γ-Graphs of Trees

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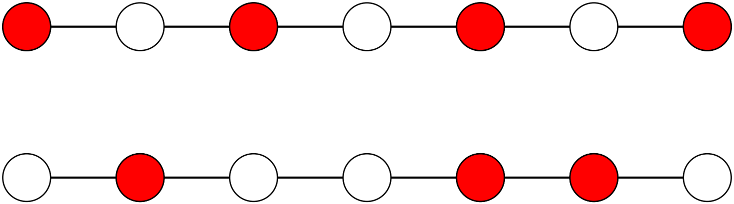
1. Introduction

2. Algorithm

3. γ-Trees of Trees

Introduction

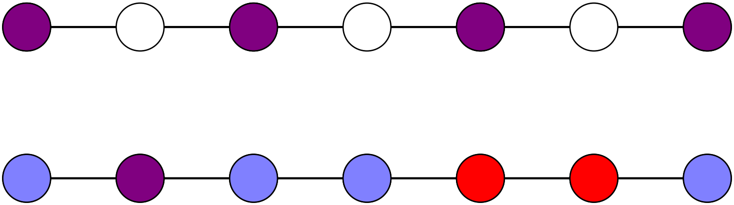
Dominating Sets



Definition

A dominating set is a subset of vertices such that each vertex not in the set is adjacent to a vertex in the set. A γ-set is a dominating set of minimum cardinality.

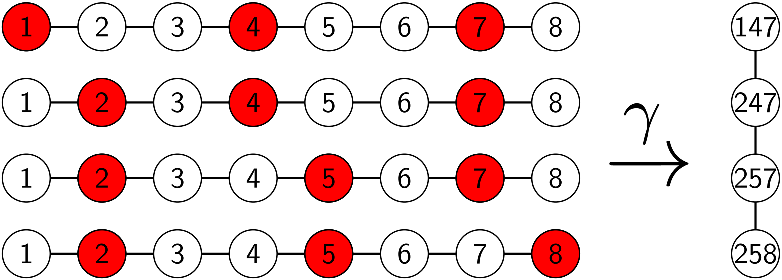
Private Neighbours



Definition

A vertex v is a private neighbour (pn) of a vertex $x \in D$ if $v \in N[x]$ and $\forall y \in D, y \neq x \implies v \notin N[y]$. If $v = x$ it is a self private neighbour (spn), otherwise it is an external private neighbour (epn).

γ-Graph



Definition

If G is a graph, let $G(\gamma)$ be the graph whose vertices are γ-sets of G, and two γ-sets D and F are adjacent if $F = D - \{v\} \cup \{w\}$ for some $vw \in E(G)$.

Questions (Fricke et al., 2011)

1. Is $\Delta(T(\gamma)) = O(n)$ for every tree T of order n?
2. Is $\mathrm{diam}(T(\gamma)) = O(n)$ for every tree T of order n?
3. Is $|V(T(\gamma))| \leq 2^{\gamma(T)}$ for every tree T?
4. Which graphs are γ-graphs of trees?
5. Which graphs are γ-graphs? Can you construct a graph H that is not a γ-graph of any graph G?
6. For which graphs G is $G(\gamma) \cong G$?
7. Under what conditions is $G(\gamma)$ a disconnected graph?

Questions (Fricke et al., 2011)

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7. ? Under what conditions is $G(\gamma)$ a disconnected graph?

Answers

Theorem (Edwards, 2015)

1. *If* $|V(T)| = n$, *then* $\Delta(T(\gamma)) \leq n - \gamma(T)$.

Theorem (Edwards, 2015)

2. *For any tree* T, $\mathrm{diam}(T(\gamma)) \leq 2(2\gamma(T) - s)$, *where* s *is the number of support vertices in* T.

Theorem (Edwards, 2015)

3. *For any tree* T, $|V(T(\gamma))| \leq ((1 + \sqrt{13})/2)^{\gamma(T)}$. *Moreover, there are infinitely many trees* T *for which* $|V(T(\gamma))| > 2^{\gamma(T)}$.

Theorem (Connelly et al., 2011)

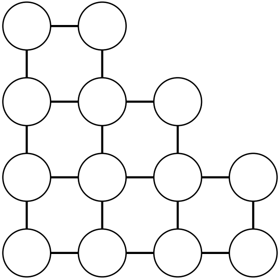
5. *For any graph* H, *there exists a graph* G *such that* $G(\gamma) \cong H$.

Examples

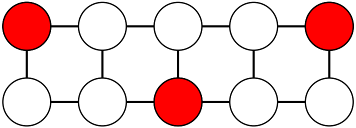
- $K_{1,n}(\gamma) \cong K_1$.
- $K_{2,n}(\gamma) \cong K_{1,2n}$ for $n \geq 3$.
- $P_{3k}(\gamma) \cong K_1$.
- $P_{3k+2}(\gamma) \cong P_{k+2}$.
- $P_{3k+1}(\gamma) \cong SG(k+1)$.

- $K_n(\gamma) \cong K_n$.
- $C_{3k+2}(\gamma) \cong C_{3k+2}$.

- $K_{m,n}(\gamma) \cong \overline{K_{mn}}$ for $m, n \geq 3$.
- $C_{3k}(\gamma) \cong \overline{K_3}$ for $k \geq 2$.
- $(P_2 \,\square\, P_{2k+1})(\gamma) \cong \overline{K_2}$ for $k \geq 2$.



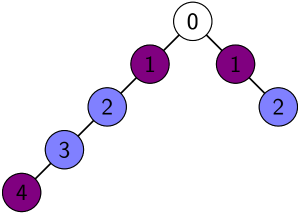
$SG(4)$



$P_2 \,\square\, P_5$

Algorithm

Highest γ-Set



$ht = 6$

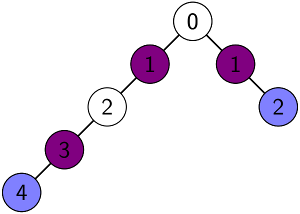
Definition

If T is a tree rooted at c and D is a γ-set, then the height of D is $ht_T(D) := \sum_{x \in D} d(x, c)$.

Proposition (Edwards, 2015)

The highest γ-set S is unique and every $x \in S$, $x \neq c$, has a child as a private neighbour.

Highest γ-Set



$ht = 5$

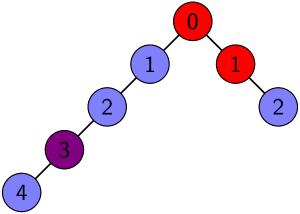
Definition

If T is a tree rooted at c and D is a γ-set, then the height of D is $ht_T(D) := \sum_{x \in D} d(x, c)$.

Proposition (Edwards, 2015)

The highest γ-set S is unique and every $x \in S$, $x \neq c$, has a child as a private neighbour.

Highest γ-Set



$ht = 4$

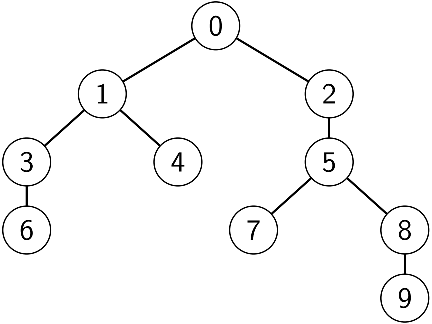
Definition

If T is a tree rooted at c and D is a γ-set, then the height of D is $ht_T(D) := \sum_{x \in D} d(x, c)$.

Proposition (Edwards, 2015)

The highest γ-set S is unique and every $x \in S$, $x \neq c$, has a child as a private neighbour.

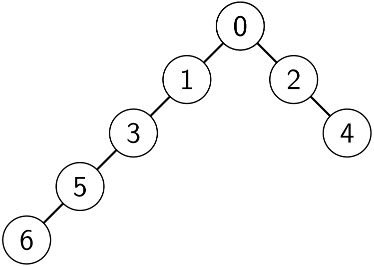
Breadth-First Search Order



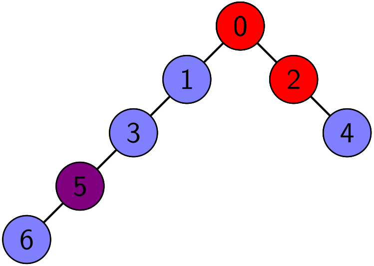
Definition

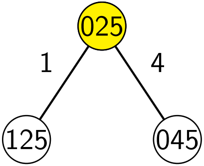
- Label the root 0. Add it to a queue.
- For each vertex in the queue, label its neighbours with the next available integers and add them to the queue. Remove the current vertex from the queue.

Algorithm Example



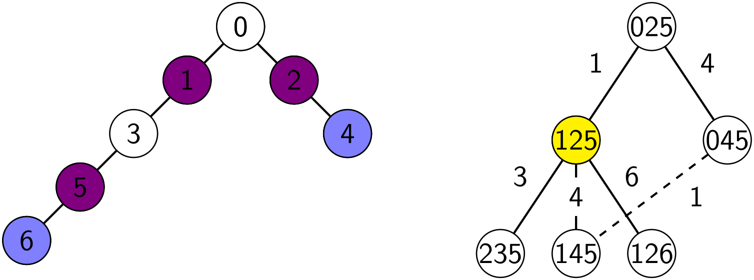
Algorithm Example





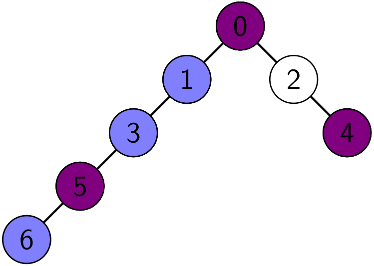
$$pn(0, 025) = \{1\}$$
$$pn(2, 025) = \{4\}$$
$$pn(5, 025) = \{3, 5, 6\}$$

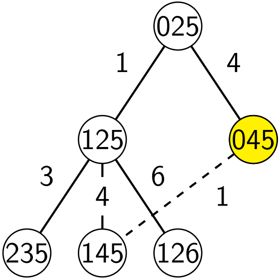
Algorithm Example



$$pn(1,125) = \{1\}$$
$$pn(2,125) = \{2,4\}$$
$$pn(5,125) = \{5,6\}$$

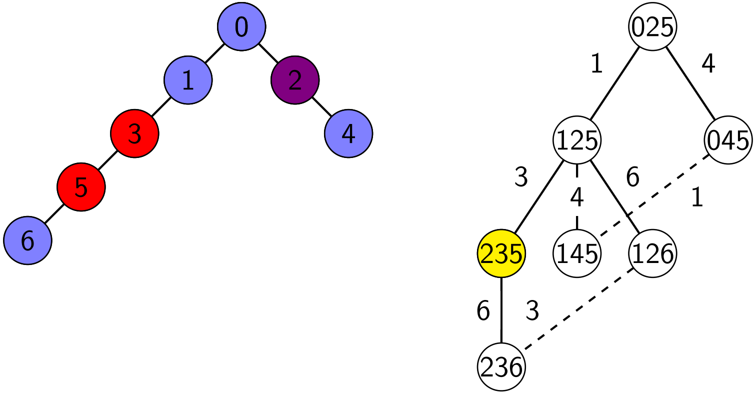
Algorithm Example





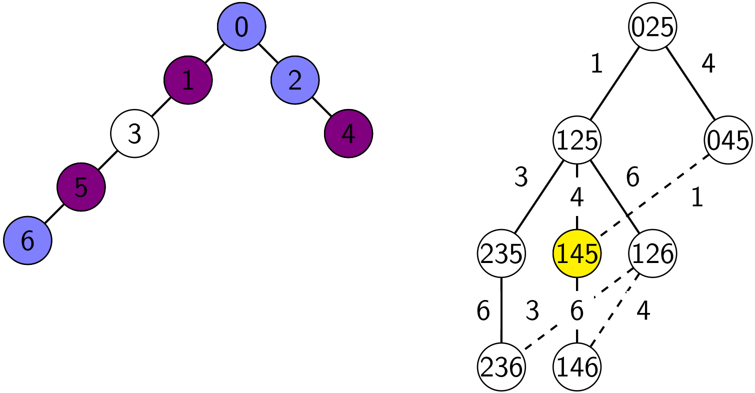
$$pn(0, 045) = \{0, 1\}$$
$$pn(4, 045) = \{4\}$$
$$pn(5, 045) = \{3, 5, 6\}$$

Algorithm Example



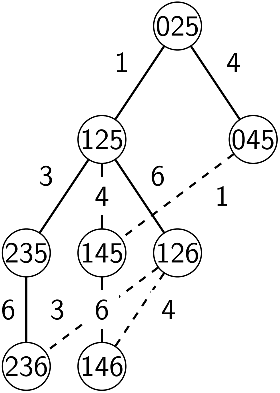
$$pn(2, 235) = \{0, 2, 4\}$$
$$pn(3, 235) = \{1\}$$
$$pn(5, 235) = \{6\}$$

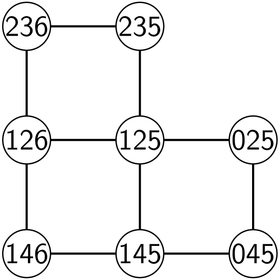
Algorithm Example



$$pn(1, 145) = \{0, 1\}$$
$$pn(4, 145) = \{2, 4\}$$
$$pn(5, 145) = \{5, 6\}$$

Algorithm Result



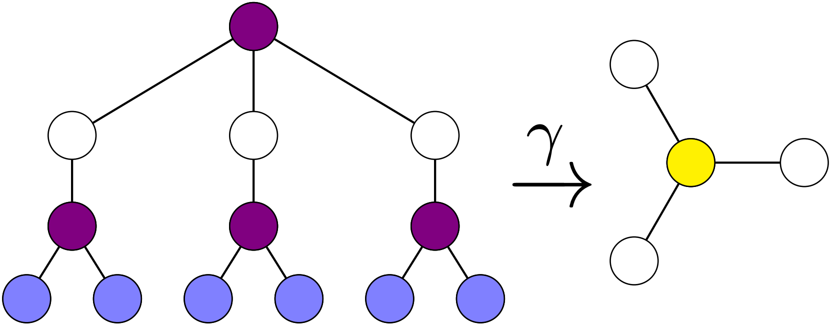


Algorithm to Determine $T(\gamma)$

- Root T at an arbitrary vertex c.
- Find the highest γ-set S of (T, c). Let $\mathrm{new}(S) = -1$.
- Assign a breadth-first search ordering rank to (T, c).
- For every γ-set D found:
    - For every vertex $x \in D$:
        - Find the private neighbours of x in D.
        - If x has exactly one external private neighbour y and $\mathrm{rank}(y) > \mathrm{new}(D)$, form $D' = D - \{x\} \cup \{y\}$.
        - If x has only a self private neighbour, then for each $y \in N(x)$ with $\mathrm{rank}(y) > \mathrm{new}(D)$, form $D' = D - \{x\} \cup \{y\}$.
    - For every D' found:
        - Add edge DD'. Let $\mathrm{new}(D') = \mathrm{rank}(y)$ and $w(DD') = y$.
        - If $\exists P \in N(D)$ and $\exists C \in N(P)$ such that $w(PC) = y$, add edge CD' and let $w(CD') = w(PD)$.

γ-Trees of Trees

γ-Stars



Leaves of γ-Trees

Lemma

If D is a leaf in $T(\gamma)$, then exactly one vertex in D has fewer than two epns.

Proof.

- If $z \in D$ has no epns, then for every $w \in N(z)$, $D - \{z\} \cup \{w\}$ is an adjacent γ-set of T.
- If $z \in D$ has exactly one epn y, then $D - \{z\} \cup \{y\}$ is an adjacent γ-set of T.
- If $z \in D$ has at least two epns v, w, then z is their only common neighbour, so every adjacent γ-set of T contains z.

□

Subtrees of γ-Trees

Theorem

If $G \cong T(\gamma)$ and L is a leaf of G, then $G - L \cong T'(\gamma)$.

Proof.

- Suppose x is the only vertex in L with fewer than two epns.
- $\exists! y$ such that $L - \{x\} \cup \{y\}$ is a γ-set of T.
- If every other γ-set of T contains y, T' is formed by adding a leaf to y.
- Otherwise, delete x, and T' is the component containing y. □

Corollary

$$G \cong T(\gamma) \implies \forall G' \subseteq G, \quad G' \cong T'(\gamma).$$

Private Neighbours

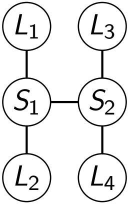
Lemma

Let D and F be adjacent γ-sets of a tree T and let $x \in D \cap F$. Then $||pn(x, D)| - |pn(x, F)|| \leq 1$.

Proof.

- Let $F = D - \{y\} \cup \{z\}$.
- Suppose instead $|pn(x, D)| - |pn(x, F)| \geq 2$.
- Then $\exists v, w \in pn(x, D)$ such that $v, w \notin pn(x, F)$.
- Hence $v, w \in N[x]$ and $v, w \in N[z]$.
- Therefore, x, z, v, w form a cycle in T, which is a contradiction. □

γ-Graph H?



Lemma: If $T(\gamma)$ is a tree, $H \not\subseteq T(\gamma)$.

- L_i is the only γ-set containing x_i, and x_i swaps with y_i.
- If $y_1 \neq y_2$, x_1 and x_2 are epns in S_1. Then y_1 and y_2 have a common neighbour z, and no other swaps are possible.
- Hence $y_1 = y_2$ has only an spn in S_1 and $y_3 = y_4$ has only an spn in S_2.
- If $y_1 \in S_2$, then y_1 has at most one epn in S_2, so it has a swap.
- Hence $y_1 \notin S_2$ and $y_3 \notin S_1$. But then y_3 is an epn of y_1 in S_1.

Characterization

Theorem

If G is a tree, $G \cong T(\gamma)$ iff $H \not\subseteq G$.

Proof.

- True if G is a star.
- Assume true for all subgraphs of G.
- Choose a degree 2 vertex x.
- Let G_1, G_2 be the components of $G - x$.
- $G_1 \cup \{x\} \cong T_1(\gamma)$ and $G_2 \cup \{x\} \cong T_2(\gamma)$.
- Each γ-set corresponding to x contains a vertex in no other γ-set of $T_1(\gamma)$ and $T_2(\gamma)$.
- Join these two vertices with a new vertex. □

What Other Graphs are γ-Graphs of Trees?

- Every Cartesian product graph whose factors are γ-graphs of trees is a γ-graph of a tree.
- $K_{2,3} \not\subseteq T(\gamma)$.
- Every edge which is not a bridge of $T(\gamma)$ is contained in a 4-cycle.
- ...?

Thank You!

Mount Allison University

AARMS