# A variant of Atanassov's method for $(t, s)$-sequences and $(t, \mathbf{e}, s)$-sequences

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**Abstract**

The term low-discrepancy sequences is widely used to refer to $s$-dimensional sequences $X$ for which the bound $D^*(N, X) \le c_s(\log N)^s + O((\log N)^{s-1})$ is satisfied, where $D^*$ denotes the usual star discrepancy. In this paper, we study such bounds for $(t, s)$-sequences and a newer class of low-discrepancy sequences called $(t, \mathbf{e}, s)$-sequences, introduced recently by Tezuka [16]. In the first case, by using a combinatorial argument coupled with a careful worst-case analysis, we are able to improve the discrepancy bounds from [5] for $(t, s)$-sequences. In the second case, an adaptation of the same pair of arguments allows us to improve the asymptotic behaviour of the bounds from [16] in the case of even bases.

*Keywords:* Discrepancy bounds, Atanassov's method, $(t, s)$-sequences.

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## 1. Introduction

Low-discrepancy sequences are designed to overcome the lack of uniformity that is inherent in random sampling. Their superiority over random sampling is often assessed by examining the behaviour of their so-called *star discrepancy* $D^*(N, X)$, where $X$ denotes a given sequence of points in $I^s := [0, 1)^s$, and $N$ is the number of points considered. The goal is then to show that for some carefully designed constructions $X$, the function $D^*(N, X)/N$ converges to 0 with $N$ faster than random sampling. However, exact calculations of this discrepancy measure are very difficult to perform.

For that reason, over the years the star discrepancy has instead been studied by providing bounds for it and studying their behaviour. More precisely, the idea is to establish bounds of the form

$$D^*(N, X) \le c_s(\log N)^s + O((\log N)^{s-1}), \tag{1}$$

where $c_s$ is a constant independent of $N$. A sequence $X$ for which such a bound can be established is usually referred to as a low-discrepancy sequence. The first constructions that were shown to satisfy these bounds were those proposed by Halton [7], Sobol' [14] and Faure [3]. The two latter belong to the general family of $(t, s)$-sequences in base $b$, which was introduced by Niederreiter in [10].

In the above mentioned work, formulas for the constant $c_s$ were also established. But over the last few years, there has been a number of improvements to these previously known bounds. The most relevant to our work are those provided by Atanassov [1], Kritzer [9], Faure and Lemieux [5], Faure and Kritzer [4], and Tezuka [16]. More precisely, Atanassov's result was very significant, as it provided an improvement by a factor of $s!$ for Halton sequences. Then, quite recently, Kritzer [9] improved the constants for $(t, s)$-sequences in the case $s \ge 2$ by a factor $1/2$ for an odd base and $b/(2(b+1))$ for an even base $b \ge 4$. In parallel, Faure and Lemieux [5] obtained an improvement by a factor of $((b-1)/b)^s$ in the case of even bases $b$, still in the case of $(t, s)$-sequences in base $b$, but using the approach proposed by Atanassov in [1] for Halton sequences. Shortly after that, Faure and Kritzer improved this result by a constant $b^2/2(b^2-1)$ for even bases [4]. Finally, and still using Atanassov's method, Tezuka was able to provide bounds with an improved constant $c_s$ for a family of constructions that he calls $(t, \mathbf{e}, s)$-sequences [16]. Note that the proofs in [3], [9] and [10] use an argument introduced by Sobol' [14] that is based on the study of the $(t, m, s)$-nets and involves a double recursion on $m$ and $s$ while [4], still based on the study of the $(t, m, s)$-nets, needs only a recursion on $s$.

In this paper, we propose a variant to Atanassov's method for deriving discrepancy bounds. It consists in replacing one of the key arguments based on diophantine geometry by an exact counting argument, together with a careful worst-case analysis of signed digits used to represent a given volume over which the local discrepancy is measured. This variant allows us to get tighter bounds for $(t, s)$-sequences in base $b$, compared to the recent bounds from [5] and [4], especially in higher dimensions. In addition, and still using the same pair of arguments, we are able to provide a smaller constant $c_s$ than in [16] in the case of $(t, \mathbf{e}, s)$-sequences in base $b$.

This paper is organized as follows: in Section 2 we provide background definitions, review known discrepancy bounds, and provide some of the key lemmas used in [5]. The next two sections are devoted to the respective cases of $(t, s)$-sequences and $(t, \mathbf{e}, s)$-sequences in base $b$, and in each section we provide our new bounds and comment on the improvement they provide over other known bounds.

## 2. Basic definitions and some key lemmas

We start with a review of the notion of discrepancy, which will be used throughout the paper. Various types exist, but here we only consider the so-called *extreme discrepancy*, which corresponds to the worst case error in the domain of complexity of multivariate problems. Assume we have a point set $\mathcal{P}_N = \{X_1, \ldots, X_N\} \subseteq I^s := [0,1]^s$ and denote by $\mathcal{J}$ (resp $\mathcal{J}^*$) the set of intervals $J$ of $I^s$ of the form $J = \prod_{j=1}^{s} [y_j, z_j)$, where $0 \leq y_j < z_j \leq 1$ (resp. $J = \prod_{j=1}^{s} [0, z_j)$). Then the *discrepancy function* of $\mathcal{P}_N$ on such an interval $J$ is the difference

$$E(J; N) = A(J; \mathcal{P}_N) - NV(J),$$

where $A(J; \mathcal{P}_N) = \#\{n; 1 \leq n \leq N, X_n \in J\}$ is the number of points in $\mathcal{P}_N$ that fall in the subinterval $J$, and $V(J) = \prod_{j=1}^{s} (z_j - y_j)$ is the volume of $J$.

Then, the *star (extreme) discrepancy* $D^*$ and the *(extreme) discrepancy* $D$ of $\mathcal{P}_N$ are defined by

$$D^*(\mathcal{P}_N) = \sup_{J \in \mathcal{J}^*} |E(J; N)| \quad \text{and} \quad D(\mathcal{P}_N) = \sup_{J \in \mathcal{J}} |E(J; N)|.$$

It is well known that $D^*(\mathcal{P}_N) \leq D(\mathcal{P}_N) \leq 2^s D^*(\mathcal{P}_N)$. For an infinite sequence $X$, we denote by $D(N, X)$ and $D^*(N, X)$ the discrepancies of its first $N$ points. Note that several authors multiply by a $1/N$ factor when defining the above quantities.

Moving on to $(t, s)$-sequences, this concept was introduced by Niederreiter [10] to give a general framework for various constructions using generating matrices applied to van der Corput sequences, including Sobol' sequences [14], Faure sequences [3], and later a more general class of constructions referred to as Niederreiter-Xing sequences [12].

**Definition 1.** Given an integer $b \geq 2$, a $b$-adic *elementary interval* in base $b$ in $I^s$ is an interval of the form $\prod_{i=1}^{s}[a_i b^{-d_i}, (a_i+1)b^{-d_i})$ where $a_i, d_i$ are nonnegative integers with $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$.
Given integers $t, m$ with $0 \leq t \leq m$, a $(t,m,s)$*-net in base* $b$ is an $s$-dimensional set with $b^m$ points such that any elementary interval in base $b$ with volume $b^{t-m}$ contains exactly $b^t$ points of the set.
An $s$-dimensional sequence $(X_n)_{n\geq 1}$ in $I^s$ is a $(t,s)$*-sequence* if the subset $\{X_n : kb^m < n \leq (k+1)b^m\}$ is a $(t,m,s)$-net in base $b$ for all integers $k \geq 0$ and $m \geq t$.

In order to give sense to new important constructions, Tezuka [15] and then Niederreiter and Xing [11, 12] introduced a new definition using the so-called truncation operator that we now define.

*Truncation* : Let $x = \sum_{i=1}^{\infty} x_i b^{-i}$ be a $b$-adic expansion of $x \in [0,1]$, with the possibility that $x_i = b-1$ for all but finitely many $i$. For every integer $m \geq 1$, we define the $m$-truncation of $x$ by $[x]_{b,m} = \sum_{i=1}^{m} x_i b^{-i}$ (depending on $x$ via its expansion). In the case where $X \in I^s$, the notation $[X]_{b,m}$ means an $m$-truncation is applied to each coordinate of $X$.

**Definition 2.** An $s$-dimensional sequence $(X_n)_{n\geq 1}$, with prescribed $b$-adic expansions for each coordinate, is a $(t,s)$*-sequence (in the broad sense)* if the subset $\{[X_n]_{b,m}; kb^m < n \leq (k+1)b^m\}$ is a $(t,m,s)$-net in base $b$ for all integers $k \geq 0$ and $m \geq t$.

The former $(t,s)$-sequences are now called $(t,s)$-sequences *in the narrow sense* and the others just $(t,s)$-sequences (Niederreiter-Xing [12], Definition 2 and Remark 1); in this paper, we will sometimes use intentionally the expression *in the broad sense* to emphasize the difference.

Going back to the discrepancy bounds of the form (1) that were mentioned in the introduction, low-discrepancy sequences $X$ that satisfy such inequalities are often compared to each other by calculating the constant $c_s$ in (1), and studying its behaviour as a function of the dimension $s$. Here is a summary of the sequence of improvements for these constants $c_s$ that have been obtained in the literature:

$$c_s^{Ni} = \begin{cases} \frac{b^t}{s}\left(\frac{b-1}{2\log b}\right)^s & \text{for } s=2, s=3 \ \& \ b=2, s=4 \ \& \ b=2 \\ \frac{b^t}{s!}\frac{b-1}{2\lfloor \frac{b}{2} \rfloor}\left(\frac{\lfloor \frac{b}{2} \rfloor}{\log b}\right)^s & \text{in all other cases} \end{cases} \qquad \text{(from [10]);}$$

$$c_s^{Kr} = \begin{cases} \frac{b^t}{s!}\frac{1}{2}\left(\frac{b-1}{2\log b}\right)^s & \text{if } b \text{ is odd} \\ \frac{b^t}{s!}\frac{b-1}{2(b+1)}\left(\frac{b}{2\log b}\right)^s & \text{if } b \text{ is even} \end{cases} \qquad \text{(from [9]);}$$

$$c_s^{FL} = \frac{b^t}{s!}\left(\frac{b-1}{2\log b}\right)^s \text{ if } b \text{ is an even base} \qquad \text{(from [5]);}$$

$$c_s^{FK} = \frac{b^t}{s!}\frac{b^2}{2(b^2-1)}\left(\frac{b-1}{2\log b}\right)^s \text{ if } b \text{ is an even base} \qquad \text{(from [4]).}$$

Hence $c_s^{Kr}$ for an odd base and $c_s^{FK}$ for an even base are currently the best constants $c_s$ for general $(t, s)$-sequences.

In the next section, we provide a new bound for $(t, s)$-sequences with associated constant $c_s$ equal to $c_s^{FL}$, but the other terms in the bounds differ from what was obtained in [5]. In moderate to large dimensions and for reasonable values of $N$, our new bound is smaller than the ones from [5] and [4], as illustrated through numerical results in [6].

In what follows, $\mathcal{P}_N(X)$ denotes the set containing the first $N$ points of a sequence $X$ and until the end, we set $n := \lfloor \log N / \log b \rfloor$. Also, several results in this section apply to the *truncated version* of the $(t, s)$-sequence under consideration, a concept that we now define.

**Definition 3.** Let $X$ be a $(t, s)$-sequence in base $b$, with its $k$th term defined as $X_k = (X_k^{(1)}, \ldots, X_k^{(s)})$, for $k \geq 1$. Let

$$[\mathcal{P}_N(X)] = \{([X_k^{(1)}]_{b,n+1}, \ldots, [X_k^{(s)}]_{b,n+1}), 1 \leq k \leq N\}.$$

We refer to $[\mathcal{P}_N(X)]$ as the *first $N$ points of a truncated version of the sequence $X$.*

We now recall three lemmas from [5], which will be useful in the forthcoming two sections. The first result would be trivial without the truncation operator.

**Lemma 1.** *Let $X$ be a $(t, s)$-sequence in base $b$ and $J = \prod_{i=1}^{s}[b_i b^{-d_i}, c_i b^{-d_i})$ with integers $b_i$, $c_i$ satisfying $0 \leq b_i < c_i \leq b^{d_i}$. Then for $N \geq b^{d_1} \cdots b^{d_s}$, $A(J; [\mathcal{P}_N(X)])$ is a nondecreasing function of $N$.*

The next lemma directly follows from the definition of $(t, s)$-sequences, but it requires some adaptation due to the truncation operator.

**Lemma 2.** *Let $X$ be a $(t,s)$-sequence and let $J=\prod_{i=1}^{s}[b_ib^{-d_i},c_ib^{-d_i})$ where $b_i,c_i$ are integers satisfying $0\le b_i<c_i\le b^{d_i}$. Then, for integers $N\ge 1$ and $u\ge 1$ we have*

$$A(J;[\mathcal{P}_N(X)])=ub^t(c_1-b_1)\cdots(c_s-b_s)\quad \text{if } N=ub^tb^{d_1}\cdots b^{d_s},$$

$$|E(J;[\mathcal{P}_N(X)])|\le b^t\prod_{i=1}^{s}(c_i-b_i)\ \text{for all } N\ge 1\quad\text{and}$$

$$A(J;[\mathcal{P}_N(X)])\le b^t\prod_{i=1}^{s}(c_i-b_i)\quad \text{if } N<b^tb^{d_1}\cdots b^{d_s}.$$

**Definition 4.** Consider an interval $J\subseteq I^s$. We call a *signed splitting* of $J$ any collection of intervals $J_1,\ldots,J_n$ and respective signs $\epsilon_1,\ldots,\epsilon_n$ equal to $\pm1$, such that for any (finitely) additive function $\nu$ on the intervals in $I^s$, we have $\nu(J)=\sum_{i=1}^{n}\epsilon_i\nu(J_i)$.

The following lemma, slightly reformulated, is taken directly from [1, Lemma 3.5] (see also [2, Lemma 3.40]), and as mentioned above, was also used in [5].

**Lemma 3.** *Let $J=\prod_{i=1}^{s}[0,z^{(i)})$ be an $s$-dimensional interval and let $n_i\ge 0$ be given integers for $1\le i\le s$. Set $z_0^{(i)}=0$, $z_{n_i+1}^{(i)}=z^{(i)}$ and, if $n_i\ge 1$, let $z_j^{(i)}\in[0,1]$ be arbitrary given numbers for $1\le j\le n_i$. Then the collection of intervals $\prod_{i=1}^{s}[\min(z_{j_i}^{(i)},z_{j_i+1}^{(i)}),\max(z_{j_i}^{(i)},z_{j_i+1}^{(i)}))$, with signs $\epsilon(j_1,\ldots,j_s)=\prod_{i=1}^{s}sgn(z_{j_i+1}^{(i)}-z_{j_i}^{(i)})$, $0\le j_i\le n_i$, is a signed splitting of $J$.*

## 3. Discrepancy bound for $(t,s)$-sequences

In this section, we provide a sharper bound than the one in [5] for $(t,s)$-sequences (in the broad sense). We first prove two new lemmas that will replace [5, Lemma 3] in the new proof of [5, Theorems 2 and 3].

**Lemma 4.** *Let $k\ge 1$ and $n\ge 0$ be two integers. The number of nonnegative integer solutions of the inequality $0\le x_1+\cdots+x_k\le n$ is equal to $\binom{n+k}{k}$.*

*Proof.* For the sake of completeness, we give an elementary proof of this well known combinatorial property:

First we consider the equation $x_1+\cdots+x_k=n$ over positive integers and show that the number of solutions is $\binom{n-1}{k-1}$ (with the convention $\binom{c}{l}=0$

if $l > c$). Indeed, a solution $(x_1, \ldots, x_k)$ is characterized by the $(k-1)$ sums $s_i = s_{i-1} + x_i$ $(1 \le i \le k-1)$, $x_k$ being given by $n - s_{k-1}$, these sums belonging to the set $\{1, 2, \ldots, n-1\}$. Hence, the number of solutions is the number of combinations of order $(k-1)$ for a set of $(n-1)$ elements.

Then, we show that the same equation in nonnegative integers has $\binom{n+k-1}{k-1}$ solutions by using new unknowns $y_i = x_i + 1$, which amounts to the preceding equation with the right hand side equal to $n + k$.

Finally, summing up the number of solutions for each possible value $m = 0, \ldots, n$, we obtain the desired result thanks to a classical relation on binomial coefficients. □

**Lemma 5.** *Let $n \ge 0$, $k \ge 1$ be integers. For integers $j \ge 0$ and $1 \le i \le k$, let $c_j^{(i)} \ge 0$ be given numbers such that $c_{2h+1}^{(i)} \le c$ and $c_{2h}^{(i)} \le c'$ for any $h \ge 0$, where $c$ and $c'$ are some fixed nonnegative numbers. Then*

$$\sum_{(j_1,\ldots,j_k)\in S} \prod_{i=1}^{k} c_{j_i}^{(i)} \le \frac{1}{k!} \left(\frac{c+c'}{2}\right)^k \prod_{l=1}^{k} (n+2l), \tag{2}$$

*where $S = \{(j_1, \ldots, j_k)\,; j_i \ge 0$ for all $i$ and $0 \le j_1 + \cdots + j_k \le n\}$.*

*Proof.* Let $N(l, k, n)$ be the number of $k$-tuples $(j_1, \ldots, j_k)$ in $S$ that have exactly $l$ even terms and $k - l$ odd terms. To determine this number, we first choose which $j_i$ are odd and which ones are even (there are $\binom{k}{l}$ possible configurations). Then, for a given configuration, the even terms have the form $j_i = 2h_i$ and the odd ones have the form $j_i = 2h_i + 1$, with $0 \le h_i \le \lfloor (n - (k-l))/2 \rfloor$. Hence, the number of $k$-tuples $(j_1, \ldots, j_k) \in S$ of the form $j_i = 2h_i$ for $l$ terms and $j_i = 2h_i + 1$ for $(k - l)$ terms is equal to the number of $k$-tuples $(h_1, \ldots, h_k)$ satisfying $0 \le h_1 + \cdots + h_k \le \lfloor (n - (k-l))/2 \rfloor$. According to Lemma 4, this number is equal to $\binom{\lfloor \frac{n-k+l}{2} \rfloor + k}{k}$, so that

$$N(l, k, n) = \binom{k}{l} \binom{\lfloor \frac{n-k+l}{2} \rfloor + k}{k}.$$

Now, our assumption on the coefficients $c_{j_i}^{(i)}$ implies the LHS of (2) is bounded by

$$\sum_{l=0}^{k} N(l, k, n) c'^l c^{(k-l)} = \sum_{l=0}^{k} \binom{k}{l} c'^l c^{(k-l)} \binom{\lfloor \frac{n-k+l}{2} \rfloor + k}{k}, \tag{3}$$

where we use the convention that $0^0 = 1$ (when exactly one of $c$ or $c'$ is zero). This can in turn be bounded by $\sum_{l=0}^{k} \binom{k}{l} c'^l c^{(k-l)} \binom{\lfloor \frac{n}{2} \rfloor + k}{k}$ since $l \leq k$. Finally, thanks to the binomial formula, the left hand side of (2) is bounded by

$$\frac{1}{k!}\,(c+c')^k \prod_{l=1}^{k} \left(\frac{n}{2} + l\right) = \frac{1}{k!} \left(\frac{c+c'}{2}\right)^k \prod_{l=1}^{k} (n+2l).$$

□

We can now prove the main result of this section.

**Theorem 1.** *For any $(t,s)$-sequence $X$ (in the broad sense) in any base $b$ and for any $N \geq 1$ we have*

$$D^*(N,X) \leq \frac{b^t}{s!} \left(\frac{b-1}{2}\right)^s \prod_{k=1}^{s} \left(\frac{\log N}{\log b} + \gamma_b k\right) \qquad (4)$$
$$+ b^t \sum_{k=0}^{s-1} \frac{b}{k!} \left(\frac{b-1}{2}\right)^k \prod_{l=1}^{k} \left(\frac{\log N}{\log b} + \gamma_b l\right),$$

*where $\gamma_b = (2 - b \pmod 2)$.*

*Proof.* As in [3] and [1], we will use special numeration systems in base $b$ — using signed digits $a_j$ bounded by $\lfloor b/2 \rfloor$ — to expand reals in $[0,1)$. That is, we write $z \in [0,1)$ as

$$z = \sum_{j=0}^{\infty} a_j b^{-j} \begin{cases} \text{with } |a_j| \leq \frac{b-1}{2} \text{ if } b \text{ is odd} \\ \text{with } |a_j| \leq \frac{b}{2} \text{ and } |a_j| + |a_{j+1}| \leq b-1 \text{ if } b \text{ is even.} \end{cases} \qquad (5)$$

The existence and uniqueness of such expansions are obtained by induction, see [1, p. 21–22] or [17, p. 12–13] where more details are given. For later use, it is worth pointing out that the expansion starts at $b^0$ and as a result, it is easy to see that $a_0$ is either 0 or 1.

We now begin the proof: Let $(z^{(1)}, \ldots, z^{(s)}) \in [0,1)^s$ and consider $b$-adic expansions $z^{(i)} = \sum_{j=0}^{\infty} a_j^{(i)} b^{-j}$ according to the numeration systems (5) above. We are going to define a signed splitting associated to $J = \prod_{i=1}^{s} [0, z^{(i)})$ using these $b$-adic expansions.

Recall that $n = \lfloor \log N / \log b \rfloor$ and define $z_0^{(i)} = 0$ and $z_{n+2}^{(i)} = z^{(i)}$. Then, consider the numbers $z_k^{(i)} = \sum_{j=0}^{k-1} a_j^{(i)} b^{-j}$ for $k = 1, \ldots, n+1$. Applying

Lemma 3 with $n_i = n+1$ for all $1 \leq i \leq s$, we split up $J$ using the numbers $z_j^{(i)}$, with $0 \leq j \leq n+2$, and obtain the signed splitting

$$I(\boldsymbol{j}) := \prod_{i=1}^{s} [\min(z_{j_i}^{(i)}, z_{j_i+1}^{(i)}), \max(z_{j_i}^{(i)}, z_{j_i+1}^{(i)})), \qquad 0 \leq j_i \leq n+1, \tag{6}$$

with signs $\epsilon(\boldsymbol{j}) = \prod_{i=1}^{s} \operatorname{sgn}(z_{j_i+1}^{(i)} - z_{j_i}^{(i)})$, where $\boldsymbol{j} = (j_1, \dots, j_s)$.

Since $V$ and $A(\,.\,; [\mathcal{P}_N(X)])$ are both additive, $A(J; [\mathcal{P}_N(X)]) - NV(J) = E(J; [\mathcal{P}_N(X)])$ may be expanded as

$$E(J; [\mathcal{P}_N(X)]) = \sum_{j_1=0}^{n+1} \cdots \sum_{j_s=0}^{n+1} \epsilon(\boldsymbol{j}) E(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) =: \Sigma_1 + \Sigma_2, \tag{7}$$

where we rearrange the terms so that in $\Sigma_1$ we put the terms such that $b^{j_1} \cdots b^{j_s} \leq N$ (i.e. such that $\boldsymbol{j} \in S$, see Lemma 5 with $k = s$) and in $\Sigma_2$ the rest. Notice that in $\Sigma_1$, the $j_i's$ are small, so the corresponding $I(\boldsymbol{j})$ is bigger. Hence, $\Sigma_1$ deals with the coarser part of $J$ whereas $\Sigma_2$ deals with the finer part of $J$. Notice also that if $j_i = n+1$ for some $i$, then $b^{j_1} \cdots b^{j_s} > N$, so that $s$-tuples $\mathbf{j}$ for which some $j_i = n+1$ are not taken into account in $\Sigma_1$. Hence, according to the definition of intervals $I(\mathbf{j})$, Lemma 2 applies to all intervals $I(\mathbf{j})$ with $\mathbf{j} \in \Sigma_1$ and gives:

$$|\, E(I(\boldsymbol{j}); [\mathcal{P}_N(X)])| \leq b^t \prod_{i=1}^{s} |\, z_{j_i+1}^{(i)} - z_{j_i}^{(i)}|\, b^{j_i} = b^t \prod_{i=1}^{s} |\, a_{j_i}^{(i)}|. \tag{8}$$

To prove Theorem 1, we are going to bound $|\Sigma_1|$ by the first term in (4) and $|\Sigma_2|$ by the second term. First, we deal with $\Sigma_1$. Using (8), we need to prove

$$\sum_{(j_1,\dots,j_s)\in S} \prod_{i=1}^{s} |a_{j_i}^{(i)}| \leq \frac{1}{s!} \left(\frac{b-1}{2}\right)^s \prod_{k=1}^{s} \left(\frac{\log N}{\log b} + \gamma_b k\right). \tag{9}$$

• Let us first consider the simpler case where $b$ is odd. Since from expansion (5) we have $|a_{j_i}^{(i)}| \leq (b-1)/2$, we get $\prod_{i=1}^{s} |a_{j_i}^{(i)}| \leq \left(\frac{b-1}{2}\right)^s$. Hence

$$\sum_{(j_1,\dots,j_s)\in S} \prod_{i=1}^{s} |a_{j_i}^{(i)}| \leq \left(\frac{b-1}{2}\right)^s \sum_{(j_1,\dots,j_s)\in S} 1 = \left(\frac{b-1}{2}\right)^s \binom{n+s}{s}$$

thanks to Lemma 4 (in the case where $k = s$) and the result follows since $\binom{n+s}{s} = (n+s)\dots(n+1)/s!$.

• The case where $b$ is even needs a more careful analysis. Consider a term in the sum on the left-hand side of (9), $\prod_{i=1}^{s} |a_{j_i}^{(i)}|$:
First, for a single coordinate $i$, we have an $m_i$-tuple $(a_0^{(i)}, \dots, a_{m_i}^{(i)})$ with $m_i$ depending on the condition $0 \le j_1 + \dots + j_s \le n$ (for instance, if $j_l = 0$ for all $l \neq i$ then $m_i = n$). We argue that any configuration for the coefficients $a_l^{(i)}$ gives a sum in (9) lower than the sum obtained for the worst case configuration occurring when, for all coordinates $i$, $|a_l^{(i)}| = b/2$ for all even $l$ — in which case, from (5), $|a_l^{(i)}| < b/2$ for odd $l$. Indeed, let $r$ be the smallest even index such that $|a_r^{(i)}| \neq b/2$ and let $g > r$ be the smallest index such that $|a_g^{(i)}| = b/2$ (if $|a_l^{(i)}| < b/2$ for all indices, clearly we have a configuration leading to a sum lower than the worst case sum). Then, switching $a_g^{(i)}$ with $a_r^{(i)}$ we get a new configuration in which there is one more even index $l$ such that $|a_l^{(i)}| = b/2$. Repeating this procedure, we arrive at a conf iguration $(\frac{b}{2}, a_1^{(i)}, \frac{b}{2}, \dots, a_{2e-1}^{(i)}, \frac{b}{2}, a_{2e+1}^{(i)} \dots, a_{m_i}^{(i)})$ in which $|a_l^{(i)}| < b/2$ for all $l > 2e$, where $e$ is some integer. Hence any configuration still gives a bound lower than the one to be deduced from the announced worst case configuration.
Now, the same process applied to each coordinate leads to a worst-case configuration of the digits $a_{j_i}^{(i)}$ to which Lemma 5 can be applied directly, with $c_j^{(i)} = |a_l^{(i)}|$, $c = (b-2)/2$, $c' = b/2$ and $k = s$. Hence we obtain the desired formula (9).

The way we deal with $\Sigma_2$ is similar to what is done in the proof of Theorem 2 in [5]. However, recall that the signed splitting of $J$ used here is such that $z^{(i)} = z_{n+2}^{(i)}$ instead of having $z^{(i)} = z_{n+1}^{(i)}$ as in [5]. This allows us to use Lemma 2 to bound the discrepancy on intervals $I(\mathbf{j})$ as long as $j_i \le n$ for all $i$, a condition that is met when dealing with $\Sigma_1$. So for $\Sigma_2$, we split the sum over $\{\mathbf{j}; b^{j_1} \dots b^{j_s} > N\}$ into subsets $B_0, \dots, B_{s-1}$ where $B_0 = \{\mathbf{j}; b^{j_1} > N\}$ and $B_k = \{\mathbf{j}; b^{j_1} \dots b^{j_k} \le N, b^{j_1} \dots b^{j_{k+1}} > N\}$ for $k = 1, \dots, s-1$. We note that $B_0 \neq \emptyset$ since we must have $j_1 = n+1$. In order to evaluate its contribution to $\Sigma_2$, we proceed as follows. Let $r \ge 1$ be the largest integer such that $b^{r-1} \le N$, so that $r = n+1$. Hence, $\mathbf{j} \in B_0$ if and only if $j_1 = r$, $j_2, \dots, j_s$ being arbitrary in $[0, n+1]$. Recall that $J = \prod_{i=1}^{s}[0, z^{(i)})$, and set

$$J' = [0, z_r^{(1)}) \times \prod_{i=2}^{s}[0, z^{(i)}) \text{ and } K = [\min(z_r^{(1)}, z^{(1)}), \max(z_r^{(1)}, z^{(1)})) \times \prod_{i=2}^{s}[0, z^{(i)}).$$

If $z^{(1)} > z_r^{(1)}$, then we have $J = J' \cup K$, and otherwise we have $J' = J \cup K$ (disjoint unions), so that

$$\operatorname{sgn}(z^{(1)} - z_r^{(1)}) E(K; [\mathcal{P}_N(X)]) = E(J; [\mathcal{P}_N(X)]) - E(J'; [\mathcal{P}_N(X)]).$$

Therefore, we have $\pm E(K; [\mathcal{P}_N(X)]) = \sum_{\mathbf{j} \in B_0} \epsilon(\boldsymbol{j}) E(I(\boldsymbol{j}); [\mathcal{P}_N(X)])$ and so, $E(K; [\mathcal{P}_N(X)])$ is the contribution of $B_0$ to $\Sigma_2$. Now since $b^{r-1} z_r^{(1)} \in \mathbb{Z}$ and

$$|\, z^{(1)} - z_r^{(1)}| = |\sum_{j=r}^{\infty} a_j^{(k+1)} b^{-j}| \le \left\lfloor \frac{b}{2} \right\rfloor \frac{1}{b^r} \frac{b}{b-1} \le \frac{1}{b^{r-1}},$$

we get $[\min(z_r^{(1)}, z^{(1)}), \max(z_r^{(1)}, z^{(1)})) \subseteq [m_1 b^{-r}, m_2 b^{-r})$ for some non-negative integers $m_1, m_2$ satisfying $0 \le m_2 - m_1 \le b$. Hence, $K \subset [m_1 b^{-r}, m_2 b^{-r}) \times [0,1)^{s-2}$ and so, using Lemma 2 (observe that $N < b^r \le b^{t+r}$) we have $A(K; [\mathcal{P}_N(X)]) \le b^t (m_2 - m_1) \le b^{t+1}$. But we also have $NV(K) \le b^{t+r}(m_2 - m_1) b^{-r} \le b^{t+1}$ which in the end gives the bound $|E(K; [\mathcal{P}_N(X)])| \le b^{t+1}$ for the contribution of $B_0$.

We can then deal with the sets $B_k$ for $1 \le k \le s-1$ in a similar fashion to what we did for $B_0$ (see [5, p. 72] for complete details), so that we get

$$|\Sigma_2| \le b^t \sum_{k=0}^{s-1} b \sum_{\{(j_1,\ldots,j_k); b^{j_1} \ldots b^{j_k} \le N\}} \prod_{i=1}^{k} |a_{j_i}^{(i)}|. \tag{10}$$

Finally, we bound the $a_{j_i}^{(i)}$'s using Lemma 4 if $b$ is odd and Lemma 5 is $b$ is even. Hence, we get the bound

$$|\Sigma_2| \le b^t \sum_{k=0}^{s-1} \frac{b}{k!} \left( \frac{b-1}{2} \right)^k \prod_{l=1}^{k} \left( \frac{\log N}{\log b} + \gamma_b l \right), \tag{11}$$

which in turn gives the last term in the bound (4) in Theorem 1. □

**Remark 1.** As alluded to in the above proof, we wish to point out that there is an inaccuracy in [5] in how the signed splitting is defined. As is done here, it should use $[z_{n+1}^{(i)}, z^{(i)})$ as the last interval for coordinate $i$, rather than using $[z_n^{(i)}, z^{(i)})$, so that when dealing with $\Sigma_1$, Lemma 2 can be applied to intervals $I(\mathbf{j})$ such that one of the $j_i$'s is $n$ and the others are 0. This inaccuracy led to the conclusion in [5] that the set $B_0$ is empty, and consequently the sum corresponding to (10) and (11) starts at $k = 1$ in [5] instead of starting at 0. Hence the term $b \cdot b^t$ is wrongly omitted in the bound given in [5, Theorem 2]. Notice that the same correction must be applied to the last part of the bound (13) of Theorem 3 in [5].

**Remark 2.** *Comparing Lemmas 4–5 and [5, Lemma 3].* In both cases, the problem is to bound $\sum_{(j_1,\dots,j_k)\in S} \prod_{i=1}^{k} c_{j_i}^{(i)} = \sum_{0\le j_1+\cdots+j_k\le n} \prod_{i=1}^{k} c_{j_i}^{(i)}$, with $c_0^{(i)} \le 1$ and $0 \le c_{j_i}^{(i)} \le c$ (in the case where $c = c'$ in Lemma 5).

• Lemma 4 immediately gives the bound $c^k \binom{n+k}{k} = c^k (n+k)\dots(n+1)/k!$, without any condition on $c_0^{(i)}$.

• Lemma 3 in [5] consists in studying subsets of integers $j_i \neq 0$:
For $0 \le m \le k$, thanks to the analog of [1, Lemma 3.2], we have

$$\sum_{j_{i_1}+\cdots+j_{i_m}\le n} \prod_{l=1}^{m} c_{j_{i_l}}^{(i_l)} \le c^m \sum_{j_{i_1}+\cdots+j_{i_m}\le n} 1 \le c^m n^m/m!.$$

Then, summing over all possible subsets, we obtain the bound

$$\sum_{(j_1,\dots,j_k)\in S} \prod_{i=1}^{k} c_{j_i}^{(i)} \le \sum_{m=0}^{k} \binom{k}{m} \frac{(cn)^m}{m!} \le \sum_{m=0}^{k} \binom{k}{m} \frac{(cn)^m k^{k-m}}{k!} = \frac{1}{k!}(cn+k)^k.$$

In this proof, we took into account the fact that $c_{j_i}^{(i)} \le 1$ when $j_i = 0$ and we used the inequality $k! \le k^{k-m} m!$.

• The first assertion in the proof of Lemma 4 gives another way to get the result of [5, Lemma 3], instead of using the analog of [1, Lemma 3.2]:
First, adding the solutions of equations with second members from $m$ to $n$, we have

$$\sum_{m\le j_{i_1}+\cdots+j_{i_m}\le n} \prod_{l=1}^{m} c_{j_{i_l}}^{(i_l)} \le c^m \left( \binom{m-1}{m-1} + \cdots + \binom{n-1}{m-1} \right) = c^m \binom{n}{m}.$$

Then, summing over all possible subsets, we get

$$\sum_{(j_1,\dots,j_k)\in S} \prod_{i=1}^{k} c_{j_i}^{(i)} \le \sum_{m=0}^{k} c^m \binom{n}{m} \binom{k}{m} \le \sum_{m=0}^{k} \binom{k}{m} \frac{(cn)^m k^{k-m}}{k!} = \frac{1}{k!}(cn+k)^k.$$

Here, we bounded $\binom{n}{m}$ by $\frac{n^m}{m!}$ and $\frac{1}{m!}$ by $\frac{k^{k-m}}{k!}$.

• Finally, in the case where $c \geq 1$, the method of [5, Lemma 3] together with the first assertion in the proof of Lemma 4 also gives the bound directly obtained by Lemma 4. Instead of using bounds for $\binom{n}{m}$ and $\frac{n^m}{m!}$, we bound $c^m$ by $c^k$:

$$\sum_{m=0}^{k} c^m \binom{n}{m}\binom{k}{m} \leq c^k \sum_{m=0}^{k} \binom{n}{m}\binom{k}{m} = c^k \sum_{m=0}^{k} \binom{n}{m}\binom{k}{k-m} = c^k \binom{n+k}{k},$$

the last equality being the Vandermonde's formula.

• In conclusion, we see that in both cases the bounds follow from the same inequality

$$\sum_{(j_1,\ldots,j_k)\in S} \prod_{i=1}^{k} c_{j_i}^{(i)} \leq \sum_{m=0}^{k} c^m \binom{n}{m}\binom{k}{m},$$

in which we can choose to bound the binomial coefficient $\binom{n}{m}$ by $n^m k^{k-m}/k!$ or to bound $c^m$ by $c^k$ (when $c \geq 1$). In both cases, these estimations seem loose but each of them has its interest: for small bases, especially 2 and 3, $c^k\binom{n+k}{k}$ is better but for large bases, especially $b \geq s$ in large dimensions, $(cn+k)^k/k!$ is preferable. Of course, it is still possible to keep the bound $\sum_{m=0}^{k} c^m \binom{n}{m}\binom{k}{m}$ in the statement of Theorem 1, at the expense of simplicity of the formulation.

## 4. New bound for $(t, \mathbf{e}, s)$-sequences

The notion of $(t, \mathbf{e}, s)$-sequences was recently introduced by Tezuka [16]. It has already had important consequences for new constructions of low-discrepancy sequences [8, 13]. This new family of sequences is defined as follows.

**Definition 5.** Given integers $t, m$ with $0 \leq t \leq m$ and an $s$-tuple of positive integers $\mathbf{e} = (e_1, \ldots, e_s)$, a $(t, m, \mathbf{e}, s)$*-net in base* $b$ is an $s$-dimensional point set with $b^m$ points such that any elementary interval $J = \prod_{i=1}^{s}[a_i b^{-d_i}, (a_i + 1)b^{-d_i})$ with $0 \leq a_i < b^{d_i}$ and $e_i | d_i$ for $1 \leq i \leq s$, and $V(J) = b^{t-m}$, contains exactly $b^t$ points of the set. (Notice that these conditions imply that $m - t$ is of the form $j_1 e_1 + \cdots + j_s e_s$).

The definition of $(t, \mathbf{e}, s)$-sequences is the same as for $(t, s)$-sequences, with $(t, m, \mathbf{e}, s)$-nets in place of the usual $(t, m, s)$-nets. We also note that $(t, s)$-sequences are $(t, \mathbf{e}, s)$-sequences with $\mathbf{e} = (1, \ldots, 1)$.

Note that a stronger version of this definition has been further introduced by Hofer and Niederreiter in [8] with the condition $V(J) \geq b^{t-m}$ (and $b^m V(J)$ points in $J$ instead of $b^t$) in order to avoid problems with propagation rules and other reasons.

In [16], Tezuka was able to sharpen discrepancy bounds for generalized Niederreiter sequences (as defined in [15, Section 3] and where $b$ is a prime power) by characterizing them as $(0, \mathbf{e}, s)$-sequences, where $e_i$ is the degree of the irreducible polynomial over $\mathbb{F}_b$ used in the definition of the $i$th generating matrix of the sequence [16, Theorem 1]. This is a first important application of the new notion of $(t, \mathbf{e}, s)$-sequences (see [16, Corollary 1] and our Corollary 1).

Now, concerning the main result in [16], it is quite remarkable that Atanassov's techniques also apply to $(t, \mathbf{e}, s)$-sequences: using an adaptation of these techniques, Tezuka was able to get the following bound for the discrepancy of an arbitrary $(t, \mathbf{e}, s)$-sequence $X$ in base $b$ when $N > b^t$ (see his Theorem 2 in [16]):

$$D^*(N, X) \leq \frac{b^t}{s!} \prod_{i=1}^{s} \left( \frac{\lfloor b^{e_i}/2 \rfloor}{e_i} \left( \frac{\log N}{\log b} - t \right) + s \right) + \sum_{k=0}^{s-1} \frac{b^{t+e_{k+1}}}{k!} \prod_{i=1}^{k} \left( \frac{\lfloor b^{e_i}/2 \rfloor}{e_i} \left( \frac{\log N}{\log b} - t \right) + k \right).$$

Not surprinsingly, using our own adaptation of Atanassov's techniques presented in the lemmas of the previous section, we can derive a bound with better constant $c_s$ for even $b$, as stated in the following result:

**Theorem 2.** *Let $b \geq 2$ be an arbitrary integer. The star discrepancy of the first $N \geq 1$ points of a $(t, \mathbf{e}, s)$-sequence $X$ in base $b$ satisfies*

$$D^*(N, X) \leq \frac{b^t}{s!} \prod_{i=1}^{s} \left( \frac{b^{e_i} - 1}{2e_i} \left( \frac{\log N}{\log b} + \sum_{i=1}^{s} e_i \right) + s \right) + \sum_{k=0}^{s-1} \frac{b^{t+e_{k+1}}}{k!} \prod_{i=1}^{k} \left( \frac{b^{e_i} - 1}{2e_i} \left( \frac{\log N}{\log b} + \sum_{i=1}^{k} e_i \right) + k \right). \quad (12)$$

To prove Theorem 2, we introduce two new lemmas corresponding to Lemmas 4 and 5 in the previous section.

**Lemma 6.** *Let $k$, $n$, and $e_1, \ldots, e_k$ be positive integers. Then the number of positive integer solutions of the inequality $e_1x_1 + \cdots + e_kx_k \leq n$ is bounded by $\frac{1}{k!}\prod_{i=1}^{k} \frac{n}{e_i}$.*

The proof is the same as for the key lemma on diophantine geometry in the paper of Atanassov [1, Lemma 3.2].

**Lemma 7.** *Let $k$, $n$, and $e_1, \ldots, e_k$ be positive integers. For integers $j \geq 0$ and $1 \leq i \leq k$, let $c_j^{(i)} \geq 0$ be given numbers such that $c_0^{(i)} \leq 1$, $c_{2h+1}^{(i)} \leq f_i$ for any $h \geq 0$ and $c_{2h}^{(i)} \leq f_i'$ for any $h \geq 1$, where $f_i$ and $f_i'$ are some fixed nonnegative numbers. Then*

$$\sum_{(j_1,\ldots,j_k)\in S'} \prod_{i=1}^{k} c_{j_i}^{(i)} \leq \frac{1}{k!}\prod_{i=1}^{k}\left(\frac{f_i+f_i'}{e_i}\left\lfloor\frac{n+\sum_{i=1}^{k} e_i}{2}\right\rfloor + k\right), \tag{13}$$

*where $S' = \{(j_1, \ldots, j_k)\,; j_i \geq 0$ for all $i$ and $e_1j_1 + \cdots + e_kj_k \leq n\}$.*

*Proof.* In the same manner as in [1, Lemma 3.3], [5, Lemma 3] and [16, Lemma 2], we split up the sum on the left-hand side (LHS) of (13) along subsets $u$ of $\{1, \ldots, k\}$ with $j_i > 0$ if $i \in u$ and $j_i = 0$ if $i \notin u$, but we add a new splitting according to the parity of the $j_i$'s. To this end, we consider subsets $\mathcal{L}$ of $u$ with $i \in \mathcal{L}$ if $j_i$ is even and $i \in u\backslash\mathcal{L}$ if $j_i$ is odd:

$$\sum_{(j_1,\ldots,j_k)\in S'} \prod_{i=1}^{k} c_{j_i}^{(i)} = \sum_{u\subseteq\{1,\ldots,k\}} \sum_{l=0}^{|u|} \sum_{\mathcal{L}\subseteq u, |\mathcal{L}|=l} \sum_{\substack{j\in S_u' \\ j_i \text{even } \Leftrightarrow i\in\mathcal{L}}} \prod_{i=1}^{k} c_{j_i}^{(i)},$$

where $S_u' = \{(j_1, \ldots, j_k) \in S'\,; e_{i_1}j_{i_1} + \cdots + e_{i_{|u|}}j_{i_{|u|}} \leq n$ and $j_i = 0 \Leftrightarrow i \notin u\}$. According to the hypothesis on the coefficients $c_{j_i}^{(i)}$, we obtain

$$\begin{aligned}\sum_{(j_1,\ldots,j_k)\in S'} \prod_{i=1}^{k} c_{j_i}^{(i)} &\leq \sum_{u\subseteq\{1,\ldots,k\}} \sum_{l=0}^{|u|} \sum_{\mathcal{L}\subseteq u, |\mathcal{L}|=l} \sum_{\substack{j\in S_u' \\ j_i \text{even } \Leftrightarrow i\in\mathcal{L}}} \left[\prod_{i\in\mathcal{L}} f_i' \prod_{i\in u\backslash\mathcal{L}} f_i\right] \\ &\leq \sum_{u\subseteq\{1,\ldots,k\}} \sum_{l=0}^{|u|} \sum_{\mathcal{L}\subseteq u, |\mathcal{L}|=l} N(u, \mathcal{L}, k, n) \prod_{i\in\mathcal{L}} f_i' \prod_{i\in u\backslash\mathcal{L}} f_i,\end{aligned}$$

where $N(u,\mathcal{L},k,n)$ is the cardinality of the set $\{\boldsymbol{j}\in S'_u\,;\, j_i \text{ even } \Leftrightarrow i\in\mathcal{L}\}$.

In order to bound $N(u,\mathcal{L},k,n)$, we use a similar idea as in the proof of Lemma 5: for $i\in\mathcal{L}$ we write $j_i = 2h_i$ while for $i\in u\backslash\mathcal{L}$ we write $j_i = 2h_i - 1$. Since $j_i > 0$ if and only if $i\in u$, we also have $h_i > 0$ for $i \in u$. We then note that

$$\sum_{i\in\mathcal{L}} 2h_i e_i + \sum_{i\in u\backslash\mathcal{L}} (2h_i - 1)e_i \le n$$

if and only if

$$\sum_{i\in u} h_i e_i \le \left\lfloor \frac{n+\sum_{i\in u\backslash\mathcal{L}} e_i}{2} \right\rfloor.$$

Hence, applying Lemma 6 we get

$$N(u,\mathcal{L},k,n) \le \frac{1}{|u|!}\prod_{i\in u}\frac{\lfloor\frac{n+\sum_{i\in u} e_i}{2}\rfloor}{e_i}$$

for all $u\subseteq\{1,\dots,k\}$ and $\mathcal{L}\subseteq u$. In addition, we note that

$$\sum_{l=0}^{|u|}\sum_{\mathcal{L}\subseteq u, |\mathcal{L}|=l}\prod_{i\in\mathcal{L}} f_i \prod_{i\in u\backslash\mathcal{L}} f'_i = \prod_{i\in u}(f_i + f'_i).$$

Putting this all together, we get

$$\begin{aligned}
\text{LHS of (13)} &\le \sum_{u\subseteq\{1,\dots,k\}} \frac{1}{|u|!}\prod_{i\in u}\frac{\lfloor\frac{n+\sum_{i\in u} e_i}{2}\rfloor}{e_i}\prod_{i\in u}(f_i+f'_i)\\
&\le \sum_{u\subseteq\{1,\dots,k\}} \frac{k^{k-|u|}}{k!}\prod_{i\in u}\frac{\lfloor\frac{n+\sum_{i\in u} e_i}{2}\rfloor(f_i+f'_i)}{e_i}\\
&\le \frac{1}{k!}\prod_{i=1}^{k}\left(\frac{\lfloor\frac{n+\sum_{i=1}^{k} e_i}{2}\rfloor(f_i+f'_i)}{e_i} + k\right).
\end{aligned}$$

This gives the desired bound (13). □

Before we proceed to the proof of Theorem 2, notice that our bound in that theorem is valid for all $N\ge 1$, rather than for $N > b^t$ as in [16]. As a tradeoff, the term $\log N/\log b - t$ found in the bound from [16] is replaced by $\log N/\log b + \sum e_i$ in our case. This difference arises in part because we choose to split the bound in two sums $\Sigma_1, \Sigma_2$ according to whether or not $(\mathbf{e}, \boldsymbol{j}) := \sum_{i=1}^{s} e_i j_i$ is smaller than $n$ rather than $n-t$ (where $n = \lfloor \log N/\log b \rfloor$). Note that this difference has an impact on the value of the bound for finite $N$, but not on its asymptotic behavior.

We are now ready to prove Theorem 2.

*Proof.* The proof starts similarly as our proof of Theorem 1, except that expansions for coordinate $i$ are done in base $b^{e_i}$ rather than $b$. More precisely, we let $\boldsymbol{z} = (z_1, \ldots, z_s)$, where

$$z_i = \sum_{j=0}^{\infty} a_j^{(i)} b^{-e_i j} \begin{cases} \text{with } |a_j| \le \frac{b^{e_i}-1}{2} \text{ if } b \text{ is odd} \\ \text{with } |a_j| \le \frac{b^{e_i}}{2} \text{ and } |a_j| + |a_{j+1}| \le b^{e_i} - 1 \text{ if } b \text{ is even.} \end{cases} \tag{14}$$

Let $n_i = \lfloor n/e_i \rfloor + 1$ and define $z_0^{(i)} = 0$ and $z_{n_i+1}^{(i)} = z^{(i)}$. We then set $z_k^{(i)} = \sum_{j=0}^{k-1} a_j^{(i)} b^{-e_i j}$, apply Lemma 3 to get a signed splitting of $J = \prod_{i=1}^{s} [0, z^{(i)})$ as in the proof of Theorem 1, and write

$$E(J; [\mathcal{P}_N(X)]) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_s=0}^{n_s} \epsilon(\boldsymbol{j}) E(I(\boldsymbol{j}); [\mathcal{P}_N(X)]) = \Sigma_1 + \Sigma_2, \tag{15}$$

where $\Sigma_1$ is the sum over all $\boldsymbol{j}$ such that $(\mathbf{e}, \boldsymbol{j}) \le n$ and $\Sigma_2$ is the sum over the remaining terms. Using Lemma 1 from [16] (which is the analog of our Lemma 2, but where the interval $J$ is split according to the $b^{e_i}$ rather than just $b$), we get

$$|E(I(\boldsymbol{j}); [\mathcal{P}_N(X)])| \le b^t \prod_{i=1}^{s} |a_{j_i}^{(i)}|. \tag{16}$$

Based on this, we can write

$$|\Sigma_1| \le \sum_{\boldsymbol{j}:(\mathbf{e},\boldsymbol{j}) \le n} |E(I(\boldsymbol{j}); [\mathcal{P}_N(X)])| \le b^t \sum_{\boldsymbol{j}:(\mathbf{e},\boldsymbol{j}) \le n} \prod_{i=1}^{s} |a_{j_i}^{(i)}|$$

and then apply Lemma 7 to $\Sigma_1$ with $k = s$, $n = \lfloor \log N/\log b \rfloor$, $c_{j_i}^{(i)} = |a_{j_i}^{(i)}|$, $f_i = \lfloor b^{e_i}/2 \rfloor$ and $f_i' = \lfloor (b^{e_i} - 1)/2 \rfloor$. When $b$ is odd, the application of Lemma 7 is straightforward with $f_i = f_i'$. When $b$ is even, $f_i = b^{e_i}/2$ and $f_i' = (b^{e_i} - 2)/2$, and we repeat the argumentation developed in the proof of Theorem 1 to bound by the worst case configuration occurring in the corresponding situation of $(t, \mathbf{e}, s)$-sequences. This gives

$$|\Sigma_1| \leq \frac{1}{s!} \prod_{i=1}^{s} \left( \frac{\lfloor \frac{n+\sum_{i=1}^{s} e_i}{2} \rfloor (b^{e_i} - 1)}{e_i} + s \right),$$

which corresponds to the first term in the bound (12).

The second term corresponds to the sum $\Sigma_2$ for the remaining vectors $\boldsymbol{j}$, which is calculated similarly as in the proof of Theorem 1 (and of Theorem 2 from [5]), but here again with the base $b$ replaced by $b^{e_i}$ for coordinate $i$, as done also in [16]. The steps we need to outline in more detail are the same as those for the proof of Theorem 1, starting from (note the similarity with the second-to-last inequality in the proof of Theorem 1, with the main difference being that $b$ is replaced by $b^{e_{k+1}}$)

$$\begin{aligned} |\Sigma_2| &\leq \sum_{k=0}^{s-1} b^{t+e_{k+1}} \sum_{\boldsymbol{j}=(j_1,\ldots,j_k):(\mathbf{e},\boldsymbol{j})\leq n} \prod_{i=1}^{k} |a_{j_i}^{(i)}| \\ &\leq \sum_{k=0}^{s-1} \frac{b^{t+e_{k+1}}}{k!} \prod_{i=1}^{k} \left( \frac{b^{e_i}-1}{e_i} \left\lfloor \frac{n+\sum_{i=1}^{k} e_i}{2} \right\rfloor + k \right), \end{aligned}$$

where the second inequality is obtained by applying Lemma 7. The result follows easily. □

We note that Sobol' and Faure sequences, as well as the more general family of digital $(t, s)$-sequences introduced in [10], are all examples of *generalized Niederreiter sequences* [15], which are proven to be $(0, \mathbf{e}, s)$-sequences by Tezuka [16, Theorem 1]. This implies that as a corollary of Theorem 2, we have:

**Corollary 1.** *For a generalized Niederreiter sequence in base $b$, where $b$ is a prime power, we have that (1) holds with*

$$c_s = c_s^{FL} := \frac{1}{s!} \prod_{i=1}^{s} \frac{b^{e_i} - 1}{2e_i \log b}.$$

We can compare this with the corresponding constant

$$c_s^{Tez} = \frac{1}{s!} \prod_{i=1}^{s} \frac{\lfloor b^{e_i}/2 \rfloor}{e_i \log b}$$

from [16]. When $b$ is odd, the two constants are equal, but in an even base, we have $c_s^{FL} < c_s^{Tez}$. In Table 4, we provide a numerical comparison of the two constants when $b = 2$ for some values of $s$. In these calculations, the integers $e_i$ are the degrees of the irreducible polynomials over the finite field $\mathbb{F}_2$ in non-decreasing order.

Table 1: Comparison of the constants $c_s^{\mathrm{FL}}$ and $c_s^{\mathrm{Tez}}$ for generalized Niederreiter sequences in base 2

| | Values of $s$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $c_s^{\mathrm{Tez}}$ | 1.44 | 1.04 | 5.00e-1 | 2.41e-1 | 9.26e-2 | 4.45e-2 | 1.84e-2 | 6.62e-3 |
| $c_s^{\mathrm{FL}}$ | 7.21e-1 | 2.60e-1 | 9.38e-2 | 3.95e-2 | 1.33e-2 | 5.99e-3 | 2.32e-3 | 7.83e-4 |
| | 9 | 10 | 15 | 20 | 25 | 30 | 40 | 50 |
| $c_s^{\mathrm{Tez}}$ | 3.40e-3 | 1.57e-3 | 1.52e-5 | 2.20e-7 | 2.74e-9 | 6.40e-11 | 3.32e-15 | 2.18e-18 |
| $c_s^{\mathrm{FL}}$ | 3.89e-4 | 1.74e-4 | 1.46e-6 | 1.96e-8 | 4.58e-10 | 5.14e-12 | 2.46e-16 | 1.55e-19 |

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