7	4.20e-5	4.39e-4	3.64e-3	2.55e-2	4.52	9.85e1	5.77e7
	3.83e-5	6.04e-4	1.23e-2	3.08e-1	5.88e3	3.98e6	-
	1.75e-5	3.45e-4	6.08e-3	9.47e-2	2.08e2	2.48e4	3.88e13
	3.25e-5	3.16e-4	2.50e-3	1.72e-2	3.29	8.15e1	1.47e8

Table A.4: Results for base $b = 5$; each pair (N, s) shows successively the bounds (2) from Theorem 1, (4) from Theorem 3, and (5) from Theorem 4

$N\backslash s$	2	3	4	5	8	10	20
10^4	1.58e-2	1.02e-1	5.40e-1	2.49	1.49e2	1.78e3	1.33e8
	1.23e-2	2.07e-1	3.10	4.13e1	5.80e4	5.33e6	-
	1.76e-2	1.20e-1	6.55e-1	3.09	1.83e2	2.06e3	6.74e7
10^5	2.14e-3	1.61e-2	9.75e-2	5.08e-1	4.13e1	5.80e2	7.28e7
	1.60e-3	2.94e-2	4.84e-1	7.10	1.34e4	1.50e6	2.53e15
	2.36e-3	1.84e-2	1.15e-1	6.14e-1	5.04e1	6.81e2	4.40e7
10^6	2.80e-4	2.38e-3	1.62e-2	9.43e-2	1.01e1	1.66e2	3.48e7
	2.01e-4	3.98e-3	7.10e-2	1.13	2.78e3	3.69e5	1.27e15
	3.04e-4	2.68e-3	1.88e-2	1.12e-1	1.22e1	1.95e2	2.38e7
10^7	3.53e-5	3.36e-4	2.54e-3	1.63e-2	2.26	4.26e1	1.48e7
	2.46e-5	5.21e-4	9.98e-3	1.71e-1	5.31e2	8.21e4	5.39e14
	3.80e-5	3.73e-4	2.90e-3	1.90e-2	2.70	5.02e1	1.11e7

Table A.5: Results comparing bounds for $(0, \mathbf{e}, s)$-sequences in base 2; each pair (N, s) shows successively the bounds (5) from Theorem 4, (6) from Theorem 5, and (7) from Theorem 6.

$N\backslash s$	2	3	4	5	8	10	20
10^4	5.03e-3	3.79e-2	4.63e-1	4.86	2.51e4	2.17e7	1.02e23
	1.43e-2	1.16e-1	1.03	6.51	2.53e3	1.93e5	7.04e13
	6.31e-3	5.46e-2	7.22e-1	8.81	9.28e4	1.66e8	2.26e26
10^5	6.86e-4	5.94e-3	8.27e-2	9.82e-1	7.00e3	7.30e6	7.00e22
	1.98e-3	1.83e-2	1.83e-1	1.31	7.37e2	7.24e4	9.11e13
	8.15e-4	7.70e-3	1.09e-1	1.39	1.58e4	2.85e7	3.85e25
10^6	8.95e-5	8.77e-4	1.37e-2	1.81e-1	1.73e3	2.14e6	3.97e22
	2.62e-4	2.70e-3	3.00e-2	2.40e-1	1.88e2	2.30e4	8.63e13
	1.02e-4	1.05e-3	1.57e-2	2.12e-1	2.59e3	4.76e6	6.45e24
10^7	1.13e-5	1.24e-4	2.14e-3	3.12e-2	3.88e2	5.61e5	1.94e22
	3.63e-5	4.23e-4	5.32e-3	4.85e-2	5.57e1	8.85e3	1.20e14
	1.33e-5	1.50e-4	2.44e-3	3.50e-2	4.78e2	9.07e5	1.26e24

Table A.6: Results comparing bounds for $(0, \mathbf{e}, s)$-sequences in base 3; each pair (N, s) shows the bounds (5) from Theorem 4, (6) from Theorem 5, and (7) from Theorem 6.

$N \backslash s$	2	3	4	5	8	10	20
10^4	7.99e-3	3.62e-2	3.91e-1	3.58	1.29e4	4.36e6	1.64e21
	8.00e-3	4.02e-2	3.62e-1	2.32	1.29e3	6.67e4	4.56e13
	1.05e-2	7.06e-2	1.05	1.41e1	2.40e5	3.08e8	9.23e26
10^5	1.09e-3	5.74e-3	7.15e-2	7.49e-1	3.86e3	1.62e6	1.39e21
	1.08e-3	1.83e-2	6.29e-2	4.58e-1	3.73e2	2.52e4	6.45e13
	1.37e-3	1.02e-2	1.62e-1	2.29	4.18e4	5.43e7	1.61e26
10^6	1.43e-4	8.54e-4	1.20e-2	1.42e-1	1.01e3	5.13e5	9.47e20
	1.40e-4	8.98e-4	1.02e-2	8.28e-2	9.43e1	8.03e3	6.49e13
	1.73e-4	1.40e-3	2.38e-2	3.54e-1	6.99e3	9.27e6	2.75e25
10^7	1.81e-5	1.21e-4	1.90e-3	2.49e-2	2.36e2	1.43e5	5.37e20
	1.76e-5	1.25e-4	1.56e-3	1.40e-2	2.14e1	2.24e3	5.04e13
	2.13e-5	1.87e-4	3.38e-3	5.28e-2	1.13e3	1.54e6	4.64e24

Table A.7: Comparisons of discrepancy bounds across bases

$N \setminus s$		2	5	10
10^4	Sobol'	4.74e-3 (4)	4.86 (5)	4.34e7 (5)
	Faure	4.74e-3 (4)	2.49 (2)	5.72e4 (2)
	Halton	7.28e-3	2.96	6.52e4
	GNied	5.03e-3 (5)	4.86 (5)	1.93e5 (6)
10^5	Sobol'	6.26e-4 (4)	9.82e-1 (5)	1.46e7 (5)
	Faure	6.26e-4 (4)	5.08e-1 (2)	2.35e4 (2)
	Halton	1.05e-3	6.39e-1	2.66e4
	GNied	6.86e-4 (5)	9.82e-1 (5)	7.24e4 (6)
10^6	Sobol'	7.96e-5 (4)	1.81e-1 (5)	4.28e6 (5)
	Faure	7.96e-5 (4)	9.43e-2 (2)	8.04e3 (2)
	Halton	1.43e-4	1.25e-1	9.15e3
	GNied	8.95e-5 (5)	1.81e-1 (5)	2.30e4 (6)
10^7	Sobol'	9.85e-6 (4)	3.12e-2 (5)	1.12e6 (5)
	Faure	9.85e-6 (4)	1.63e-2 (2)	2.38e3 (2)
	Halton	1.87e-5	2.27e-2	2.76e3
	GNied	1.13e-5 (5)	3.12e-2 (5)	8.85e3 (6)