Multiple Qubit State Transfer on Paths

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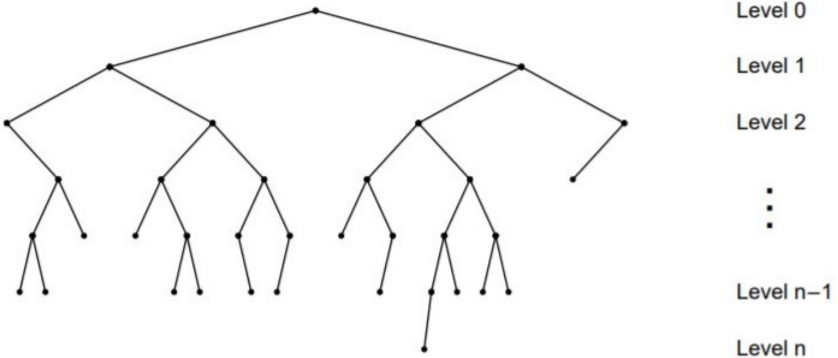
Decision Trees



E. Farhi and S. Gutmann, Quantum computation and decision trees, *Physical Review A* 58 (1998), no. 2, 915.

Continuous Random Walk

Definition

Let X be a graph. The matrix

$$M(t) := \exp(-tL) = \sum_{n\geq 0} \frac{t^n}{n!}(-L)^n$$

is such that the (a, b) entry is the **probability** that a "walker" starting on vertex a is at vertex b after time t.

Definition

A **continuous random walk** is modelled such that in a short time interval δt, the walker leaves the current vertex and moves to one of the adjacent vertices with equal probability.

Continuous Quantum Walk

Definition

Let S be a real symmetric matrix. The **transition operator** given by S is

$$U(t) := \exp(itS) = \sum_{n\geq 0} \frac{(it)^n}{n!} S^n,$$

and defines a **continuous quantum walk**.

For a graph X, the choices for S we will consider are the adjacency matrix A and the Laplacian L.

Mixing Matrix

Definition

The **mixing matrix** given by A is

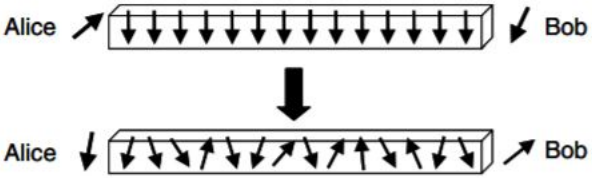
$$M(t) := U(t) \circ \overline{U(t)}$$

and is such that the (a, b) entry is the **probability** that a quantum state starting at vertex a is at vertex b after time t.

Example (P_2)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; U(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}, \; M(t) = \begin{pmatrix} \cos^2(t) & \sin^2(t) \\ \sin^2(t) & \cos^2(t) \end{pmatrix}.$$

Using Spin Chains for Quantum Communication



S. Bose, Quantum communication through an unmodulated spin chain, *Physical Review Letters* 91 (2003), no. 20, 207901.

Perfect State Transfer

Definition

A graph X has **perfect state transfer** between vertices a and b if there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that $U(\tau)\mathbf{e}_a = \gamma \mathbf{e}_b$.

Paths

$$P_2 : U_A(\pi/2)\mathbf{e}_1 = i\mathbf{e}_2, \qquad U_L(\pi/2)\mathbf{e}_1 = \mathbf{e}_2;$$
$$P_3 : U_A(\pi/\sqrt{2})\mathbf{e}_1 = -\mathbf{e}_3.$$

Perfect State Transfer on Paths

Theorem (Christandl et al. 2005)

With respect to the adjacency matrix, P_n has perfect state transfer **between the end vertices** *if and only if $n = 2, 3$.*

Theorem (Stevanović 2011; Godsil 2012)

With respect to the adjacency matrix, P_n has perfect state transfer if and only if $n = 2, 3$.

Theorem (Coutinho & Liu, 2015)

With respect to the Laplacian, if T is a tree, then T has perfect state transfer if and only if $T = P_2$.

Pretty Good State Transfer (PGST)

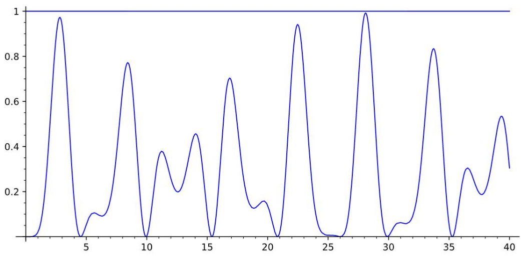
Definition

A graph X has **pretty good state transfer** between vertices a and b if, for every $\epsilon > 0$, there exists $\tau \in \mathbb{R}$ and a complex scalar γ such that

$$||U(\tau)\mathbf{e}_a - \gamma \mathbf{e}_b|| < \epsilon.$$

Example (P_4)

$||U(t)_{1,4}||^2$



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Pretty Good State Transfer on Paths (End Vertices)

Theorem (Godsil, Kirkland, Severini, Smith; 2012)

With respect to the adjacency matrix, there is pretty good state transfer between the end vertices of P_n if and only if:

1. $n = 2^t - 1$, $t \in \mathbb{Z}_+$;
2. $n = p - 1$, p *a prime; or,*
3. $n = 2p - 1$, p *a prime.*

Moreover, when pretty good state transfer occurs between the end vertices of P_n, then it occurs between vertices a and $n+1-a$ for all $a \neq (n+1)/2$.

Theorem (Banchi, Coutinho, Godsil, Severini; 2017)

With respect to the Laplacian, there is pretty good state transfer between the end vertices of P_n if and only if n is a power of 2. Moreover, when pretty good state transfer occurs between the end vertices of P_n, then it occurs between vertices a and $n+1-a$ for all $a \neq (n+1)/2$.

Spectral Decomposition and Eigenvalue Support

Fact

If A is symmetric with distinct eigenvalues $\theta_1, \dots, \theta_m$, and if E_r is the orthogonal projection onto the eigenspace belonging to θ_r, then A and has **spectral decomposition**

$$A = \sum_r \theta_r E_r,$$

and moreover

$$U(t) = \exp(itA) = \sum_r e^{i\theta_r t} E_r.$$

Definition

If $a \in V(X)$, then the **eigenvalue support** of a, denoted Θ_a, is the set

$$\{\theta_r : E_r \mathbf{e}_a \neq 0\}.$$

Demonstrating Pretty Good State Transfer on Paths

$$\|U(\tau)\mathbf{e}_a - \gamma\mathbf{e}_b\| < \epsilon$$

$$\left\| \sum_{\theta_r \in \Theta_a} \exp(i\tau\theta_r)(E_r)\mathbf{e}_a - \gamma\mathbf{e}_b \right\| < \epsilon$$

$$\left\| \sum_{\theta_r \in \Theta_a} \left(\exp(i\tau\theta_r) - (-1)^r \exp\left(i\delta\right)\right) E_r\mathbf{e}_a \right\| < \epsilon$$

$$|\tau\theta_r - (\delta + \sigma_r\pi)| < \epsilon' \pmod{2\pi}, \quad (r : \theta_r \in \Theta_a), \qquad (*)$$

where σ_r is 0 if r is odd and 1 if r is even.

Kronecker's Theorem

Let $\theta_1, \ldots, \theta_n$ and $\sigma_1, \ldots, \sigma_n$ be arbitrary real numbers. For an arbitrarily small ϵ, the system of inequalities

$$|\theta_r \tau - \sigma_r| < \epsilon \pmod{2\pi}, \quad (r = 1, \ldots, n),$$

admits a solution for τ if and only if, for integers $\ell_1, \ldots, \ell_n$, if

$$\sum_{r=1}^{n} \ell_r \theta_r = 0,$$

then

$$\sum_{r=1}^{n} \ell_r \sigma_r \equiv 0 \pmod{2\pi}.$$

Pretty Good State Transfer of Internal Vertices on P_{11}

- Vertices 2 and 10 are strongly cospectral.
- We observe that $\theta_6 \notin \Theta_2$ and $\theta_r = -\theta_{12-r}$.
- Choosing $\delta = 0$, it suffices by $(*)$ to demonstrate

$$|\tau\theta_r - \sigma_r\pi| < \epsilon \pmod{2\pi}, \quad \sigma_r = (1 + (-1)^r)/2 \qquad (1 \leq r \leq 5).$$

- By Kronecker's Theorem, it suffices to demonstrate that for integers $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$, we have

$$\sum_{r=1}^{5} \ell_r\theta_r = 0 \implies \ell_2 + \ell_4 \equiv 0 \pmod{2\pi},$$

which is easily verified since θ_2, θ_4 are rationally independent of $\{\theta_1, \theta_3, \theta_5\}$.

Pretty Good State Transfer with Internal Vertices of Paths

Theorem (Coutinho, Guo, van Bommel; 2017)

Given any odd prime p and positive integer t, there is pretty good state transfer in $P_{2^t p-1}$ between vertices a and $2^t p - a$, whenever $2^{t-1} \mid a$.

Theorem

There is pretty good state transfer on P_n between vertices a and b if and only if $a + b = n + 1r$:

- $n = 2^t - 1$, $t \in \mathbb{Z}_+$;
- $n = p - 1$, p a prime; or,
- $n = 2^t p - 1$, $t \in \mathbb{N}$, p an odd prime, and $2^{t-1} \mid a$.

Theorem

With respect to the Laplacian, there is pretty good state transfer on P_n between vertices a and b if and only if $a + b = n + 1$ and n is a power of 2.

Extending Pretty Good State Transfer to Multiple Qubits

Definition

A graph X has **pretty good state transfer** of the state $\mathbf{v}$, given by

$$\sum_{j=1}^{m} \beta_j \mathbf{e}_j, \qquad \sum_{j=1}^{m} |\beta_j|^2 = 1,$$

to the state $\mathbf{w}$ if for every $\epsilon > 0$, there exist $\tau \in \mathbb{R}$ and a complex number γ with $|\gamma| = 1$, such that

$$\|U(\tau)\mathbf{v} - \gamma\mathbf{w}\| < \epsilon.$$

Proposition

For all $\mathbf{v}$ and τ, if $\mathbf{w} = U(\tau)\mathbf{v}$, then there is pretty good state transfer between $\mathbf{v}$ and $\mathbf{w}$.

Aiming for Symmetry

Problem

We are interested in pretty good state transfer in X between states $\mathbf{v}$ and $\mathbf{v}^\sigma$, where σ is an automorphism of X and $\mathbf{v}^\sigma$ is given by

$$\mathbf{v}^\sigma = \sum_{x \in V(X)} \beta_x \mathbf{e}_{\sigma(x)}.$$

On P_n, we assume $\sigma(x) = n + 1 - x$.

Proposition

Let $\mathbf{v}$ be a state of P_n and suppose for each $a \in V(P_n)$ such that $\beta_a \neq 0$, there is pretty good state transfer between a and $n+1-a$. Then there is pretty good state transfer between $\mathbf{v}$ and $\mathbf{v}^\sigma$.

Pretty Good State Transfer of Multiple Qubits on P_{11}

- Consider states $\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_3)$ and $\mathbf{v}^\sigma = \frac{1}{\sqrt{2}}(\mathbf{e}_{11} + \mathbf{e}_9)$.
- We observe that $E_6\mathbf{v} = 0$.
- Analogously to the single qubit case, we see that

$$\|U(\tau)\mathbf{v} - \gamma\mathbf{v}^\sigma\| < \epsilon$$

  implies it suffices to demonstrate

$$|\tau\theta_r - \sigma_r\pi| < \epsilon \pmod{2\pi}, \quad \sigma_r = (1 + (-1)^r)/2 \qquad (1 \le r \le 5).$$

- By Kronecker's Theorem, it again suffices to demonstrate that for integers $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$, we have

$$\sum_{r=1}^{5} \ell_r\theta_r = 0 \implies \ell_2 + \ell_4 \equiv 0 \pmod{2\pi},$$

  which is easily verified since θ_2, θ_4 are rationally independent of $\{\theta_1, \theta_3, \theta_5\}$.

Parity States & Eigenvalue Support

Definition

Let $\mathbf{v}$ be a state. If $\mathbf{v}$ is such that $\beta_a = 0$ for all even a, we say that $\mathbf{v}$ is an **odd state**. If $\mathbf{v}$ is such that $\beta_a = 0$ for all odd a, we say that $\mathbf{v}$ is an **even state**. We say $\mathbf{v}$ is a **parity state** if it is an odd state or an even state.

Definition

The **eigenvalue support** of $\mathbf{v}$, denoted $\Theta_{\mathbf{v}}$, is the set

$$\{\theta_r : E_r\mathbf{v} \neq 0\}.$$

Lemma

Let $\mathbf{v}$ be a parity state of P_n. If $\theta_j \notin \Theta_{\mathbf{v}}$, then $\theta_{n+1-j} \notin \Theta_{\mathbf{v}}$.

PGST of Parity States on Odd Paths

Theorem

Let $m = 2^t p^s$, *where* p *is an odd prime and* $s, t > 0$, *and let* $\mathbf{v}$ *be a parity state of* P_{m-1}. *Define*

$$S_c := \{\theta_j : 1 \leq j < m,\ j \equiv c \pmod{m/p}\},\ 1 \leq c < m/p;$$
$$S_0 := \{\theta_{m/2}\} = \{0\}.$$

With respect to the adjacency matrix, there is pretty good state transfer in P_{m-1} *between* $\mathbf{v}$ *and* $\mathbf{v}^\sigma$ *if and only if there does not exist* S_c *with* c *odd and* $S_{c'}$ *with* c' *even such that* $S_c \cup S_{c'} \subseteq \Theta_{\mathbf{v}}$.

PGST of Two Qubit Parity States on Odd Paths

Corollary

Given any odd prime p and positive integer $t \geq 2$, there is pretty good state transfer in $P_{2^t p-1}$ between states

$$\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{e}_a + \alpha \mathbf{e}_b) \quad \textit{and} \quad \mathbf{v}^\sigma = \frac{1}{\sqrt{2}}(\mathbf{e}_{2^t p-a} + \alpha \mathbf{e}_{2^t p-b})$$

whenever $a \neq b$, $\alpha = \pm 1$, and $a + \alpha b \equiv 0 \pmod{2^t}$.

PGST of Parity States on Even Paths

Theorem

Let $m = p^s$, where p is an odd prime and $s > 0$, and let $\mathbf{v}$ be a parity state of P_{m-1}. Define

$$R_c := \{\theta_j : 1 \leq j < m,\ j \equiv c,\ m/p - c \pmod{m/p}\},\ 1 \leq c \leq m/(2p).$$

With respect to the adjacency matrix, there is pretty good state transfer in P_{m-1} between $\mathbf{v}$ and $\mathbf{v}^\sigma$ if and only if there does not exist R_c such that $R_c \subseteq \Theta_{\mathbf{v}}$.

Example

For P_8, there is pretty good state transfer of $\alpha\mathbf{e}_1 + \beta\mathbf{e}_3$ to $\alpha\mathbf{e}_8 + \beta\mathbf{e}_6$, where

$$\alpha = \frac{\sin\left(\frac{\pi}{3}\right)}{\sqrt{\sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{9}\right)}}, \qquad \beta = -\frac{\sin\left(\frac{\pi}{9}\right)}{\sqrt{\sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{9}\right)}}$$

Applying Kronecker's Theorem

Lemma

Let m be a positive integer of the form $2^t p^s$, where p is an odd prime and $s > 0$, and let $\theta_j = 2\cos(j\pi/m)$, $1 \leq j < m$. If there is a linear combination satisfying

$$\sum_{j=1}^{m-1} \ell_j \theta_j = 0,$$

where each ℓ_j is an integer, then if $1 \leq j \leq m - m/p$, and we let $j := q(m/p) + r$, $0 \leq r < m/p$, we have

$$\ell_j = \begin{cases} \ell_{m-j} + (-1)^q(\ell_{m-m/p+r} - \ell_{m/p-r}), & r \neq 0; \\ \ell_{m-j}, & r = 0. \end{cases}$$

Future Directions

- What time interval is required to ensure state transfer with a particular probability?
- When does perfect state transfer or pretty good state transfer occur on trees?
- What is the characterization of eigenvalue supports that permit pretty good state transfer of multiple qubit states on paths?
- Are there other interesting forms of multiple qubit state transfer that could be considered?

Thank you!