Quantum Walks, State Transfer, and Entanglement

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May 28, 2019

Continuous Quantum Walk

Definition

Let H be a real symmetric matrix. The transition operator given by H is

$$U(t) := \exp(itH) = \sum_{n \geq 0} \frac{(it)^n}{n!} H^n,$$

and defines a continuous quantum walk.

For a graph X, the most interesting choices for H are the adjacency matrix A and the Laplacian L.

Perfect State Transfer

Definition

A graph X has perfect state transfer between vertices a and b if there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that $U(\tau)\mathbf{e}_a = \gamma \mathbf{e}_b$.

Paths

$$P_2 : U_A(\pi/2)\mathbf{e}_1 = i\mathbf{e}_2, \qquad U_L(\pi/2)\mathbf{e}_1 = \mathbf{e}_2;$$

$$P_3 : U_A(\pi/\sqrt{2})\mathbf{e}_1 = -\mathbf{e}_3.$$

Perfect State Transfer on Paths

Theorem (Christandl et al. 2005)

With respect to the adjacency matrix, P_n has perfect state transfer between the end vertices if and only if $n = 2, 3$.

Theorem (Stevanović 2011; Godsil 2012)

With respect to the adjacency matrix, P_n has perfect state transfer if and only if $n = 2, 3$.

Theorem (Countinho & Liu, 2015)

With respect to the Laplacian, if T is a tree, then T has perfect state transfer if and only if $T = P_2$.

Pretty Good State Transfer (PGST)

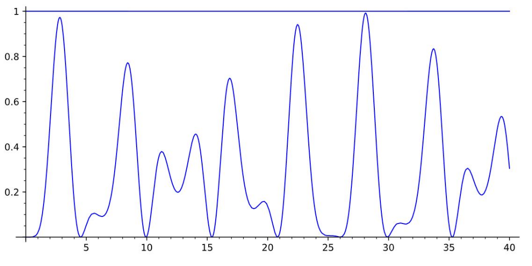
Definition

A graph X has pretty good state transfer between vertices a and b if, for every $\epsilon > 0$, there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that

$$||U(\tau)\mathbf{e}_a - \gamma \mathbf{e}_b|| < \epsilon.$$

Example (P_4)

$||U(t)_{1,4}||^2$



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Pretty Good State Transfer on Paths (Adjacency Matrix)

Theorem (Godsil, Kirkland, Severini, Smith; 2012)

With respect to the adjacency matrix, there is pretty good state transfer on P_n between the end vertices if and only if either:

1. $n = 2^t - 1$, $t \in \mathbb{Z}_+$;
2. $n = p - 1$, p *a prime; or,*
3. $n = 2p - 1$, p *a prime.*

Moreover, when pretty good state transfer occurs between the end vertices of P_n, then it occurs between vertices a and $n+1-a$ for all $a \neq (n+1)/2$.

Theorem (van Bommel, 2018+)

With respect to the adjacency matrix, there is pretty good state transfer on P_n between vertices a and b if and only if $a + b = n + 1$ and either:

- $n = 2^t - 1$, $t \in \mathbb{Z}_+$; *or,*
- $n = 2^t p - 1$, $t \in \mathbb{Z}_{\geq 0}$, p *an odd prime, and* $2^{t-1} \mid a$.

Pretty Good State Transfer on Paths (Laplcian)

Theorem (Banchi, Countinho, Godsil, Severini; 2017)

With respect to the Laplacian, there is pretty good state transfer on P_n between the end vertices if and only if n is a power of 2. Moreover, when pretty good state transfer occurs between the end vertices of P_n, then it occurs between vertices a and $n+1-a$ for all $a \neq (n+1)/2$.

Theorem (van Bommel, 2019+)

With respect to the Laplacian, there is pretty good state transfer on P_n between vertices a and b if and only if $a+b=n+1$ and n is a power of 2.

PGST of Entangled States on Paths

Definition

$$\mathbf{v} := \sum_{a=1}^{n} \beta_a \mathbf{e}_a, \quad \sum_{a=1}^{n} |\beta_a|^2 = 1$$

$$\mathbf{v}' := \sum_{a=1}^{n} \beta_{n+1-a} \mathbf{e}_a$$

Definition

A graph X has pretty good state transfer between states $\mathbf{v}$ and $\mathbf{v}'$ if, for every $\epsilon > 0$, there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that

$$||U(\tau)\mathbf{e}_{\mathbf{v}} - \gamma \mathbf{e}_{\mathbf{v}'}|| < \epsilon.$$

Examples of PGST of Entangled States on Paths

Theorem (Sousa & Omar, 2014)

With respect to the adjacency matrix, there is pretty good state transfer between $\mathbf{v}$ *and* $\mathbf{v}'$ *on* P_n *if* $n = p - 1, 2p - 1$, *or* $2^k - 1$, *where* p *is a prime and* $k \in \mathbb{N}$.

Proposition

Let $\mathbf{v}$ *be a state of* P_n *and suppose for each* $a \in V(P_n)$ *such that* $\beta_a \neq 0$, *there is pretty good state transfer between* a *and* $n + 1 - a$. *Then there is pretty good state transfer between* $\mathbf{v}$ *and* $\mathbf{v}'$.

Spectral Decomposition and Eigenvalue Support

Fact

If A is symmetric with distinct eigenvalues $\theta_1, \dots, \theta_m$ and if E_i is the orthogonal projection onto the eigenspace belonging to θ_i, then A and has spectral decomposition

$$A = \sum_r \theta_r E_r,$$

and moreover

$$U(t) = \exp(itA) = \sum_r e^{i\theta_r t} E_r.$$

Definition

The eigenvalue support of $\mathbf{v}$, denoted $\Theta_{\mathbf{v}}$, is the set

$$\{\theta_r : E_r \mathbf{v} \neq 0\}.$$

Parity-Entangled States

Definition

Let $\mathbf{v}$ be a state. If $\mathbf{v}$ is such that $\beta_a = 0$ for all even a, we say that $\mathbf{v}$ is an odd-entangled state. If $\mathbf{v}$ is such that $\beta_a = 0$ for all odd a, we say that $\mathbf{v}$ is an even-entangled state. We say $\mathbf{v}$ is a parity-entangled state if it is an odd-entangled state or an even-entangled state.

Lemma

Let $\mathbf{v}$ be a parity-entangled state of P_n. If $\theta_j \notin \Theta_{\mathbf{v}}$, then $\theta_{n+1-j} \notin \Theta_{\mathbf{v}}$.

PGST of Parity-Entangled States on Odd Paths

Theorem

Let $m = 2^t p^s$, where p is an odd prime and $s, t > 0$, and let $\mathbf{v}$ be a parity-entangled state of P_{m-1}. Define

$$S_c := \{\theta_j : 1 \leq j < m,\ j \equiv c \pmod{m/p}\},\ 1 \leq c < m/p;$$
$$S_0 := \{\theta_{m/2}\} = \{0\}.$$

With respect to the adjacency matrix, there is pretty good state transfer in P_{m-1} between $\mathbf{v}$ and $\mathbf{v}'$ if and only if there does not exist S_c with c odd and $S_{c'}$ with c' even such that $S_c \cup S_{c'} \subseteq \Theta_{\mathbf{v}}$.

Example

For P_{11}, there is pretty good state transfer of $\mathbf{v} := \mathbf{e}_1 + \mathbf{e}_3$ to $\mathbf{e}_{11} + \mathbf{e}_9$.

PGST of Parity-Entangled States on Even Paths

Theorem

Let $m = p^s$, where p is an odd prime and $s > 0$, and let $\mathbf{v}$ be a parity-entangled state of P_{m-1}. Define

$$R_c := \{\theta_j : 1 \leq j < m,\ j \equiv c, m/p - c \pmod{m/p}\},\ 1 \leq c \leq m/(2p).$$

With respect to the adjacency matrix, there is pretty good state transfer in P_{m-1} between $\mathbf{v}$ and $\mathbf{v}'$ if and only if there does not exist R_c such that $R_c \subseteq \Theta_{\mathbf{v}}$.

Example

For P_8, there is pretty good state transfer of $\mathbf{v} := \alpha \mathbf{e}_1 + \beta \mathbf{e}_3$ to $\alpha \mathbf{e}_8 + \beta \mathbf{e}_6$,

$$\alpha = \frac{\sin\left(\frac{\pi}{3}\right)}{\sqrt{\sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{9}\right)}}, \qquad \beta = -\frac{\sin\left(\frac{\pi}{9}\right)}{\sqrt{\sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{9}\right)}}$$

Kronecker's Theorem

Let $\theta_1, \ldots, \theta_n$ and $\sigma_1, \ldots, \sigma_n$ be arbitrary real numbers. For an arbitrarily small ϵ, the system of inequalities

$$|\theta_r \tau - \sigma_r| < \epsilon \pmod{2\pi}, \quad (r = 1, \ldots, n),$$

admits a solution for τ if and only if, for integers $\ell_1, \ldots, \ell_n$, if

$$\sum_{r=1}^{n} \ell_r \theta_r = 0,$$

then

$$\sum_{r=1}^{n} \ell_r \sigma_r \equiv 0 \pmod{2\pi}.$$

Applying Kronecker's Theorem

Lemma

Let m be a positive integer of the form $2^t p^s$, where p is an odd prime and $s > 0$, and let $\theta_j = 2\cos(j\pi/m)$, $1 \leq j < m$. If there is a linear combination satisfying

$$\sum_{j=1}^{m-1} \ell_j \theta_j = 0,$$

where each ℓ_j is an integer, then if $1 \leq j \leq m - m/p$, and we let $j := q(m/p) + r$, $0 \leq r < m/p$, we have

$$\ell_j = \begin{cases} \ell_{m-j} + (-1)^q(\ell_{m-m/p+r} - \ell_{m/p-r}), & r \neq 0; \\ \ell_{m-j}, & r = 0. \end{cases}$$

Open Problems

- What time interval is required to ensure state transfer at a particular probability?
- When does perfect state transfer or pretty good state transfer occur on trees?
- What other examples of pretty good state transfer can we find involving entangled states?

Thank you!