An adaptive premium policy with a Bayesian motivation in the classical risk model

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Abstract

In this paper, we consider an extension of the classical risk model in which the premium rate policy is adaptive to claims experience. We assume that the premium rate is reviewed each time the surplus reaches a new descending ladder height. A choice between a finite number m of rates is then made depending on the time elapsed between successive ladder heights. We derive explicit expressions for the probability of ruin in this model, assuming claim sizes have a mixed Erlang distribution. We then motivate further the idea behind this adaptive premium rate policy by using a mixed Poisson process for the claims arrival, and propose a method to fix the parameters of the policy in this setting. Finally, we discuss other applications of this method.

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1 Introduction

The main topic of this paper is to study models that consider an insurer's surplus process and its associated probability of ruin. More precisely, we are looking at ruin models of the form

$$U(t) = u + \int_0^t c(s)ds - \sum_{k=1}^{N(t)} X_k,$$

where $u \geq 0$ is the initial surplus, the stochastic process $\{N(t), t \geq 0\}$ denotes the number of claims up to time t and the rv's $\{X_k, k \geq 1\}$ represent the corresponding claim amounts. In the classical ruin model, $\{N(t), t \geq 0\}$ is assumed to be a Poisson process with arrival rate $\lambda > 0$, the claim amounts $\{X_k, k \geq 1\}$ are independent and identically distributed (iid) rv's with probability density function (pdf) $p(x)$, cumulative distribution function (cdf) $P(x) = 1 - \bar{P}(x)$, and mean μ, and the process $c(t)$ is fixed to a constant $c > 0$.

In this paper, we consider two directions for generalization: (1) non-constant and non-deterministic premium rates; (2) Bayesian setting for the parameters of the model. More precisely, we propose an adaptive premium policy that chooses among a certain number of possible rates based on the behavior

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of the surplus, as opposed to strictly looking at the current surplus *level*. Instead, we review the premium rate every time the surplus drops below its previously reached minimum, and then choose a rate based on the time elapsed since the last drop. By assuming a Bayesian setup for the arrival process, we can then link the selection of the new rate to an assessment of the most likely regions for the (unknown) parameter Λ representing the arrival rate of the claim process. Thus the premium income is in fact stochastic rather than deterministic, but it is convenient notationally to express it in the form $\int_0^t c(s)ds$ as there is no impact of this representation in the ensuing analysis.

In the past, there has been several studies in which the premium rate has been allowed to take different values. One such line of study is to work with a surplus model in which the company pays a certain level of dividends depending on the current surplus level. See for example [3, 6, 18] and the notes and references in [4, p.221]. As for Bayesian models in risk theory, the use of mixed Poisson processes to model the claim arrival process has been studied in detail in [12, 23, 25], among others. More recently, in [2] the authors have considered Bayesian models for both the claim arrival process and the claim amounts. The idea of connecting the Bayesian setting with some kind of adaptive policy for the premium charged has been considered in [1, 20, 21] using a Bühlmann-type credibility set-up, and is also discussed in [4, p.407]. In these previous papers, the premium policy adapts itself in such a way that the rate is not restricted to specific values. As a result, exact expressions for the probability of ruin cannot be derived and one must resort to numerical methods (as in [1, 21]) or asymptotic analyses (as in [4, 20]). Another closely related idea is explored in [19], where additional contributions to the surplus can be made at each control point, in addition to changing the premium rate. However, the ensuing analysis (of finite-time ruin probabilities) is done based on the assumption that the rate will remain fixed thereafter.

In this work, we propose to fix the premium rates ahead of time, as this has the clear advantage of yielding a closed-form expression for the probability of ruin. Note that in practice, a company might find it desirable from a marketing point of view to have premium rates that are known ahead of time, and thus we believe our chosen set-up may be of practical interest.

The rest of this paper is organized as follows. In Section 2, we introduce our model when the parameters for the inter-arrival times and claim amounts distribution are assumed to be known, and derive the probability of ruin in this case. In Section 3, we use a Bayesian framework to introduce and compare different strategies that can be used by the insurer to determine the premium rate. Two examples including numerical results are presented in Section 4. Concluding remarks and ideas for future work are provided in Section 5.

Before we proceed to Section 2, we wish to emphasize that the primary motivation for introducing our new premium rate policy is as a mean to manage uncertainty on the parameters of the model. This point of view will become clear in Section 3, when we introduce a Bayesian framework for our model and propose a method to choose the parameters of the policy. In order to perform this type of analysis, we first need to study the model when the parameters are fixed, i.e., we first need to condition on the value taken by these parameters. This is what is done in Section 2. It should be noted, however, that the results of Section 2 are also of interest on their own, in the following sense. Even if an insurer has reasons to believe that there is no uncertainty about the model parameters, it is still of interest to adopt a premium rate policy that can better weather the turbulence caused by a string of bad (unlucky) experience. In that case, the trigger that forces the premium rate to increase is perceived as an indication that bad claim experience has struck, rather than one suggesting that the parameters have been misestimated.

2 Model under fixed parameter setting

In this section, we assume that the arrival rate λ of the Poisson process $\{N(t), t \geq 0\}$ is fixed. We also assume that the claim sizes are iid mixed Erlang with parameter β and mixing weights $\{q_k\}_{k=1}^{\infty}$. That is, the pdf of a claim size rv has the form

$$p(x) = \sum_{k=1}^{\infty} q_k e_{\beta,k}(x), \qquad x \geq 0, \tag{1}$$

where $\{q_k\}_{k=1}^{\infty}$ is a discrete probability measure and $e_{\beta,k}(x) = \beta^k x^{k-1} e^{-\beta x}/(k-1)!$ is the Erlang pdf with mean $k\beta^{-1}$ and variance $k\beta^{-2}$ (with corresponding cdf denoted $E_{\beta,k}(x)$). Hence the mean is given by $\mu = \sum_{k=1}^{\infty} kq_k/\beta$. We fix m premium rates $c_1, \ldots, c_m$ where $c_1 > \ldots > c_m > 0$ and $m-1$ thresholds $x_1, \ldots, x_{m-1}$, where $0 < x_1 < \ldots < x_{m-1}$, whose use will be explained shortly. Let τ_i be the time of the ith (descending) ladder height of the surplus process. That is, for $i \geq 1$ we have recursively

$$\tau_i = \inf_{t \geq \tau_{i-1}} \{U(t) < U(\tau_{i-1})\},$$

beginning with $\tau_0 = 0$. Also, let T_i be the time elapsed between the $(i-1)$th and ith ladder heights, i.e., $T_i = \tau_i - \tau_{i-1}$. The $m-1$ thresholds $x_1, \ldots, x_{m-1}$ are then used to determine intervals of values for T_i associated with a given premium rate. That is, the premium rate review policy is such that if $T_i \in (x_{j-1}, x_j]$, then the premium rate is fixed to c_j, for $j = 1, \ldots, m-1$ and if $T_i > x_{m-1}$, then the premium rate is fixed to c_m. Hence, together the $2m-1$ parameters $(c_1, \ldots, c_m, x_1, \ldots, x_{m-1})$ characterize this *deficit-adaptive premium policy.*

Our goal is then to compute the probability of ruin $\psi_i(u)$ for $i = 1, \ldots, m$, defined as

$$\psi_i(u) := \psi(u, c_i, \lambda) = P(T < \infty | c(0) = c_i),$$

where T is the time of ruin for the surplus process $\{U(t), t \geq 0\}$. In what follows, two joint distributions will be particularly relevant to the analysis. The first one is the joint defective pdf of the time t and size y of the first drop given a premium rate c_i, denoted by $k_i(t, y)$, which is derived in, e.g., [17]. The behavior of these two variables influence the choice of premium rate in our model, which is why we need their joint distribution. For mixed Erlang claim sizes and for any $\lambda > 0$, we have that $k_i(t, y) = k_{i,1}(t, y) + k_{i,2}(t, y)$, where

$$k_{i,1}(t, y) = \lambda e^{-\lambda t} \sum_{k=1}^{\infty} q_k e_{\beta,k}(c_i t + y),$$
$$k_{i,2}(t, y) = \lambda e^{-\lambda t} \sum_{m=1}^{\infty} \frac{(\lambda t)^m}{m!} \int_0^{c_i t} \frac{x}{c_i t} \sum_{k=1}^{\infty} q_k^{*m} e_{\beta,k}(c_i t - x) \sum_{r=1}^{\infty} q_r e_{\beta,r}(x + y) dx.$$

Note that $\{q_k^{*m}\}_{k=1}^{\infty}$ are the mixing weights associated with the m-fold convolution of the pdf $p(x)$, i.e.,

$$p^{*m}(x) = \sum_{k=1}^{\infty} q_k^{*m} e_{\beta,k}(x),$$

where $q_k^{*1} = q_k$ for each $k \geq 1$, while for $m > 1$, we have

$$q_k^{*m} = \sum_{j=1}^{k-1} q_j^{*(m-1)} q_{k-j}$$

if $k \geq m$ and $q_k^{*m} = 0$ if $k < m$. See [9] for more details. As mentioned above, the expression for $k_i(t, y)$ holds regardless of the value of $\lambda > 0$, and is therefore valid for both positive and negative security loadings.

Next, we introduce $h_{i,j}(y)$, the defective pdf of the drop y leading to premium rate c_j when the premium rate effective just before the drop is c_i. In order to ease the analysis involving this pdf, we make use of a technique called *Erlangization* in risk theory [4, Chap. IX, Sect. 8],[5], and which was originally proposed for American option valuation in [8]. In our case, this amounts to the replacement of the threshold differences $g_j := x_j - x_{j-1}$ by independent Erlang rv's G_j with parameters $(n_j, (n_j/g_j))$, $j = 1, \ldots, m-1$, where $n_j \in \mathbb{N}$. Note that by increasing n_j, we can approximate the constant g_j by G_j with arbitrary precision. More precisely, we have that $\mathrm{E}(G_j) = g_j$ and $\mathrm{Var}(G_j) = g_j^2/n_j$ and thus $\lim_{n_j \to \infty} \mathrm{Var}(G_j) = 0$, which justifies how this model can be used to approximate a model in which the thresholds are fixed. Furthermore, we assume without loss of generality that for $j = 1, \ldots, m-1$, g_j can be written as $g_j = \gamma_j x$ for $\gamma_j \in \mathbb{N}$ and some $x > 0$. That is, we assume that the threshold differences g_j are all multiples of a common quantity x. Hence we can choose a positive integer n and set $n_j = \gamma_j n$ so that the scale parameter of G_j is $1/\nu$, where $\nu := n_j/g_j = n/x$ and n_j is a positive integer for all $j = 1, \ldots, m-1$.

We can now determine $h_{i,j}(y)$, assuming the thresholds x_j have been replaced by Erlang rv's as described above. Observe that in order for a ladder height to cause the premium rate to switch to c_j, the time since the last drop must be between D_{j-1} and D_j, where $D_j = G_1 + \ldots + G_j$ for $j = 1, \ldots, m-1$ and we define $D_0 = 0, D_m = \infty$. Note that D_j is Erlang with parameters $(\tilde{n}_j = n(\gamma_1 + \ldots + \gamma_j), n/x)$ for $j = 1, \ldots, m$, with the convention that $\tilde{n}_0 = 0$ and $\tilde{n}_m = \infty$. Hence, we have

$$\begin{aligned} h_{i,j}(y) &= \int_0^\infty k_i(t, y) \Pr\left(D_{j-1} < t \leq D_j\right) dt \\ &= \frac{1}{\nu} \sum_{k=\tilde{n}_{j-1}+1}^{\tilde{n}_j} \int_0^\infty k_i(t, y) e_{\nu,k}(t) dt. \end{aligned} \tag{2}$$

Before going further, we want to point out that the forthcoming analysis does not require that $c_i > \lambda\mu$ for all $i = 1, \ldots, m$. Hence, in what follows we consider both cases, i.e., $c_i > \lambda\mu$ or $c_i \leq \lambda\mu$. Obviously, if $c_i \leq \lambda\mu$ for all i, then ruin occurs with certainty regardless of the initial premium rate, and the model becomes uninteresting.

2.1 Renewal equation and Laplace transform

Suppose that $U(0) = u$ and $c(0) = c_i$. By first conditioning on the relevant characteristics of the first drop in surplus, we have

$$\psi_i(u) = \sum_{j=1}^m \int_0^u \psi_j(u - y) h_{i,j}(y) dy + v_i(u),$$

where

$$v_i(u) = \int_u^\infty h_i(y) dy$$

and $h_i(y) = \sum_{j=1}^m h_{i,j}(y)$ is the defective pdf of the drop y when the premium rate is fixed to c_i. We can then take the Laplace transform of $\psi_i(u)$ for $i = 1, \ldots, m$. In what follows, we use the notation for an arbitrary function $g(x)$,

$$\tilde{g}(z) = \int_0^\infty e^{-zx} g(x)dx.$$

Hence for $i = 1, \ldots, m$ we get

$$\tilde{\psi}_i(z) = \sum_{j=1}^{m} \tilde{h}_{i,j}(z)\tilde{\psi}_j(z) + \tilde{v}_i(z).$$

In order to solve for $\tilde{\psi}_i(z)$, $i = 1, \ldots, m$, it is convenient to rewrite the above system of equations in matrix form as

$$\tilde{\boldsymbol{\psi}}(z) = \tilde{\mathbf{H}}(z)\tilde{\boldsymbol{\psi}}(z) + \tilde{\mathbf{v}}(z),$$

where the matrix $\tilde{\mathbf{H}}(z)$ has element $\tilde{h}_{i,j}(z)$ on its ith row and jth column, and

$$\tilde{\boldsymbol{\psi}}(z) = (\tilde{\psi}_1(z), \ldots, \tilde{\psi}_m(z))^T$$
$$\tilde{\mathbf{v}}(z) = (\tilde{v}_1(z), \ldots, \tilde{v}_m(z))^T.$$

Solving for $\tilde{\boldsymbol{\psi}}(z)$ we get

$$\tilde{\boldsymbol{\psi}}(z) = [\mathbf{I} - \tilde{\mathbf{H}}(z)]^{-1}\tilde{\mathbf{v}}(z), \tag{3}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix.

The components of the vector $\tilde{\mathbf{v}}(z)$ are easy to obtain, as they do not involve the potential change of premium rates, and are thus identical to the corresponding quantities in the classical model with a fixed rate of c_i. More precisely, we have

$$\tilde{v}_i(z) = \int_0^\infty e^{-zu} \int_u^\infty h_i(y)dydu. \tag{4}$$

Now, for a general claim size pdf $p(x)$, the defective density $h_i(y)$ is given by [11]

$$h_i(y) = \frac{\lambda}{c_i} \int_0^\infty e^{-\rho_i x} p(x+y)dx,$$

where ρ_i is the largest real solution of Lundberg's fundamental equation

$$\lambda - c_i\rho = \lambda\tilde{p}(\rho). \tag{5}$$

When the security loading is positive (i.e., $c_i > \lambda\mu$), then $\rho_i = 0$ and $h_i(y) = \lambda\bar{P}(y)/c_i$. Otherwise, ρ_i is known to be the strictly positive root of (5), which is generally found numerically (or analytically, in simple cases such as the exponential distribution).

When the claim size distribution is mixed Erlang of the form (1), we have

$$\begin{aligned}
h_i(y) =& \frac{\lambda}{c_i} \int_0^\infty e^{-\rho_i x} \sum_{k=1}^{\infty} q_k e_{\beta,k}(x+y)dx \\
=& \frac{\lambda}{\beta c_i} \sum_{k=1}^{\infty} q_k \int_0^\infty e^{-\rho_i x} \sum_{s=1}^{k} e_{\beta,s}(y)e_{\beta,k-s+1}(x)dx \\
=& \frac{\lambda}{\beta c_i} \sum_{s=1}^{\infty} q_{s,\rho_i} e_{\beta,s}(y),
\end{aligned} \tag{6}$$

where

$$q_{s,\rho_i} = \sum_{k=s}^{\infty} q_k \left(\frac{\beta}{\beta+\rho_i}\right)^{k-s+1}.$$

Note that the second equality above is obtained by choosing $k = 0$ in the easily established algebraic identity

$$x^k e_{\beta,r}(x+y) = \frac{1}{\beta^{k+1}} \sum_{s=1}^{r} \frac{(s+k-1)!}{(s-1)!} e_{\beta,s+k}(x) e_{\beta,r-s+1}(y). \tag{7}$$

Substituting (6) into (4) yields

$$\begin{aligned}
\tilde{v}_i(z) =& \frac{\lambda}{\beta c_i} \int_0^{\infty} e^{-zu} \sum_{s=1}^{\infty} \int_u^{\infty} q_{s,\rho_i} e_{\beta,s}(y) dy du \\
=& \frac{\lambda}{\beta c_i} \int_0^{\infty} e^{-zu} \sum_{s=1}^{\infty} \frac{1}{\beta} \sum_{r=0}^{s-1} q_{s,\rho_i} e_{\beta,r+1}(u) du \\
=& \frac{\lambda}{\beta c_i} \sum_{s=1}^{\infty} \sum_{r=0}^{s-1} \frac{q_{s,\rho_i}}{\beta} \left(\frac{\beta}{\beta+z}\right)^{r+1} \\
=& \frac{\lambda}{\beta^2 c_i} \sum_{r=0}^{\infty} \bar{Q}_{r,\rho_i} \left(\frac{\beta}{\beta+z}\right)^{r+1},
\end{aligned}$$

where

$$\bar{Q}_{r,\rho_i} = \sum_{s=r+1}^{\infty} q_{s,\rho_i}. \tag{8}$$

Similarly, we find that

$$\tilde{h}_i(z) = \frac{\lambda}{\beta c_i} \sum_{r=1}^{\infty} q_{r,\rho_i} \left(\frac{\beta}{\beta+z}\right)^{r}. \tag{9}$$

Remark 1 *It is well known that in the classical model with a constant premium rate c_i, the Laplace transform of the probability of ruin $\psi_{cl}(u, c_i, \lambda)$ is given by*

$$\tilde{\psi}_{cl}(z, c_i, \lambda) = \frac{\tilde{v}_i(z)}{1 - \tilde{h}_i(z)}.$$

Hence we can rewrite (3) as

$$\tilde{\boldsymbol{\psi}}(z) = [\mathbf{D}(z)(\mathbf{I} - \tilde{\mathbf{H}}(z))]^{-1} \tilde{\boldsymbol{\psi}}_{cl}(z), \tag{10}$$

where $\mathbf{D}(z)$ is a diagonal matrix with its ith element given by $(1 - \tilde{h}_i(z))^{-1}$, and $\tilde{\boldsymbol{\psi}}_{cl}(z)$ is a vector whose ith component is $\tilde{\psi}_{cl}(z, c_i, \lambda)$, for $i = 1, \ldots, m$. Note that the sum of the elements on the ith row of $\mathbf{I} - \tilde{\mathbf{H}}(z)$ is precisely $(1 - \tilde{h}_i(z))$, and thus (10) amounts to writing $\tilde{\psi}_i(z)$ as a weighted average of the Laplace transforms of the classical probabilities of ruin, namely $\tilde{\psi}_{cl}(z, c_i, \lambda)$, $i = 1, \ldots, m$.

Remark 2 *From the Laplace transform (9) at $z = 0$, it follows that*

$$\psi_{cl,i}(0) := \psi_{cl}(0, c_i, \lambda) = \frac{\lambda}{\beta c_i} \sum_{r=1}^{\infty} q_{r,\rho_i}.$$

It is not difficult to show that $\psi_{cl,i}(0) = \min(\lambda\mu/c_i, 1)$. *Indeed, we have that*

$$\begin{aligned}\psi_{cl,i}(0) =& \frac{\lambda}{\beta c_i}\sum_{r=1}^{\infty} q_{r,\rho_i} = \frac{\lambda}{\beta c_i}\sum_{r=1}^{\infty}\sum_{s=r}^{\infty} q_s \left(\frac{\beta}{\beta+\rho_i}\right)^{s-r+1} \\ =& \frac{\lambda}{\beta c_i}\sum_{w=1}^{\infty}\left(\frac{\beta}{\beta+\rho_i}\right)^{w}\sum_{s=w}^{\infty} q_s \\ =& \frac{\lambda}{\beta c_i}\sum_{w=1}^{\infty}\left(\frac{\beta}{\beta+\rho_i}\right)^{w}\bar{Q}_{w-1},\end{aligned} \tag{11}$$

where $\bar{Q}_w := \bar{Q}_{w,0}$. *Now, if* $\lambda < c_i/\mu$, *then* $\rho_i = 0$ *and from (11) we get*

$$\psi_{cl,i}(0) = \frac{\lambda}{\beta c_i}\sum_{w=1}^{\infty}\bar{Q}_{w-1} = \frac{\lambda\mu}{c_i},$$

since $\mu = \sum_{w=1}^{\infty}\bar{Q}_{w-1}/\beta$. *Note that in this case, the corresponding density*

$$h_i(y) = \frac{\lambda}{\beta c_i}\sum_{s=1}^{\infty} e_{\beta,s}(y)\bar{Q}_{s-1}$$

is consistent with the result stated in [24, pp.192–193]. On the other hand, if the security loading is negative, i.e., $\lambda \geq c_i/\mu$, *then* $\rho_i > 0$ *and from Lundberg's fundamental equation we have*

$$\lambda - c_i\rho_i = \lambda\tilde{p}(\rho_i) \Leftrightarrow \frac{1-\tilde{p}(\rho_i)}{\mu\rho_i} = \frac{c_i}{\lambda\mu}. \tag{12}$$

Now, in the mixed Erlang case, we have [22, Sect. 4]

$$\frac{1-\tilde{p}(\rho_i)}{\mu\rho_i} = \sum_{k=1}^{\infty}\frac{\bar{Q}_{k-1}}{\mu\beta}\left(\frac{\beta}{\beta+\rho_i}\right)^{k}. \tag{13}$$

Therefore, combining (11), (12), and (13), when $\lambda \geq c_i/\mu$ *we get* $\psi_{cl,i}(0) = 1$. *Note that the relation* $\psi_{cl,i}(0) = \min(\lambda\mu/c_i, 1)$ *is consistent with the fact that* $h_i(y)$ *is a defective pdf when* $\lambda < c_i/\mu$, *and is a proper pdf otherwise.*

2.2 Identifying the probability of ruin from its Laplace transform

In order to get an explicit expression for the probability of ruin $\psi_i(u)$, we need to express $\tilde{\boldsymbol{\psi}}(z)$ in a form amenable to inversion. The first step toward reaching this goal is to determine $\tilde{h}_{i,j}(z)$, which is given in the following lemma. In the statement of this result, we use the notation $f_{NB}(\cdot; r, p)$ ($F_{NB}(\cdot; r, p)$) for the pmf (cdf) of a Negative Binomial rv with mean $r(1-p)/p$ and variance $r(1-p)/p^2$, with the convention that $F_{NB}(\cdot; 0, \cdot) = 1$ and $F_{NB}(\cdot; \infty, \cdot) = 0$.

Lemma 1 *We have that*

$$\tilde{h}_{i,j}(z) = \sum_{v=1}^{\infty}\zeta_{i,j,v}\left(\frac{\beta}{\beta+z}\right)^{v},$$

where

$$\zeta_{i,j,v}=\frac{\lambda}{\beta c_i}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}q_k^{*m}q_{v+s-1}\frac{s}{k+s}f_{NB}\left(m;k+s,\frac{\beta c_i}{\lambda+\beta c_i}\right)\eta_{i,j}(m+k+s-1),$$

and

$$\eta_{i,j}(s)=F_{NB}\left(s;\tilde{n}_{j-1},\frac{\nu}{\lambda+\beta c_i+\nu}\right)-F_{NB}\left(s;\tilde{n}_j,\frac{\nu}{\lambda+\beta c_i+\nu}\right).$$

Proof. As seen in (2), $h_{i,j}(y)$ depends on the joint density $k_i(t,y)$, which is decomposed into two parts as $k_i(t,y)=k_{i,1}(t,y)+k_{i,2}(t,y)$. Consequently, we also write $\tilde{h}_{i,j}(z)=\tilde{h}_{i,j,1}(z)+\tilde{h}_{i,j,2}(z)$, where

$$\tilde{h}_{i,j,r}(z)=\int_0^{\infty}e^{-zy}\left\{\frac{1}{\nu}\int_0^{\infty}k_{i,r}(t,y)\sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j}e_{\nu,p}(t)dt\right\}dy,\qquad \text{for } r=1,2.$$

Using (7) at $k=0$, one finds

$$\begin{aligned}
\tilde{h}_{i,j,1}(z)=&\frac{1}{\nu}\int_0^{\infty}\int_0^{\infty}e^{-zy}\lambda e^{-\lambda t}\sum_{k=1}^{\infty}q_k e_{\beta,k}(c_it+y)\sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j}e_{\nu,p}(t)dtdy\\
=&\frac{\lambda}{\beta\nu}\sum_{k=1}^{\infty}\sum_{l=1}^{k}q_k\left[\int_0^{\infty}e^{-zy}e_{\beta,k+1-l}(y)dy\right]\int_0^{\infty}e^{-\lambda t}e_{\beta,l}(c_it)\sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j}e_{\nu,p}(t)dt\\
=&\frac{\lambda}{\beta\nu}\sum_{k=1}^{\infty}\sum_{l=1}^{k}q_k\left(\frac{\beta}{\beta+z}\right)^{k+1-l}\sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j}\frac{\beta(\beta c_i)^{l-1}\nu^p}{(\lambda+\beta c_i+\nu)^{l+p-1}}\binom{l+p-2}{l-1}\\
=&\frac{\lambda}{\lambda+\beta c_i}\sum_{k=1}^{\infty}\sum_{l=1}^{k}q_k\left(\frac{\beta}{\beta+z}\right)^{k+1-l}\left(\frac{\beta c_i}{\lambda+\beta c_i}\right)^{l-1}\frac{\lambda+\beta c_i}{\nu}\sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j}\frac{(\lambda+\beta c_i)^{l-1}\nu^p}{(\lambda+\beta c_i+\nu)^{l+p-1}}\binom{l+p-2}{l-1}\\
=&\frac{\lambda}{\lambda+\beta c_i}\sum_{k=1}^{\infty}\sum_{l=1}^{k}q_k\left(\frac{\beta}{\beta+z}\right)^{k+1-l}\left(\frac{\beta c_i}{\lambda+\beta c_i}\right)^{l-1}\eta_{i,j}(l-1),
\end{aligned}$$

which holds because

$$\begin{aligned}
&\frac{\lambda+\beta c_i}{\nu}\sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j}f_{NB}\left(s;p,\frac{\nu}{\lambda+\beta c_i+\nu}\right)\\
=&\frac{\lambda+\beta c_i}{\nu}\left[\sum_{p=1}^{\tilde{n}_j}f_{NB}\left(s;p,\frac{\nu}{\lambda+\beta c_i+\nu}\right)-\sum_{p=1}^{\tilde{n}_{j-1}}f_{NB}\left(s;p,\frac{\nu}{\lambda+\beta c_i+\nu}\right)\right]\\
=&F_{NB}\left(s;\tilde{n}_{j-1},\frac{\nu}{\lambda+\beta c_i+\nu}\right)-F_{NB}\left(s;\tilde{n}_j,\frac{\nu}{\lambda+\beta c_i+\nu}\right)\\
=&\eta_{i,j}(s),
\end{aligned}$$

by definition of $\eta_{i,j}(s)$. In the previous display, we used a result stated in [15, Prob. 6.34(c)] to go from the second to the third line.

Changing the order of summation and using the change of variable $v = k - l + 1$, we get

$$\tilde{h}_{i,j,1}(z) = \sum_{v=1}^{\infty} \zeta_{i,j,v,1} \left(\frac{\beta}{\beta + z} \right)^{v},$$

where

$$\zeta_{i,j,v,1} = \frac{\lambda}{\beta c_i} \sum_{l=1}^{\infty} q_{v+l-1} \left(\frac{\beta c_i}{\lambda + \beta c_i} \right)^{l} \eta_{i,j}(l-1).$$

Moving to the second piece, we first need to compute the integral inside the definition of $k_{i,2}(t,y)$. Using (7) at $k = 1$, we get

$$\begin{aligned}
\sum_{r=1}^{\infty} q_r \int_0^{c_i t} \frac{x}{c_i t} e_{\beta,k}(c_i t - x) e_{\beta,r}(x+y) dx =& \frac{1}{c_i t \beta^2} \sum_{r=1}^{\infty} q_r \sum_{s=1}^{r} s e_{\beta,r+1-s}(y) \int_0^{c_i t} e_{\beta,s+1}(x) e_{\beta,k}(c_i t - x) dx \\
=& \frac{1}{c_i t \beta^2} \sum_{r=1}^{\infty} q_r \sum_{s=1}^{r} s e_{\beta,r+1-s}(y) e_{\beta,k+1+s}(c_i t) \\
=& \frac{1}{\beta} \sum_{r=1}^{\infty} q_r \sum_{s=1}^{r} \frac{s}{k+s} e_{\beta,r+1-s}(y) e_{\beta,k+s}(c_i t) \\
=& \frac{1}{\beta} \sum_{s=1}^{\infty} \sum_{v=1}^{\infty} q_{v+s-1} \frac{s}{k+s} e_{\beta,v}(y) e_{\beta,k+s}(c_i t).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\tilde{h}_{i,j,2}(z) =& \frac{\lambda}{\beta \nu} \int_0^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-zy} \sum_{m=1}^{\infty} \frac{(\lambda t)^m}{m!} \sum_{k=1}^{\infty} q_k^{*m} \sum_{s=1}^{\infty} \sum_{v=1}^{\infty} q_{v+s-1} \frac{s}{k+s} e_{\beta,v}(y) e_{\beta,k+s}(c_i t) \sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j} e_{\nu,p}(t) dt dy \\
=& \frac{\lambda}{\beta \nu} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} q_k^{*m} \lambda^m \sum_{s=1}^{\infty} \sum_{v=1}^{\infty} q_{v+s-1} \frac{s}{k+s} \left(\frac{\beta}{\beta+z} \right)^{v} \sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j} \int_0^{\infty} e^{-\lambda t} \frac{t^m}{m!} e_{\beta,k+s}(c_i t) e_{\nu,p}(t) dt \\
=& \frac{\lambda}{c_i \beta} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \sum_{v=1}^{\infty} q_k^{*m} q_{v+s-1} \frac{s}{k+s} \left(\frac{\beta}{\beta+z} \right)^{v} \frac{\lambda^m \beta^{k+s} c_i^{k+s}}{(\lambda + \beta c_i)^{m+k+s}} \binom{m+k+s-1}{m} \\
& \times \frac{\lambda + \beta c_i}{\nu} \sum_{p=\tilde{n}_{j-1}+1}^{\tilde{n}_j} \left(\frac{\nu}{\lambda + \beta c_i + \nu} \right)^{p} \left(\frac{\lambda + \beta c_i}{\lambda + \beta c_i + \nu} \right)^{m+k+s-1} \binom{m+k+s+p-2}{p-1} \\
=& \frac{\lambda}{c_i \beta} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \sum_{v=1}^{\infty} q_k^{*m} q_{v+s-1} \frac{s}{k+s} \left(\frac{\beta}{\beta+z} \right)^{v} f_{NB} \left(m; k+s, \frac{\beta c_i}{\lambda + \beta c_i} \right) \eta_{i,j}(m+k+s-1).
\end{aligned}$$

Thus

$$\tilde{h}_{i,j,2}(z) = \sum_{v=1}^{\infty} \zeta_{i,j,v,2} \left(\frac{\beta}{\beta + z} \right)^{v},$$

where

$$\zeta_{i,j,v,2} = \frac{\lambda}{\beta c_i} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} q_k^{*m} q_{v+s-1} \frac{s}{k+s} f_{NB} \left(m; k+s, \frac{\beta c_i}{\lambda + \beta c_i} \right) \eta_{i,j}(m+k+s-1).$$

It is easy to see that

$$\zeta_{i,j,v,1}+\zeta_{i,j,v,2}=\frac{\lambda}{\beta c_i}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}q_k^{*m}q_{v+s-1}\frac{s}{k+s}f_{NB}\left(m;k+s,\frac{\beta c_i}{\lambda+\beta c_i}\right)\eta_{i,j}(m+k+s-1)=\zeta_{i,j,v}.$$

That is, the sum of $\zeta_{i,j,v,1}$ and $\zeta_{i,j,v,2}$ is the same as $\zeta_{i,j,v,2}$ but with the summation over m and k starting at 0 instead of 1, and with the convention that $q_k^{*0}=\mathbf{1}_{k=0}$. The result easily follows. $\square$

Now that we have computed $\tilde{h}_{i,j}(z)$, we would like to replace the Erlang rv's G_j by the threshold values that they approximate. As pointed out before, this can be achieved by letting each of the parameters $n_j=\gamma_j n$ used in the definition of the rv's G_j become arbitrarily large. Conveniently, this parameter n_j appears in $\tilde{h}_{i,j}(z)$ only in its term $\eta_{i,j}(s)$. We recall the well-known result that when we take the limit as n goes to ∞ of a Negative Binomial with parameters (n,p), where p is of the form $p=1/(1+\phi/n)$, then it converges to a Poisson rv with mean ϕ (see, e.g., [14]). Hence, we have

$$\lim_{n\to\infty}\eta_{i,j}(s)=F_P(s,x(\lambda+\beta c_i)\Gamma_{j-1})-F_P(s,x(\lambda+\beta c_i)\Gamma_j), \tag{14}$$

where $F_P(\cdot,\lambda)$ is the cdf of a Poisson rv with mean λ, and $\Gamma_j=\sum_{l=1}^{j}\gamma_l$ (with the convention $\Gamma_0=0,\Gamma_m=\infty$, $F_P(s,0)=1$, and $F_P(s,\infty)=0$). Note that since $F_P(n,\lambda)$ is decreasing in λ, we have that $\lim_{n\to\infty}\eta_{i,j}(s)>0$ for all $i,j=1,\ldots,m$, which in turn implies that the $\zeta_{i,j,v}$ are all positive. From now on, we assume $\eta_{i,j}(s)$ is calculated by taking the limit $n\to\infty$, and is thus given by (14). Note also that $\sum_{j=1}^{m}\eta_{i,j}(s)=1$, which means we can interpret $\eta_{i,\cdot}(s)$ as a probability distribution over $\{1,\ldots,m\}$ for each $i=1,\ldots,m$.

Going back to our goal of finding $\tilde{\boldsymbol{\psi}}(z)$ and then inverting it, we will use the identity

$$(\mathbf{I}-\mathbf{A})^{-1}=\sum_{i=0}^{\infty}\mathbf{A}^i,$$

which holds when the spectral radius of $\mathbf{A}$ is less than one. In our case, the matrix $\mathbf{A}$ is given by $\mathbf{H}(z)$, and to get a better insight of its properties, we rewrite each entry $\tilde{h}_{ij}(z)$ of $\mathbf{H}(z)$ as

$$\tilde{h}_{i,j}(z)=C_{ij}\left(\frac{\beta}{\beta+z}\right),$$

where $C_{ij}(z)=\sum_{l=1}^{\infty}\zeta_{i,j,l}z^l$ is a defective pgf. More precisely, it is easy to see that $\zeta_{i,j,l}>0$, and we also have $\sum_{l=1}^{\infty}\zeta_{i,j,l}<1$. This is because $\sum_{j=1}^{m}\tilde{h}_{i,j}(z)=\tilde{h}_i(z)$ for all $i=1,\ldots,m$, and in turn $\tilde{h}_i(z)$ is of the form

$$\tilde{h}_i(z)=\sum_{r=1}^{\infty}\frac{\lambda q_{r,\rho_i}}{\beta c_i}\left(\frac{\beta}{\beta+z}\right)^r,$$

where, as discussed in Remark 2, $\sum_{r=1}^{\infty}\frac{\lambda}{\beta c_i}q_{r,\rho_i}=\psi_{cl,i}(0)=\min(\lambda\mu/c_i,1)$.

Putting all this together, we have that

$$\sum_{j=1}^{m}|\tilde{h}_{i,j}(z)|<\sum_{r=1}^{\infty}\frac{\lambda}{\beta c_i}q_{r,\rho_i}\leq 1$$

for any $z > 0$, and thus we conclude that the spectral radius of $\mathbf{H}(z)$ is less than 1 [13], which means we can write

$$\tilde{\boldsymbol{\psi}}(z) = \left(\sum_{l=0}^{\infty} \mathbf{H}^l(z)\right) \tilde{\mathbf{v}}(z)$$

for any $z > 0$. We then write the (i,j)th entry of $\mathbf{H}^l(z)$ as $C_{i,j}^{*l}\left(\frac{\beta}{\beta+z}\right)$, where

$$C_{i,j}^{*l}(z) = \sum_{k=0}^{\infty} \zeta_{i,j,k}^{*l} z^k$$

and the $\zeta_{i,j,k}^{*l}$'s for $l \geq 0$ are defined as

$$\begin{aligned}
\zeta_{i,j,k}^{*0} &= \mathbf{1}_{i=j,k=0}, \\
\zeta_{i,j,k}^{*1} &= \zeta_{i,j,k}, \qquad k \geq 1 \\
\zeta_{i,j,k}^{*l} &= \sum_{r=1}^{m} \sum_{v=1}^{k-1} \zeta_{i,r,v}^{*(l-1)} \zeta_{r,j,k-v}, \qquad l > 1, k \geq 1.
\end{aligned} \tag{15}$$

Note that $\zeta_{i,j,0}^{*l} = 0$ for all $l \geq 1$. Hence,

$$\begin{aligned}
\tilde{\psi}_i(z) &= \sum_{l=0}^{\infty} \sum_{j=1}^{m} \sum_{k=0}^{\infty} \zeta_{i,j,k}^{*l} \left(\frac{\beta}{\beta+z}\right)^k \frac{\lambda}{\beta^2 c_j} \sum_{r=0}^{\infty} \bar{Q}_{r,\rho_i} \left(\frac{\beta}{\beta+z}\right)^{r+1} \\
&= \sum_{l=0}^{\infty} \sum_{j=1}^{m} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{\lambda}{\beta^2 c_j} \zeta_{i,j,k}^{*l} \bar{Q}_{r,\rho_i} \left(\frac{\beta}{\beta+z}\right)^{k+r+1} \\
&= \sum_{w=0}^{\infty} \left(\frac{\beta}{\beta+z}\right)^{w+1} \sum_{l=0}^{\infty} \sum_{j=1}^{m} \sum_{r=0}^{w} \frac{\lambda}{\beta^2 c_j} \bar{Q}_{r,\rho_i} \zeta_{i,j,w-r}^{*l}.
\end{aligned} \tag{16}$$

Equation (16) immediately leads to the main result of this paper, which is stated next.

Theorem 1 *For a deficit-adaptive premium policy characterized by parameters* $(c_1, \ldots, c_m, x_1, \ldots, x_{m-1})$, *the probability of ruin for an initial premium rate* c_i *is given by*

$$\psi_i(u) = \sum_{w=0}^{\infty} \kappa_{i,w} e_{\beta,w+1}(u), \qquad i = 1, \ldots, m,$$

where

$$\kappa_{i,w} = \sum_{l=0}^{\infty} \sum_{j=1}^{m} \sum_{r=0}^{w} \frac{\lambda}{\beta^2 c_j} \bar{Q}_{r,\rho_i} \zeta_{i,j,w-r}^{*l}, \qquad w \geq 0$$

with $\bar{Q}_{r,\rho_i}$ *and* $\zeta_{i,j,k}^{*l}$ *are given by (8) and (15), respectively.*

In the following example, we use Lemma 1 to derive an expression for the probability of ruin when claim sizes are exponentially distributed and the adaptive policy involves only two premium rates ($m = 2$).

Example 1 *In the exponential case, we have that $q_1 = 1$ and $q_r = 0$ for all $r > 1$ in the mixed Erlang representation. This implies that q_k^{*m} is 1 if $k = m$ and 0 otherwise. Therefore,*

$$\zeta_{i,j,v} = \frac{\lambda}{\beta c_i} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} q_k^{*m} q_{v+s-1} \frac{s}{k+s} f_{NB}\left(m; k+s, \frac{\beta c_i}{\lambda + \beta c_i}\right) \eta_{i,j}(m+k+s-1)$$

is non-zero only when $v = 1$, $s = 1$, and $k = m$ and is then given by

$$\zeta_{i,j,1} = \frac{\lambda}{\lambda + \beta c_i} \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} \left(\frac{\lambda \beta c_i}{(\lambda + \beta c_i)^2}\right)^m \eta_{i,j}(2m).$$

Hence,

$$\tilde{h}_{i,j}(z) = \zeta_{i,j,1} \frac{\beta}{\beta + z}, \qquad i, j = 1, 2.$$

Since our matrix $\tilde{\mathbf{H}}(z)$ is only 2×2 and each term is of the form $\beta\zeta_{i,j,1}/(\beta + z)$, it is fairly easy to directly compute $(\mathbf{I} - \tilde{\mathbf{H}}(z))^{-1}$ in (3). More precisely, we get

$$(\mathbf{I} - \tilde{\mathbf{H}}(z))^{-1} = \frac{\beta + z}{p_H(\beta + z)} \begin{bmatrix} \beta + z - \beta\zeta_{2,2,1} & \beta\zeta_{1,2,1} \\ \beta\zeta_{2,1,1} & \beta + z - \beta\zeta_{1,1,1} \end{bmatrix},$$

where $p_H(\beta + z) = (\beta + z - \beta\zeta_{1,1,1})(\beta + z - \beta\zeta_{2,2,1}) - \beta^2\zeta_{1,2,1}\zeta_{2,1,1}$ is the characteristic polynomial of $\mathbf{H}$, where $h_{i,j} = \beta\zeta_{i,j,1}, i, j = 1, 2$.

Next, we need to determine $\tilde{v}_i(z)$. Recall that

$$\tilde{v}_i(z) = \frac{\lambda}{\beta^2 c_i} \sum_{r=0}^{\infty} \bar{Q}_{r,\rho_i} \left(\frac{\beta}{\beta + z}\right)^{r+1},$$

where

$$\bar{Q}_{r,\rho_i} = \sum_{s=r+1}^{\infty} \sum_{k=s}^{\infty} q_k \left(\frac{\beta}{\beta + \rho_i}\right)^{k-s+1}.$$

In the exponential case, it is easy to see that when $c_i \geq \lambda/\beta$, then $\rho_i = \lambda/c_i - \beta$. Thus we have that

$$\bar{Q}_{0,\rho_i} = \frac{\beta}{\beta + \rho_i} = \begin{cases} 1 & \text{if } \lambda < \beta c_i \\ \frac{\beta c_i}{\lambda} & \text{if } \lambda \geq \beta c_i, \end{cases}$$

and $\bar{Q}_{l,\rho_i} = 0$ for all $l \geq 1$. Thus

$$\tilde{v}_i(z) = \frac{\min(\lambda/\beta c_i, 1)}{\beta + z}.$$

Hence, we get

$$\tilde{\psi}_2(z) = \frac{\beta\zeta_{2,1,1} \min\left(\frac{\lambda}{\beta c_1}, 1\right) + (\beta + z - \beta\zeta_{1,1,1}) \min\left(\frac{\lambda}{\beta c_2}, 1\right)}{(\beta + z - \phi_1)(\beta + z - \phi_2)},$$

where ϕ_1 and ϕ_2 are the roots of $p_H(\beta + z)$, which we assume are distinct. These roots also happen to be the eigenvalues of the matrix $\mathbf{H}$. (Note that the more general infinite series approach stated in Theorem 1—based on the powers of $\mathbf{H}(z)$—does not require us to identify eigenvalues and hence make assumptions as to their distinctness, as is done here. In fact, the approach used in this example can *only be used when the claim size distribution is a mixed Erlang of finite (ideally small) order. We use it here as it allows us to provide a more compact representation for the probability of ruin, as we will see shortly.)*

Using partial fractions, it follows that

$$\tilde{\psi}_2(z) = \frac{a}{\beta + z - \phi_1} + \frac{b}{\beta + z - \phi_2}$$

where

$$a = \frac{\beta\zeta_{2,1,1} \min\left(\frac{\lambda}{\beta c_1}, 1\right) + (\phi_1 - \beta\zeta_{1,1,1}) \min\left(\frac{\lambda}{\beta c_2}, 1\right)}{\phi_1 - \phi_2}$$

$$b = \frac{\beta\zeta_{2,1,1}\frac{\lambda}{\beta c_1} + (\phi_2 - \beta\zeta_{1,1,1})\frac{\lambda}{\beta c_2}}{\phi_2 - \phi_1} = \min\left(\frac{\lambda}{\beta c_2}, 1\right) - a.$$

Finally, one obtains

$$\psi_2(u) = ae^{-u(\beta-\phi_1)} + be^{-u(\beta-\phi_2)}. \tag{17}$$

Similar steps can be used to determine an expression for $\psi_1(u)$.

Remark 3 *Had we proceeded as in Theorem 1 to get an expression for $\psi_i(u)$, we would have obtained a different (equal) representation given by an infinite sum of Erlang terms with* common *scale parameter β. Here we get an expression (17) that is a linear combination of two exponential terms with* different *scale parameters. This comes from the fact that a mixed Erlang with different scale parameters can also be represented using a common scale parameter [26].*

3 Bayesian framework and strategies for the insurer

The goal of this section is to show that by using a Bayesian framework together with the adaptive premium policy described in the previous section, the insurer is able to achieve a sounder risk management approach than by fixing the premium rate to a certain constant. Before we describe this idea in detail, we first review two possible ways of fixing the premium rate.

Strategy 1: The Classical Approach

In the classical setting, the arrival rate of the Poisson process is assumed to be fixed and known. The insurer determines the premium rate so that a certain probability of ruin is achieved. More precisely, for $\lambda = \lambda_0$, the premium rate c_0 is defined as

$$c_0 = \inf\{c \geq 0 : \psi_{cl}(u, c, \lambda_0) \leq \psi_0\}.$$

for a given ruin probability threshold $\psi_0 < 1$. For instance, when the claims are exponentially distributed with mean β^{-1}, c_0 is the unique solution to

$$\frac{\lambda}{c_0\beta} e^{-u(\beta-\lambda/c_0)} = \psi_0.$$

Strategy 2: Fixed premium rate and mixed Poisson process

In practice, the arrival rate λ of the claim arrival Poisson process is usually not known with certainty. Hence fixing c to achieve a certain threshold ψ_0 for a fixed λ_0 can be highly misleading. As $\lambda \geq \lambda_0$ increases, the probability of ruin increases to reach 1 when $\lambda = c_0/\mu$. In the second approach, we propose to still consider a constant premium rate policy, but acknowledge the uncertainty about λ when determining an appropriate premium rate.

More precisely, we now assume $\{N_t, t \geq 0\}$ is a mixed Poisson process, as in [12, 23, 25]. That is, given $\Lambda = \lambda$, $\{N_t, t \geq 0\}$ is an ordinary Poisson process with rate λ and Λ has a certain pdf that we denote by $dB(\lambda)$. As discussed in [23], the probability of ruin is given by

$$\psi_b(u,c) := \int_0^{c/\mu} \psi_{cl}(u,c,\lambda) dB(\lambda) + \left(1 - B\left(\frac{c}{\mu}\right)\right),$$

or equivalently

$$\psi_b(u,c) = \mathrm{E}_\Lambda(\psi_{cl}(u,c,\Lambda)).$$

This highlights the fact that in this setting, we think of the classical probability of ruin $\psi_{cl}(u,c,\Lambda)$ as a random variable rather than a fixed quantity.

For exponential claim sizes and when the mixing rv Λ is Erlang distributed, the probability of ruin $\psi_b(u,c)$ can be obtained explicitly, as illustrated in the following example.

Example 2 *For exponential claim sizes and $\Lambda \sim$ Erlang(k,a), we have [12, pp.224–225]*

$$\begin{aligned}\psi_b(u,c) =& \int_0^{\beta c} \frac{\lambda}{\beta c} e^{-u(\beta - \lambda/c)} e_{a,k}(\lambda) d\lambda + (1 - E_{a,k}(\beta c)) \\ =& k e^{-u\beta} \frac{a^k}{\beta c} (a - u/c)^{-(k+1)} E_{a-u/c,k+1}(\beta c) + (1 - E_{a,k}(\beta c)).\end{aligned}$$

With this model, we can then pursue a strategy similar to the one used in the classical setting. Namely, we find the value $c_{b,0}$ such that

$$c_{b,0} = \inf\{c \geq 0 : \psi_b(u,c) \leq \psi_0\}.$$

If we compare these two strategies, we have the following result.

Proposition 1 *With exponential claim sizes, if $\lambda_0 = \mathrm{E}(\Lambda)$ then $c_0 \leq c_{b,0}$ and $\psi_b(u,c_0) \geq \psi_0$.*

Proof. We first note that

$$\psi_{cl}(u,c,\lambda) = \begin{cases} \frac{\lambda}{c\beta} e^{-u(\beta - \lambda/c)} & \text{if } \lambda < c\beta \\ 1 & \text{otherwise} \end{cases}$$

is a convex function of λ. Hence by Jensen's inequality, we have that

$$\psi_{cl}(u,c,\lambda_0) = \psi_{cl}(u,c,\mathrm{E}(\Lambda)) \leq \mathrm{E}(\psi_{cl}(u,c,\Lambda)) = \psi_b(u,c).$$

Since $\psi_{cl}(u,c,\lambda)$ is decreasing with c, we must necessarily have $c_0 \leq c_{b,0}$. □

What this result means is that if, in Strategy 1, we make the assumption that λ is equal to its mean under the model used in Strategy 2, then we will be choosing a premium rate that is smaller than $c_{b,0}$. Thus if the reality is that we do not know what the true value of λ is but are fixing the premium rate as if $\lambda = \mathrm{E}(\Lambda)$, our actual probability of ruin is greater than ψ_0.

Strategy 3: Deficit-adaptive premium rate and mixed Poisson process

Continuing with the mixed Poisson process presented in Strategy 2, we now want to incorporate our deficit-adaptive premium policy, as described in Section 2 and in the special case where there are $m = 2$ different premium levels and the claim sizes are exponentially distributed. The Bayesian framework from which the above mixed Poisson process arises provides a nice motivation for this type of premium policy. For instance, one interpretation is to think that c_2 ($c_2 < c_1$) represents our initial "guess" as to what the rate should be, and that it is associated with the assumption that Λ is "not too large". For example, one may choose a threshold λ_0 and think of c_2 as an appropriate premium rate when $\Lambda \leq \lambda_0$. However, given that the true value of Λ may be larger than λ_0, we propose to observe the time-to-drop variable T_i: a relatively small T_i can be viewed as an indication that our belief about Λ might be wrong, and that we should switch to a higher rate c_1 that is more conservative in the context.

Now we need to formalize this into a strategy to choose the three parameters (c_1, c_2, x_1) that characterize our adaptive premium policy. In addition to a target probability of ruin ψ_0, we also require an error level ϵ_0 that gives the maximum allowed probability of having a "bad" risk, conditioned on having observed the case $T > x_1$ which leads to choose the lower premium rate. We then propose the following strategy:

1. Determine $c_2 = \inf\{c \geq 0 : \psi_{cl}(u, c, \lambda_0) \leq \psi_0\}$.
2. Determine $x_1 = \inf\{x \geq 0 : P(\Lambda > \lambda_0 | T_i > x) \leq \epsilon_0\}$ for a surplus process based on the premium rate c_2.
3. Determine $c_1 = \inf\{c \geq 0 : \mathrm{E}_\Lambda(\psi_2(u, \Lambda)) \leq \psi_0\}$, where $\psi_2(u, \lambda)$ is the probability of ruin for our deficit-adaptive premium policy based on the parameters (c, c_2, x_1) given an ordinary Poisson process with rate λ (and as established in Section 2).

For the second step, one needs to compute

$$P(\Lambda > \lambda_0 | T_i > x_1) = \frac{\int_{\lambda_0}^{\infty} \int_{x_1}^{\infty} g(t, \lambda) dt dB(\lambda)}{\int_0^{\infty} \int_{x_1}^{\infty} g(t, \lambda) dt dB(\lambda)},$$

where $g(t, \lambda)$ is the (defective) density of the time until the first drop below the surplus for an ordinary Poisson process with rate λ. As shown in [16] (see also [7, 10]), for exponential claim sizes and premium rate c, the defective density $g(t, \lambda)$ is given by

$$g(t, \lambda) = e^{-\beta(u+ct)} \lambda e^{-\lambda t} + \sum_{n=2}^{\infty} \frac{nu + ct}{n(n-1)} e_{\beta,n-1}(u + ct) \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}.$$

As for the third step, since λ appears in a non-straightforward way in the expression $\psi_2(u, \lambda)$, the expectation $\mathrm{E}_\Lambda(\psi_2(u, \Lambda))$ must be computed numerically, and is then fed into another numerical routine that determines the value of $c_1 = \inf\{c \geq 0 : \mathrm{E}_\Lambda(\psi_2(u, \Lambda)) \leq \psi_0\}$ (this is implemented in Matlab in the experiments of Section 4).

Comparison of Strategies 2 and 3

There is no obvious and easy way to compare Strategy 2 and Strategy 3. One possible approach would be to compute some measure of expected average premium paid (over time) for Strategy 3 and compare it with the rate c from Strategy 2. However, at first sight, computing this appears to be a difficult problem. An easier alternative is to derive an upper bound for

$$\mathrm{E} \int_0^T \frac{c(t)}{T} dt,$$

where T is the time to ruin and $c(0) = c_2$. In the above equation, the expectation is taken over the claim sizes, inter-arrival times, and also over the parameter Λ. A naive upper bound can be obtained using the following argument: if the surplus never drops below u, then we will always stay at the rate c_2. If it is not the case and we do drop at some point, then the premium rate is no larger than c_1. Hence using the fact that

$$P(\text{no drop below } u \text{ is ever observed}) = 1 - \psi_{cl}(0, c_2, \lambda),$$

we get

$$\mathrm{E} \int_0^T \frac{c(t)}{T} dt \leq \bar{c}, \qquad \text{where} \qquad \bar{c} = c_2 + (c_1 - c_2)\mathrm{E}_\Lambda \left(\psi_{cl}(0, c_2, \Lambda)\right).$$

4 Examples

We now illustrate the above strategies with two simple examples.

Example 3 *For this first example, we use the parameters $\beta = 1$, $u = 5$, $\lambda_0 = 2$, $\Lambda \sim$ Erlang$(2, 1)$, $\psi_0 = 0.05$, $\epsilon_0 = 0.05$.*

For Strategy 1, we find that $c = 3.78$, while for Strategy 2, we have $c = 6.23$. For Strategy 3, we find $x_1 = 1.9721$ and $c_1 = 7.06$ (note that by definition, c_2 is fixed to the value 3.78 used for Strategy 1).

To compare the three strategies, we plot the probability of ruin in each case as a function of λ, as shown in Figure 1. For Strategy 3, we plot $\psi_2(u)$, that is, we plot the probability of ruin when the initial premium rate is fixed to 3.78. The upper bound on the average premium paid is $\bar{c} = 5.40$, which is smaller than the rate $c = 6.23$ from Strategy 2. However, as we can see on the RHS of Figure 3, Strategy 3 provides a lower ruin probability than Strategy 2 as λ gets large. We also show the probability of ruin obtained with $c = \bar{c}$, which appears to be larger than the one corresponding to Strategy 3 for most values of λ.

It is worth pointing out that in this figure, when λ becomes larger than the threshold $\beta c_2 = 3.78$ at which the security loading corresponding to c_2 becomes negative, the function $\psi_2(u)$ starts increasing much more slowly, in a way that violates convexity. In fact, it is precisely this behaviour that allows $\psi_2(u)$ to become smaller than the probability of ruin corresponding to Strategy 2. Consequently it reaches 1 at a larger value of λ, corresponding to $\beta c_1 = 7.06$. The intuition is that when $\lambda > \beta c_2$, our premium rate policy can "detect" that the initial premium rate c_2 is too small, and will correctly choose to charge the higher rate of c_1 in that case.

Example 4 *In this second example, we use the parameters $\beta = 1$, $u = 7$, $\lambda_0 = 2$, $\Lambda \sim$ Erlang$(2, 1)$, $\psi_0 = 0.01$, $\epsilon_0 = 0.05$.*

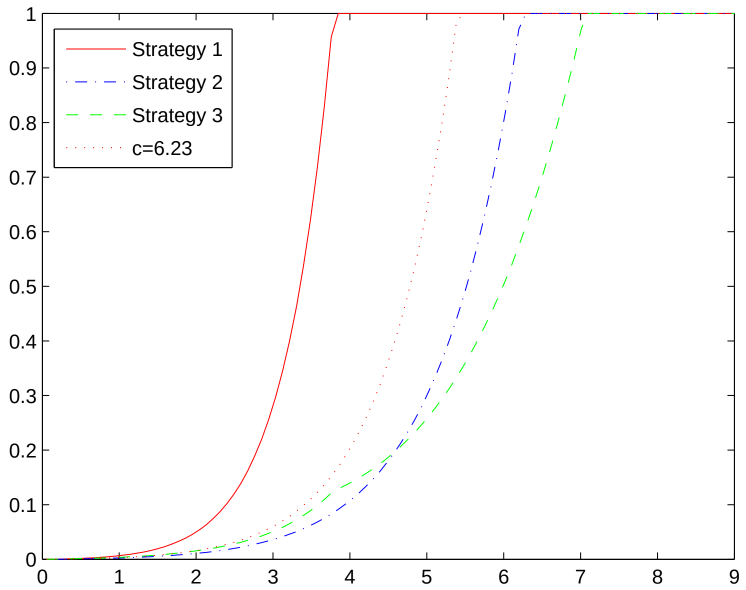

Figure 1: Probability of ruin as a function of λ for Example 3.

For Strategy 1, we find that $c = 4.40$, while for Strategy 2, we have $c = 8.39$. For Strategy 3, we find $x_1 = 1.6301$ and $c_1 = 9.40$, and the upper bound on the average premium paid is $\bar{c} = 6.58$, which is significantly smaller than the rate $c = 8.39$ from Strategy 2. We also see on Figure 4 that the probability of ruin for Strategy 3 appears to be dominated by that from Strategy 2 as λ gets large. As in the previous example, the probability of ruin obtained with $c = \bar{c}$ appears to be larger than the one corresponding to Strategy 3 for most values of λ.

5 Conclusion

In this paper, we have proposed a deficit-adaptive premium policy that chooses between a finite number of premium rates in an adaptive way, using the time elapsed between two consecutive drops below the minimum surplus as a decision variable. We have provided exact probabilities of ruin in the case of a Poisson arrival process and mixed Erlang claim sizes. We have then proposed a strategy for choosing the parameters describing the adaptive premium rate policy and compared it with two other strategies based on fixed premium rates. For the two examples considered, the proposed strategy appears to be superior. We remark that the Bayesian framework discussed in Section 3 could easily be modified to include uncertainty about the parameters of the claim size distribution rather than the arrival rate λ of the claim arrival process. This is because the time elapsed between consecutive drops depends on those parameters as well.

For future work, we would like to extend our results to a more general claim arrival process and find a better bound on the expected average rate paid with our proposed strategy, in order to have a more precise way of evaluating the usefulness of our approach.

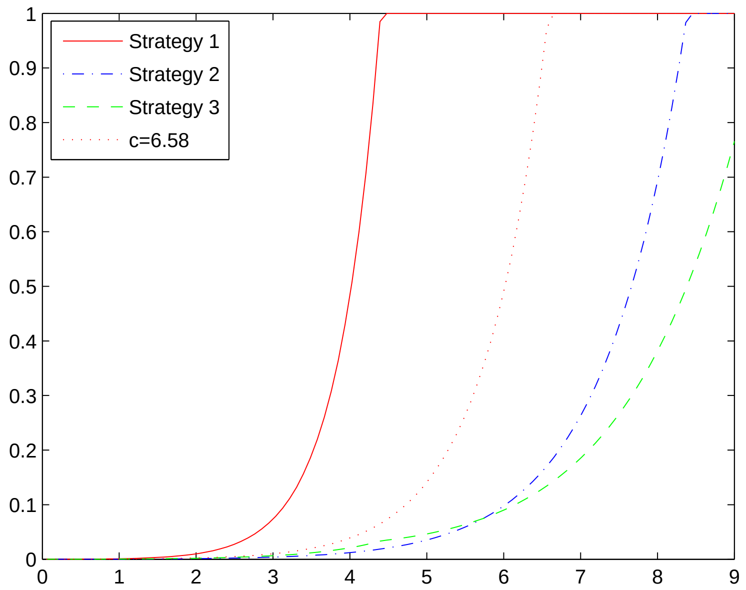

Figure 2: Probability of ruin as a function of λ for Example 4.

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