Mutually Orthogonal Latin Squares with Large Holes

Christopher M. van Bommel & Peter Dukes
University of Victoria

A Completed Sudoku

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Latin Square

- $v \times v$ array.
- Entries from a set of v symbols, often $[v]$.
- Each row and each column contains every symbol.

Playing Card Problem

A♠	K♡	Q♢	J♣
K♢	A♣	J♠	Q♡
Q♣	J♢	A♡	K♠
J♡	Q♠	K♣	A♢

Mutually Orthogonal Latin Squares (MOLS)

- Multiple $v \times v$ Latin squares.
- For any two squares, every ordered pair formed by superimposing the two squares is distinct.

Euler's 36 Officer Problem

Is it possible to arrange six regiments, each with six officers of different ranks, in a 6×6 array so that each row and each column contains one officer from each regiment and one officer of each rank?

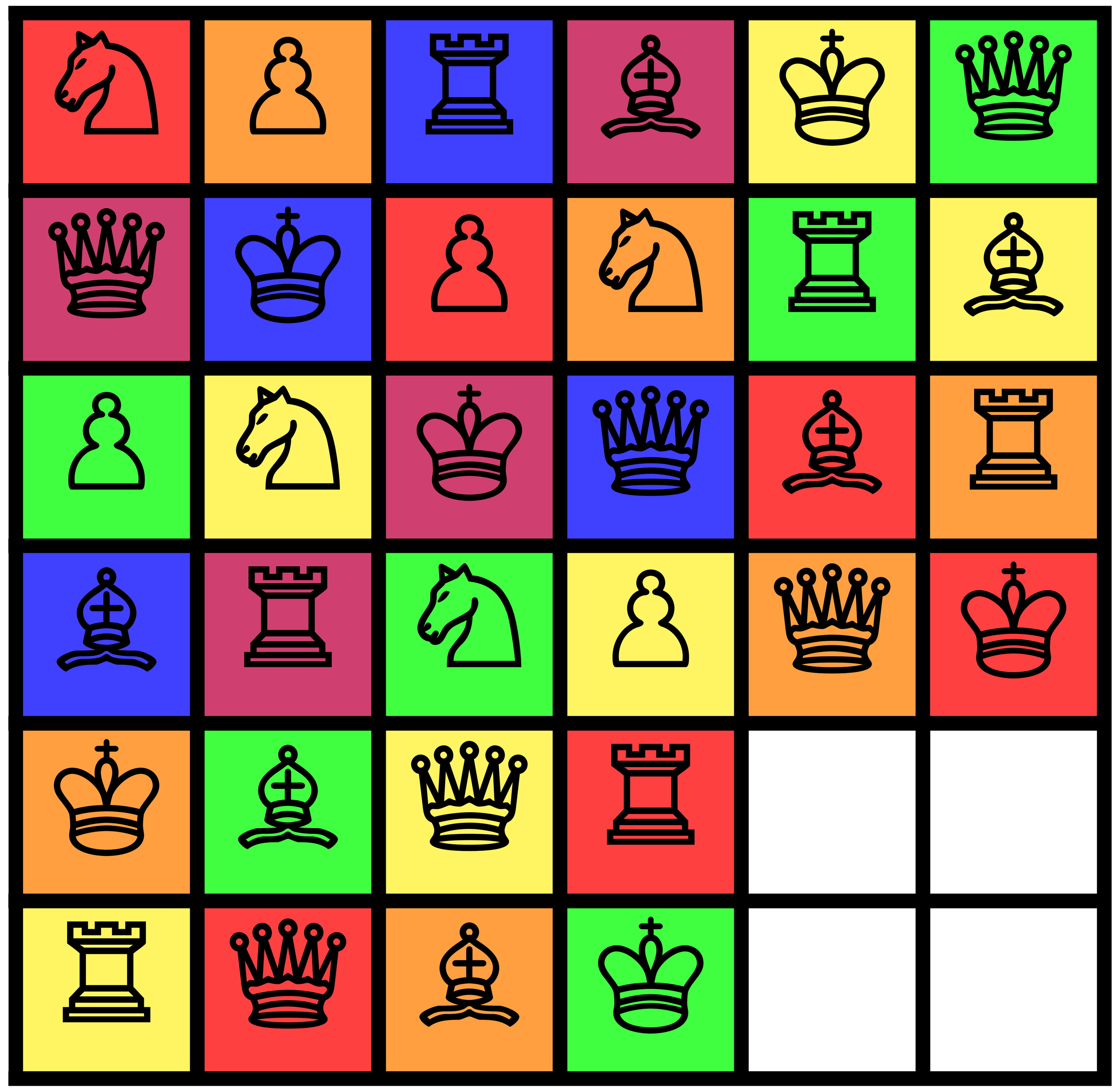
Theorem (Tarry)

Orthogonal Latin Squares of order 6 do not exist.

Theorem (Bose, Shrikhande, Parker)

Orthogonal Latin Squares exist for all $v \neq 2, 6$.

$2-\mathrm{IMOLS}(6;2)$



Incomplete MOLS (IMOLS)

- $v \times v$ arrays each with empty $n \times n$ subarray.
- Each row or column contains each symbol at most once.
- For any two squares, every ordered pair formed by superimposing the two squares is distinct.
- Can be completed with MOLS of order n (if they exist).

Theorem (Beth)

If n is sufficiently large, there exist at least $n^{\frac{1}{14.8}}$ MOLS of order n.

Theorem (Horton)

If $t-IMOLS(v;n)$ exist, then $v \geq (t+1)n$.

Main Theorem

There exist $t-IMOLS(v;n)$ for all sufficiently large v, n satisfying $v \geq 8(t+1)^2 n$.

Construction

Suppose there exists an $IPBD((v;n),K)$ and, for each $k \in K$, there exist t idempotent $MOLS$ of order k. Then there exist $t-IMOLS(v;n)$.

Incomplete Pairwise Balanced Designs

Let $K_0 \subseteq K$ with $\alpha(K_0) = \alpha(K)$.
There exists an $IPBD((v;w),K)$ for all sufficiently large admissible v, w satisfying $v \geq (\Pi_{k \in K_0} k)w$.

Proof Sketch of Main Theorem

- Take $K = \{2^f, 2^{f+1}, 3^{2f+1}\}$, where $t+1 < 2^f \leq 2(t+1)$.
- Apply Construction with IPBDs ($K_0 = \{2^f, 2^{f+1}\}$) and idempotent MOLS (existence known for prime powers).
- Can improve inequality for **specific** values of t with more carefully chosen prime powers.

Acknowledgments

CMS SMC
STUDC CMS SMC
NSERC CRSNG

Our Paper

http://qrs.ly/hp4phhp