Quantum Walks, State Transfer, and Modified Paths

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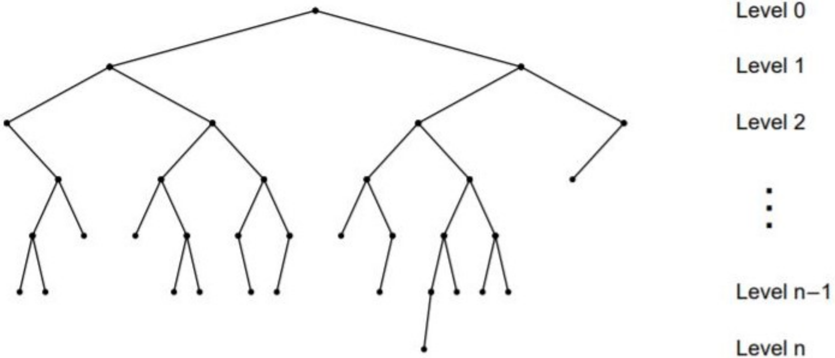
December 7, 2019

Decision Trees



E. Farhi and S. Gutmann, Quantum computation and decision trees, *Physical Review A* 58 (1998), no. 2, 915.

Continuous Random Walk

Definition

Let X be a graph. The matrix

$$M(t) := \exp(-tL) = \sum_{n\geq 0} \frac{t^n}{n!}(-L)^n$$

is such that the (a, b) entry is the **probability** that a "walker" starting on vertex a is at vertex b after time t.

Definition

A **continuous random walk** is modelled such that in a short time interval δt, the walker leaves the current vertex and moves to one of the adjacent vertices with equal probability.

Continuous Quantum Walk

Definition

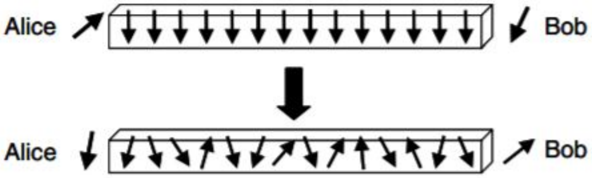
Let S be a real symmetric matrix. The **transition operator** given by S is

$$U(t) := \exp(itS) = \sum_{n \geq 0} \frac{(it)^n}{n!} S^n,$$

and defines a **continuous quantum walk**.

For a graph X, the choices for S we will consider are the adjacency matrix A and the Laplacian L.

Using Spin Chains for Quantum Communication



S. Bose, Quantum communication through an unmodulated spin chain, *Physical Review Letters* 91 (2003), no. 20, 207901.

Perfect State Transfer

Definition

A graph X has **perfect state transfer** between vertices a and b if there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that $U(\tau)\mathbf{e}_a = \gamma \mathbf{e}_b$.

Paths

$$P_2 : U_A(\pi/2)\mathbf{e}_1 = i\mathbf{e}_2, \qquad U_L(\pi/2)\mathbf{e}_1 = \mathbf{e}_2;$$
$$P_3 : U_A(\pi/\sqrt{2})\mathbf{e}_1 = -\mathbf{e}_3.$$

Perfect State Transfer on Paths

Theorem (Christandl et al. 2005)

With respect to the adjacency matrix, P_n has perfect state transfer **between the end vertices** *if and only if $n = 2, 3$.*

Theorem (Stevanović 2011; Godsil 2012)

With respect to the adjacency matrix, P_n has perfect state transfer if and only if $n = 2, 3$.

Theorem (Coutinho & Liu, 2015)

With respect to the Laplacian, if T is a tree, then T has perfect state transfer if and only if $T = P_2$.

Pretty Good State Transfer (PGST)

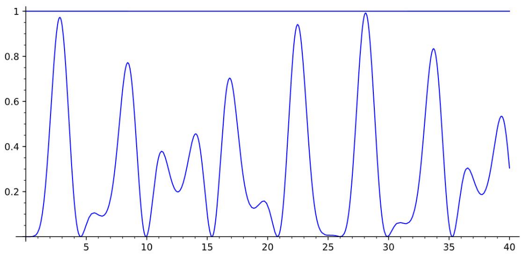
Definition

A graph X has **pretty good state transfer** between vertices a and b if, for every $\epsilon > 0$, there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that

$$||U(\tau)\mathbf{e}_a - \gamma\mathbf{e}_b|| < \epsilon.$$

Example (P_4)

$||U(t)_{1,4}||^2$



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Pretty Good State Transfer on Paths (End Vertices)

Theorem (Godsil, Kirkland, Severini, Smith; 2012)

With respect to the adjacency matrix, there is pretty good state transfer between the end vertices of P_n if and only if:

1. $n = 2^t - 1$, $t \in \mathbb{Z}_+$;
2. $n = p - 1$, p *a prime; or,*
3. $n = 2p - 1$, p *a prime.*

Moreover, when pretty good state transfer occurs between the end vertices of P_n, then it occurs between vertices a and $n+1-a$ for all $a \neq (n+1)/2$.

Theorem (Banchi, Coutinho, Godsil, Severini; 2017)

With respect to the Laplacian, there is pretty good state transfer between the end vertices of P_n if and only if n is a power of 2. Moreover, when pretty good state transfer occurs between the end vertices of P_n, then it occurs between vertices a and $n+1-a$ for all $a \neq (n+1)/2$.

Spectral Decomposition and Eigenvalue Support

Fact

If A is symmetric with distinct eigenvalues $\theta_1, \ldots, \theta_m$, and if E_r is the orthogonal projection onto the eigenspace belonging to θ_r, then A and has **spectral decomposition**

$$A = \sum_r \theta_r E_r,$$

and moreover

$$U(t) = \exp(itA) = \sum_r e^{i\theta_r t} E_r.$$

Definition

If $a \in V(X)$, then the **eigenvalue support** of a, denoted Θ_a, is the set

$$\{\theta_r : E_r \mathbf{e}_a \neq 0\}.$$

Kronecker's Theorem

Let $\theta_1, \ldots, \theta_n$ and $\sigma_1, \ldots, \sigma_n$ be arbitrary real numbers. For an arbitrarily small ϵ, the system of inequalities

$$|\theta_r \tau - \sigma_r| < \epsilon \pmod{2\pi}, \quad (r = 1, \ldots, n),$$

admits a solution for τ if and only if, for integers $\ell_1, \ldots, \ell_n$, if

$$\sum_{r=1}^{n} \ell_r \theta_r = 0,$$

then

$$\sum_{r=1}^{n} \ell_r \sigma_r \equiv 0 \pmod{2\pi}.$$

Eigenvalue Support Determines Pretty Good State Transfer

Lemma (Kempton, Lippner, Yau; 2017)

Let u, v be vertices of G, and H the Hamiltonian. Then pretty good state transfer from u to v occurs at some time if and only if:

1. *Every eigenvector x of H satifies either $x(u) = x(v)$ or $x(u) = -x(v)$. [i.e. u and v are strongly cospectral]*
2. *Let $\{\lambda_i\}$ be the eigenvalues of H corresponding to eigenvectors with $x(u) = x(v) \neq 0$ and $\{\mu_j\}$ the eigenvalues for eigenvectors with $x(u) = -x(v) \neq 0$. Then if there exists integers ℓ_i, m_j such that if*

$$\left.\begin{aligned} \sum_i \ell_i \lambda_i + \sum_j m_j \mu_j = 0 \\ \sum_i \ell_i + \sum_j m_j = 0 \end{aligned}\right\} \quad \textit{then} \quad \sum_j m_j \textit{ is even.}$$

Pretty Good State Transfer with Internal Vertices of Paths

Theorem (Coutinho, Guo, van Bommel; 2017)

Given any odd prime p and positive integer t, there is pretty good state transfer in $P_{2^t p-1}$ between vertices a and $2^t p - a$, whenever $2^{t-1} \mid a$.

Theorem

There is pretty good state transfer on P_n between vertices a and b if and only if $a + b = n + 1$ and:

- $n = 2^t - 1$, $t \in \mathbb{Z}_+$;
- $n = p - 1$, p a prime; or,
- $n = 2^t p - 1$, $t \in \mathbb{Z}_+$, p an odd prime, and $2^{t-1} \mid a$.

Theorem

With respect to the Laplacian, there is pretty good state transfer on P_n between vertices a and b if and only if $a + b = n + 1$ and n is a power of 2.

Extending Pretty Good State Transfer to Multiple Qubits

Definition

A graph X has **pretty good state transfer** of the state $\mathbf{v}$, given by

$$\sum_{j=1}^{m} \beta_j \mathbf{e}_j, \qquad \sum_{j=1}^{m} |\beta_j|^2 = 1,$$

to the state $\mathbf{w}$ if for every $\epsilon > 0$, there exist $\tau \in \mathbb{R}$ and a complex number γ with $|\gamma| = 1$, such that

$$\|U(\tau)\mathbf{v} - \gamma \mathbf{w}\| < \epsilon.$$

Proposition

For all $\mathbf{v}$ *and* τ, *if* $\mathbf{w} = U(\tau)\mathbf{v}$, *then there is pretty good state transfer between* $\mathbf{v}$ *and* $\mathbf{w}$.

Aiming for Symmetry

Problem

We are interested in pretty good state transfer in X between states $\mathbf{v}$ and $\mathbf{v}^\sigma$, where σ is an automorphism of X and $\mathbf{v}^\sigma$ is given by

$$\mathbf{v}^\sigma = \sum_{x \in V(X)} \beta_x \mathbf{e}_{\sigma(x)}.$$

On P_n, we assume $\sigma(x) = n + 1 - x$.

Proposition

Let $\mathbf{v}$ be a state of P_n and suppose for each $a \in V(P_n)$ such that $\beta_a \neq 0$, there is pretty good state transfer between a and $n+1-a$. Then there is pretty good state transfer between $\mathbf{v}$ and $\mathbf{v}^\sigma$.

Parity States & Eigenvalue Support

Definition

Let $\mathbf{v}$ be a state. If $\mathbf{v}$ is such that $\beta_a = 0$ for all even a, we say that $\mathbf{v}$ is an **odd state**. If $\mathbf{v}$ is such that $\beta_a = 0$ for all odd a, we say that $\mathbf{v}$ is an **even state**. We say $\mathbf{v}$ is a **parity state** if it is an odd state or an even state.

Definition

The **eigenvalue support** of $\mathbf{v}$, denoted $\Theta_{\mathbf{v}}$, is the set

$$\{\theta_r : E_r\mathbf{v} \neq 0\}.$$

Lemma

Let $\mathbf{v}$ be a parity state of P_n. If $\theta_j \notin \Theta_{\mathbf{v}}$, then $\theta_{n+1-j} \notin \Theta_{\mathbf{v}}$.

PGST of Parity States on Odd Paths

Theorem

Let $m = 2^t p^s$, where p is an odd prime and $s, t > 0$, and let $\mathbf{v}$ be a parity state of P_{m-1}. Define

$$S_c := \{\theta_j : 1 \leq j < m,\ j \equiv c \pmod{m/p}\},\ 1 \leq c < m/p;$$
$$S_0 := \{\theta_{m/2}\} = \{0\}.$$

With respect to the adjacency matrix, there is pretty good state transfer in P_{m-1} between $\mathbf{v}$ and $\mathbf{v}^\sigma$ if and only if there does not exist S_c with c odd and $S_{c'}$ with c' even such that $S_c \cup S_{c'} \subseteq \Theta_{\mathbf{v}}$.

PGST of Two Qubit Parity States on Odd Paths

Corollary

Given any odd prime p and positive integer $t \geq 2$, there is pretty good state transfer in $P_{2^t p-1}$ between states

$$\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{e}_a + \alpha \mathbf{e}_b) \quad \textit{and} \quad \mathbf{v}^\sigma = \frac{1}{\sqrt{2}}(\mathbf{e}_{2^t p-a} + \alpha \mathbf{e}_{2^t p-b})$$

whenever $a \neq b$, $\alpha = \pm 1$, and $a + \alpha b \equiv 0 \pmod{2^t}$.

PGST of Parity States on Even Paths

Theorem

Let $m = p^s$, where p is an odd prime and $s > 0$, and let $\mathbf{v}$ be a parity state of P_{m-1}. Define

$$R_c := \{\theta_j : 1 \leq j < m,\ j \equiv c,\ m/p - c \pmod{m/p}\},\ 1 \leq c \leq m/(2p).$$

With respect to the adjacency matrix, there is pretty good state transfer in P_{m-1} between $\mathbf{v}$ and $\mathbf{v}^\sigma$ if and only if there does not exist R_c such that $R_c \subseteq \Theta_{\mathbf{v}}$.

Example

For P_8, there is pretty good state transfer of $\alpha \mathbf{e}_1 + \beta \mathbf{e}_3$ to $\alpha \mathbf{e}_8 + \beta \mathbf{e}_6$, where

$$\alpha = \frac{\sin\left(\frac{\pi}{3}\right)}{\sqrt{\sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{9}\right)}}, \qquad \beta = -\frac{\sin\left(\frac{\pi}{9}\right)}{\sqrt{\sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{9}\right)}}$$

Modified Paths

Claim (Chen, Mereau, Feder; 2015)

Let $P_N^{(3,w)}$ denote the path of length N, vertices labeled 1 to N, with additional vertices joined to vertices 3 and $N-2$ by edges of weight w. There exists a $w \sim \sqrt{N}$ such that the fidelity approaches unity as N approaches infinity, with error $1 - F \propto N^{-1}$. The time scales efficiently with N, $t \propto N^{3/2}$.

Theorem (Kempton, Lippner, Yau; 2017)

Given a path P_N of any length, there is some choice of Q such that by placing the value Q as a potential on each endpoint of P_N there is pretty good state transfer between the endpoints.

Characteristic Polynomials

Let $P_N^{(M,w)}$ denote the path of length N, vertices labeled 1 to N, with additional vertices joined to vertices M and $N+1-M$ by edges of weight w. Then the characteristic polynomial $\phi(P_N^{(M,w)}, t)$ can be factored into P_+P_-, where the eigenvectors corresponding to eigenvalues of P_+ are symmetric and those of P_- are antisymmetric. Let $p_n = \phi(P_n, t)$. If $N = 2n$, we have

$$P_+(t) = t(p_n(t) - p_{n-1}(t)) - w^2 p_{M-1}(t)(p_{n-M}(t) - p_{n-M-1}(t)),$$
$$P_-(t) = t(p_n(t) + p_{n-1}(t)) - w^2 p_{M-1}(t)(p_{n-M}(t) + p_{n-M-1}(t)),$$

and if $N = 2n+1$, we have

$$P_+(t) = t(p_{n+1}(t) - p_{n-1}(t)) - w^2 p_{M-1}(t)(p_{n-M+1}(t) - p_{n-M-1}(t)),$$
$$P_-(t) = tp_n(t) - w^2 p_{M-1}(t)p_{n-M}(t).$$

Pretty Good State Transfer with Additional Vertices

Theorem

$P_N^{(M,w)}$ has pretty good state transfer between vertices 1 and N if $P_{\pm}(t)$ are irreducible.

Lemma

Let $P_{\pm}(t) = A(t) + w^2 B(t)$. If $\gcd(A, B) = 1$ and w is transcendental*, then $P_{\pm}(t)$ is irreducible over $\mathbb{Q}(w^2)$.

Lemma

If $N = 2n$ and $\gcd(2n+1, M) = 1$, then $P_{\pm}(t)$ are irreducible.
If $N = 2n+1$, then $t \mid P_+(t)$ or $t \mid P_-(t)$.

Examples of No PGST

If $M = 3$ and $N \equiv 2 \pmod 6$, then for every w, 1 is a root of $P_+(t)$ and -1 is a root of $P_-(t)$. Moreover, $\sum \lambda_i = 1$ and $\sum \mu_j = -1$. Consider the assignments

$$N = 12k + 2: \quad \ell_i = \begin{cases} 6k+1, & \lambda_i = 1; \\ -2, & \lambda_i \neq 1; \end{cases} \quad m_j = \begin{cases} 6k+1, & \mu_j = -1; \\ 0, & \mu_j \neq -1. \end{cases}$$

$$N = 24k + 20: \quad \ell_i = \begin{cases} 6k+5, & \lambda_i = 1; \\ -1, & \lambda_i \neq 1; \end{cases} \quad m_j = \begin{cases} 6k+5, & \mu_j = -1; \\ 0, & \mu_j \neq -1. \end{cases}$$

Then we have $\sum \ell_i \lambda_i + \sum m_j \mu_j = 0$ and $\sum \ell_i + \sum m_j = 0$ but $\sum m_j$ is odd, and hence we cannot have pretty good state transfer.

Future Directions

- What is the characterization of modified paths that permit pretty good state transfer?
- What time interval is required to ensure state transfer with a particular probability? How does modifying the path affect this time interval?
- What is the characterization of eigenvalue supports that permit pretty good state transfer of multiple qubit states on paths?
- Are there other interesting forms of multiple qubit state transfer that could be considered?
- When does perfect state transfer or pretty good state transfer occur on trees?

Thank you!