Pretty Good State Transfer on Paths

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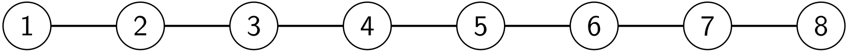
Continuous Random Walk



Definition

Let X be a graph with adjacency matrix A and degree matrix D. The matrix

$$M(t) := \exp(t(A - D)) = \sum_{n \geq 0} \frac{t^n}{n!}(A - D)^n$$

is such that the (a, b) entry is the probability that a "walker" starting on vertex a is at vertex b after time t.

Definition

A continuous random walk is modelled such that in a short time interval δt, the walker leaves the current vertex and moves to one of the adjacent vertices with equal probability.

Continuous Quantum Walk

Definition

Let X be a graph with adjacency matrix A. The transition matrix given by A is

$$U(t) := \exp(itA) = \sum_{n \geq 0} \frac{(it)^n}{n!} A^n,$$

and defines a continuous quantum walk.

Definition

The mixing matrix given by A is

$$M(t) := U(t) \circ \overline{U(t)}$$

and is such that the (a, b) entry is the probability that a quantum state starting at vertex a is at vertex b after time t.

Example: P_2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{aligned} U(t) &= \sum_{n\geq 0} \frac{(it)^n}{n!} A^n \\ &= \sum_{k\geq 0} \frac{(it)^{2k}}{(2k)!} I + \sum_{k\geq 0} \frac{(it)^{2k+1}}{(2k+1)!} A \\ &= \cos(t) I + i \sin(t) A \end{aligned}$$

$$M(t) = \cos^2(t) I + \sin^2(t) A$$

Perfect State Transfer (PST)

Definition

A graph X has perfect state transfer (PST) between vertices a and b if there exists $\tau \in \mathbb{R}$ such that $||U(\tau)_{a,b}|| = 1$ (i.e. $M(\tau)_{a,b} = 1$).

Motivation

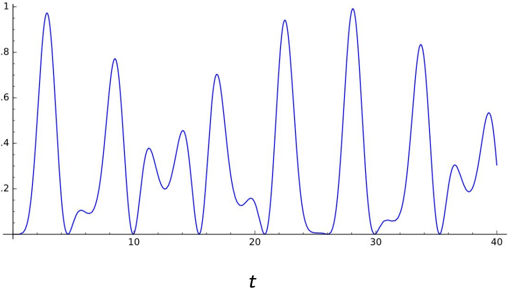
- Quantum algorithms typically requiring transferring quantum states.
- By the No-Cloning Theorem, it is impossible to copy states.
- Controlling interactions betweeen qubits is a challenge. Hence we want a quantum walk which transfers the information by design after a certain length of time.
- Paths resemble quantum wires.

PST on P_2

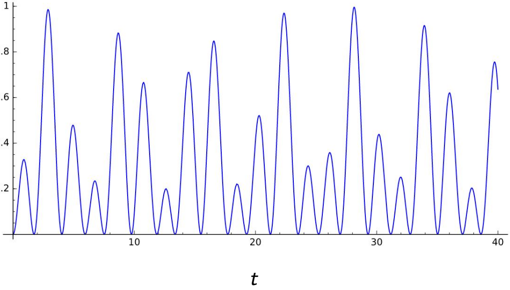
$$\begin{aligned} M(t) &= \cos^2(t)I + \sin^2(t)A \\ M(\pi/2) &= \cos^2(\pi/2)I + \sin^2(\pi/2)A = A \\ M(\pi/2)_{1,2} &= 1. \end{aligned}$$

PST on P_4?

$M(t)_{1,4}$



$M(t)_{2,3}$



PST on Paths

Theorem (Christandl et al. 2005)

P_n has PST between the end vertices if and only if $n = 2, 3$.

Theorem (Stevanović 2011; Godsil 2012)

P_n has PST if and only if $n = 2, 3$.

Pretty Good State Transfer (PGST)

Definition

A graph X has pretty good state transfer (PGST) between vertices a and b if, for every $\epsilon > 0$, there exists $\tau \in \mathbb{R}$ such that $||U(\tau)_{a,b}|| > 1 - \epsilon$.

Kronecker's Theorem

Let $\theta_1, \dots, \theta_n$ and $\sigma_1, \dots, \sigma_n$ be arbitrary real numbers. For an arbitrarily small ϵ, the system of inequalities

$$|\theta_r \tau - \sigma_r| < \epsilon \pmod{2\pi}, \quad (r = 1, \dots, n),$$

admits a solution for τ if and only if, for integers $\ell_1, \dots, \ell_n$, if

$$\sum_{r=1}^{n} \ell_r \theta_r = 0,$$

then

$$\sum_{r=1}^{n} \ell_r \sigma_r \equiv 0 \pmod{2\pi}.$$

PGST on P_4

Using spectral decomposition, we can write

$$\begin{aligned} U(t) = {} & \exp(\frac{i}{2}(\sqrt{5}+1)\tau)E_1 + \exp(\frac{i}{2}(\sqrt{5}-1)\tau)E_2 \\ & + \exp(\frac{i}{2}(-\sqrt{5}+1)\tau)E_3 + \exp(\frac{i}{2}(-\sqrt{5}-1)\tau)E_4. \end{aligned}$$

If we take $\theta_1\tau \approx \pi/2$ and $\theta_2\tau \approx 3\pi/2$ then we obtain

$$U(\tau) \approx iE_1 - iE_2 + iE_3 - iE_4 = iF.$$

Now, if there exist integers ℓ_1, ℓ_2 such that $\ell_1\theta_1 + \ell_2\theta_2 = 0$, then we have

$$\mathbb{Q} \not\ni \frac{3+\sqrt{5}}{2} = \frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{\theta_1}{\theta_2} = -\frac{\ell_2}{\ell_1} \in \mathbb{Q},$$

a contradiction. Hence, by Kronecker's Theorem, P_4 has PGST between its end vertices.

PGST on Paths

Theorem (Godsil, Kirkland, Severini, Smith; 2012)

There is PGST on P_n between the end vertices if and only if either:

1. $n = 2^t - 1$, $t \in \mathbb{Z}_+$;
2. $n = p - 1$, *p a prime; or,*
3. $n = 2p - 1$, *p a prime.*

Moreover, when PGST occurs between the end vertices of P_n, then it occurs between vertices a and $n + 1 - a$ for all $a \neq (n + 1)/2$.

PGST on Paths

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Main Result [van Bommel, 2018+]

There is PGST on P_n between vertices a and b if and only if $a + b = n + 1$ and either:

1. $n = 2^t - 1$, $t \in \mathbb{Z}_+$; *or,*
2. $n = 2^t p - 1$, $t \in \mathbb{Z}_{\geq 0}$, *p an odd prime, and* $2^{t-1} \mid a$.

Eigenvalues and Eigenvectors

Let $m = n + 1$.

- The eigenvalues of P_n are

$$\theta_r = 2\cos\left(\frac{r\pi}{m}\right) = \zeta_{2m}^r + \zeta_{2m}^{-r}, 1 \leq r \leq n,$$

  and belong to the cyclotomic field $\mathbb{Q}(\zeta_{2m})$.

- The eigenvector β^r corresponding to θ_r is given by

$$(\beta_1^r, \ldots, \beta_n^r), \quad \beta_v^r = \sin(v\pi r/m).$$

- The spectral idempotent E_j corresponding to θ_j is given by

$$E_j = \beta^j {\beta^j}^T / {\beta^j}^T \beta^j.$$

- Let $\Theta_a := \{\theta_r : E_r \mathbf{e}_a \neq 0\}$.

Strong Cospectrality

Definition

Vertices a and b are cospectral if the characteristic polynomials of $X \setminus a$ and $X \setminus b$ are equal. Equivalently, $(E_r)_{a,a} = (E_r)_{b,b}$ for all r.

Definition

Vertices a and b are strongly cospectral if $E_r \mathbf{e}_a = \pm E_r \mathbf{e}_b$ for all r.

Lemma

If PGST occurs between a and b, then they are strongly cospectral vertices.

Corollary

Vertices a and b are strongly cospectral if and only if $a + b = n + 1$.

A Kronecker Condition for PGST

Theorem (Coutinho, Guo, van Bommel; 2017)

PGST happens between vertices a and b if and only if both conditions below hold.

(i) $a + b = n + 1$. *In this case, for all* $\theta_r \in \Theta_a$, *define* $\sigma_r = 0$ *if* $E_r\mathbf{e}_a = E_r\mathbf{e}_b$, *and* $\sigma_r = 1$ *if* $E_r\mathbf{e}_a = -E_r\mathbf{e}_b$.

(ii) *For any set of integers* $\{\ell_r : \theta_r \in \Theta_a\}$ *such that*

$$\sum_{\theta_r \in \Theta_a} \ell_r \theta_r = 0 \quad \textit{and} \quad \sum_{\theta_r \in \Theta_a} \ell_r = 0,$$

then

$$\sum_{\theta_r \in \Theta_a} \ell_r \sigma_r \textit{ is even.}$$

PGST between Internal Vertices of Paths

Theorem (Coutinho, Guo, van Bommel; 2017)

Given any odd prime p and positive integer t, there is PGST in $P_{2^t p-1}$ between vertices a and $2^t p - a$, whenever $2^{t-1} \mid a$.

Proof Sketch

- Suppose for contradiction that we have $\{\ell_r\}_{r=1}^n$ such that $\sum_{r=1}^n \ell_r \theta_r = 0$. Then let

$$P(x) = \sum_{r=1}^{n} \ell_r x^r + \sum_{r=n+2}^{2n+1} \ell_{2n+2-r} x^r,$$

- We see $P(\zeta_{2(n+1)}) = 0$, so $\Phi(x) = \sum_{i=0}^{p-1} (-1)^i x^{2^t i}$ divides $P(x)$.
- The remainder of $P(x)/\Phi(x)$ is 0, so we obtain $\ell_r = \ell_{2^t p-r}$ for all even r.

A Kronecker Condition for No PGST

Lemma (van Bommel, 2018+)

Let a and b be vertices of P_n *such that* $a + b = n + 1$*. If there is a set of integers* $\{\ell_r : \theta_r \in \Theta_a,\ r \text{ odd}\}$ *such that*

$$\sum_{\substack{\theta_r \in \Theta_a \\ r \text{ odd}}} \ell_r \theta_r = 0 \quad \text{and} \quad \sum_{\substack{\theta_r \in \Theta_a \\ r \text{ odd}}} \ell_r \text{ is odd}$$

and there is a set of integers $\{\ell_r : \theta_r \in \Theta_a,\ r \text{ even}\}$ *such that*

$$\sum_{\substack{\theta_r \in \Theta_a \\ r \text{ even}}} \ell_r \theta_r = 0 \quad \text{and} \quad \sum_{\substack{\theta_r \in \Theta_a \\ r \text{ even}}} \ell_r \text{ is odd}$$

then PGST does not occur between vertices a and b.

Proof Sketch of Necessity

Fact

Let $n = km$, where m is an odd integer, and $0 \le a < k$ be an integer. Then

$$\sum_{j=0}^{m-1} (-1)^j \cos\left(\frac{(a+jk)\pi}{n}\right) = 0.$$

Case 1: If $n = 2^t r - 1$, r odd composite, and $r \nmid a$, choose $p \mid r$, $p \nmid a$ and take $\sum_{i=0}^{r/p-1} (-1)^i \theta_{c+i2^t p} = 0$.

Case 2: If $n = 2^t r - 1$, r composite, and $r \mid a$, take $\sum_{i=0}^{r-1} (-1)^i \theta_{c+i2^t} = 0$.

Case 3: If $n = 2^t p - 1$ and $2^{t-1} \nmid a$, take $\sum_{i=0}^{r-1} (-1)^i \theta_{c+i2^t} = 0$.

Open Problems and Future Research

- For a given $\epsilon > 0$, what time interval is required to achieve $||U(\tau)_{a,b}|| > 1 - \epsilon$?
- Is there PGST between internal vertices of paths with respect to the Laplacian matrix?
- When does PST or PGST occur on trees?
- What if the initial state is entangled?

Thank You!

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