Mutually Orthogonal Latin Squares with Large Holes

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Latin Squares

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Definition

- $v \times v$ array.
- Entries from a set of v symbols, often $[v]$.
- Each row and each column contains every symbol.

Orthgonal Latin Squares

Definition

- Two $v \times v$ Latin squares.
- Every ordered pair formed by superimposing the two squares are distinct.

A♠	K♡	Q♢	J♣
K♢	A♣	J♠	Q♡
Q♣	J♢	A♡	K♠
J♡	Q♠	K♣	A♢

Euler's 36 Officers Problem

Problem

Is it possible to arrange six regiments, each with six officers of different ranks, in a 6×6 array so that each row and each column contains one officer from each regiment and one officer of each rank?

Theorem (Bose, Shirkhande, & Parker, 1960)

Orthogonal Latin Squares exist for all $v \neq 2, 6$.

So Close!

Incomplete Latin Squares

4	5	6	1	2	3
6	4	5	2	3	1
5	6	4	3	1	2
1	2	3			
3	1	2			
2	3	1			

Definition

- $v \times v$ array with empty $n \times n$ subarray.
- Each row and each column contains each symbol at most once.
- Rows and columns containing empty cells do not contain the symbols that index them.

Mutually Orthogonal Incomplete Latin Squares

5	6	3	4	1	2
2	1	6	5	3	4
6	5	1	2	4	3
4	3	5	6	2	1
1	4	2	3		
3	2	4	1		

1	2	5	6	3	4
6	5	1	2	4	3
4	3	6	5	1	2
5	6	4	3	2	1
2	4	3	1		
3	1	2	4		

Definition (Orthogonal)

- Two incomplete Latin squares of order v with common hole size n.
- Every ordered pair formed by superimposing the two squares are distinct.
- No such ordered pair indexes an empty cell.

Necessary and Sufficient Conditions

Theorem (Horton, 1974)

If $t-IMOLS(v,n)$ exist, then $v \geq (t+1)n$.

Theorem

There exist $t-IMOLS(v,n)$ for

t	v	n	Reference
1	$\geq 2n$		
2	$\geq 3n$	$\neq (6,1)$	*Heinrich & Zhu, 1986*
3	$\geq 4n$	$\neq (6,1)$, $(10,1)$?	*Abel et al., 1997*
4	$\geq 7n$	≥ 98	*Colbourn & Zhu, 1995*
5	$\geq 7n+7$	≥ 571	*Drake & Lenz, 1980*
6	$\geq 7n$	≥ 781	*Colbourn & Zhu, 1995*

Sufficient Asymptotic Condition

Main Theorem (Dukes & van Bommel, 2014+)

There exist t−$IMOLS(v, n)$ for all sufficiently large v, n satisfying $v \geq 8(t+1)^2 n$.

Construction

Suppose there exists an $IPBD((v; n), K)$ and, for each $k \in K$, there exist t idempotent MOLS of order k. Then there exist t−$IMOLS(v, n)$.

Idempotent Mutually Orthogonal Latin Squares

Definition (Idempotent)

Each cell (i, i) in each square contains the symbol i.

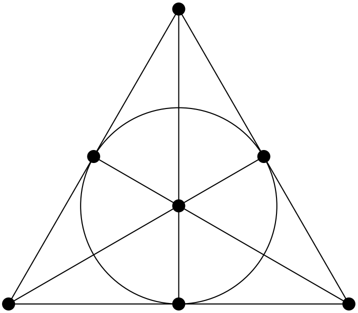
1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

Proposition

For prime powers q, there exist $q-2$ idempotent MOLS of order q.

Pairwise Balanced Designs, $PBD(v, K)$



Example

$PBD(7, \{3\})$:

$$\{1,2,4\},\{2,3,5\},\{3,4,6\},$$
$$\{4,5,7\},\{5,6,1\},\{6,7,2\},$$
$$\{7,1,3\}.$$

Definition

- v points, collection of blocks.
- Size of each block is in K.
- Each pair of distinct points appear together in exactly one block.

PBDs: Necessary and Sufficient Conditions

Proposition

The existence of an PBD(v, K) implies

$$v(v-1) \equiv 0 \pmod{\beta(K)} \qquad \text{(global)}$$
$$v-1 \equiv 0 \pmod{\alpha(K)} \qquad \text{(local)}$$

Definition

$$\beta(K) := \gcd\{k(k-1) : k \in K\}$$
$$\alpha(K) := \gcd\{k-1 : k \in K\}$$

Theorem (Wilson's Theorem)

Given any $K \subseteq \mathbb{Z}_{\geq 2}$, there exist PBD$(v, K)$ for all sufficiently large v satisfying (global) *and* (local).

Incomplete Pairwise Balanced Designs, $IPBD((v;w),K)$

Example

$IPBD((11;2),\{3,4\})$:

$$\{1,2,3,10\},\{4,5,6,10\},\{7,8,9,10\},\{1,4,7\},\{2,5,8\},\{3,6,9\},$$
$$\{1,5,9,11\},\{2,6,7,11\},\{3,4,8,11\},\{1,6,8\},\{2,4,9\},\{3,5,7\}.$$

Definition

- v points, w in the hole, collection of blocks.
- Size of each block is in K.
- No two points in the hole are in a common block.
- Each pair of distinct points not both in the hole appear together in exactly one block.

IPBDs: Necessary Conditions

Proposition

The existence of an IPBD$((v; w), K)$ implies

$$v(v-1) - w(w-1) \equiv 0 \pmod{\beta(K)} \qquad \text{(global)}$$

$$v - 1 \equiv w - 1 \equiv 0 \pmod{\alpha(K)} \qquad \text{(local)}$$

$$v \geq (\min K - 1)w + 1 \qquad \text{(inequality)}$$

Definition

$$\beta(K) := \gcd\{k(k-1) : k \in K\}$$

$$\alpha(K) := \gcd\{k - 1 : k \in K\}$$

Resolvable Designs

Example

Resolvable $PBD(9, \{3\})$:

$$\begin{gathered}\{1,2,3\},\{4,5,6\},\{7,8,9\},\\ \{1,4,7\},\{2,5,8\},\{3,6,9\},\\ \{1,5,9\},\{2,6,7\},\{3,4,8\},\\ \{1,6,8\},\{2,4,9\},\{3,5,7\}.\end{gathered}$$

Definition

- Partition the blocks into *parallel classes*.
- Each point occurs in exactly one block in each parallel class.

Resolvable Designs $\Longleftrightarrow$ IPBDs

Example

$IPBD((13,4),\{4\})$:

$$\{1,2,3,\mathbf{10}\},\{4,5,6,\mathbf{10}\},\{7,8,9,\mathbf{10}\},$$
$$\{1,4,7,\mathbf{11}\},\{2,5,8,\mathbf{11}\},\{3,6,9,\mathbf{11}\},$$
$$\{1,5,9,\mathbf{12}\},\{2,6,7,\mathbf{12}\},\{3,4,8,\mathbf{12}\},$$
$$\{1,6,8,\mathbf{13}\},\{2,4,9,\mathbf{13}\},\{3,5,7,\mathbf{13}\}.$$

Proposition

If $v=(k-1)w+1$, then there exists an $IPBD((v;w),\{k\})$ if and only if there exists a resolvable $PBD(v-w,\{k-1\})$.

IPBDs: Sufficient Conditions

Theorem (Dukes, Lamken, & Ling, 2014+)

For every real number $\epsilon > 0$, there exists an IPBD$((v; w), \{k\})$ for all sufficiently large v, w satisfying (global), (local), *and* $v \geq (k - 1 + \epsilon)w$.

Theorem (Dukes & van Bommel, 2014+)

Let $K_0 \subseteq K$ with $\alpha(K_0) = \alpha(K)$. There exists an IPBD$((v; w), K)$ for all sufficiently large v, w satisfying (global), (local), *and* $v \geq (\prod_{k \in K_0} k)w$.

IMOLS Construction

Example

IPBD$((7;3),\{3\})$:

$$\{1,2,5\},\{1,3,6\},$$
$$\{1,4,7\},\{2,3,7\},$$
$$\{2,4,5\},\{\mathbf{3},\mathbf{4},\mathbf{5}\}.$$

+

1	3	2
3	2	1
2	1	3

$\rightarrow$

1	5	6	7	2	3	4
5	2	7	6	1	4	3
6	7	**3**	**5**	**4**	1	2
7	6	**5**	**4**	**3**	2	1
2	1	**4**	**3**			
3	4	1	2			
4	3	2	1			

Proof Sketch

Main Theorem (Dukes & van Bommel, 2014+)

There exist $t\text{−}IMOLS(v, n)$ for all sufficiently large v, n satisfying $v \geq 8(t+1)^2 n$.

Proof.

- Remains to find a suitable block set K.
- To ensure idempotent MOLS exist, use prime powers.
- To ensure $\alpha(K) = 1$, need an even value.
- Hence, choose $t + 1 < 2^f \leq 2(t + 1)$.
- Also take 2^{f+1} and 3^{2f+1}.
- Then $K_0 = \{2^f, 2^{f+1}\}$.

□

Product Construction

1	2	3	7	8	9	4	5	6
2	3	1	8	9	7	5	6	4
3	1	2	9	7	8	6	4	5
7	8	9	4	5	6	1	2	3
8	9	7	5	6	4	2	3	1
9	7	8	6	4	5	3	1	2
4	5	6	1	2	3			
5	6	4	2	3	1			
6	4	5	3	1	2			

9	8	7	6	5	4	3	2	1
7	9	8	4	6	5	1	3	2
8	7	9	5	4	6	2	1	3
3	2	1	9	8	7	6	5	4
1	3	2	7	9	8	4	6	5
2	1	3	8	7	9	5	4	6
6	5	4	3	2	1			
4	6	5	1	3	2			
5	4	6	2	1	3			

Conjectures

Conjecture

Let $K \subseteq \mathbb{Z}_{\geq 2}$. For any real $\epsilon > 0$, there exist IPBD$((v; w), K)$ for all sufficiently large v, w satisfying (global), (local), *and $v \geq (\min K - 1 + \epsilon)w$.*

Conjecture

There exist $t-$IMOLS(v, n) for all sufficiently large v, n satisfying $v \geq 2(t + 1)n$.

Thank You!

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