Vertex Colouring Problems with Restrictions Given by Faces

By

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to my parents

Abstract

A vertex colouring φ of a graph is an assignment of a colour to each of the vertices. A vertex colouring is proper if no two adjacent vertices are assigned the same colour. The study of proper ℓ-residue colourings is established with upper bounds on the proper ℓ-residue chromatic number for all plane graphs and all toroidal graphs. Proper ℓ-residue colourings are colourings of graphs embedded in a surface that are proper and satisfy the condition that for each face and each colour, the number of times the colour is incident with the face is zero or is congruent to $1 \pmod{\ell}$. The development of vertex colouring problems is first examined, from the foundation with the Four Colour Theorem, to the restriction to cyclic colourings, and to the more relaxed constraints of parity vertex colourings.

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Chapter 1

Introduction

Since the middle of the nineteeth century, the problem of colouring a graph has been examined. The first and simplest problem of this form was that of determining how many colours are required to colour a map such that adjacent regions are assigned distinct colours; this problem is known as the Map-Colouring Problem. Throughout this work, the problems studied are variations of vertex colourings of plane graphs that are in some way restricted by the faces of the graph. A *vertex colouring* φ of a graph G is an assignment of colours to the vertices of G. Other definitions standard to graph theory will be outlined in the next section.

The main goal for each of these problems is to determine an upper bound on the number of colours required to colour every plane graph. In Chapter 2, the problem of proper colourings is considered, which is unrestricted by the faces of the graph. Establishing the validity of the Four Colour Conjecture, that every plane graph can be properly coloured with at most four colours, is the major focus of the problem. The history of the problem is outlined and the Discharging Method is introduced, which was the method used to prove the Four Colour Theorem.

In Chapter 3, the problem of cyclic colourings is considered, in which two vertices cannot be assigned the same colour if they are incident with a common face. The progression of results on the upper bound for both connected and 3-conencted graphs are outlined, the conjectures of the best possible results are stated, and graphs meeting the bounds of these conjectures are given. In the final section, other perspectives used in considering this problem are described.

A recently introduced variation is examined in Chapter 4. Problems of parity vertex colourings restrict the number of times a colour can be used on each face, and may also require that the colourings be proper. Results are outlined for both weak parity vertex colourings, in which at least one colour must be used an odd number of times on each face, and strong parity vertex colourings, in which every colour used on a face must occur an odd number of times. For proper strong parity vertex colourings, which have the additional requirement that such a colouring be proper, a graph is presented which improves the largest known proper strong parity chromatic number from 10 to 12. Upper bounds on the proper strong parity chromatic number for certain classes of graphs are outlined in the final section.

In Chapter 5, a vertex colouring is introduced which extends the concept of parity vertex colouring by using an arbitrary arithmetic sequence whose first term is 1 as the set of values a colour is allowed to occur on a face. This vertex colouring is referred to a proper ℓ-residue colouring, where ℓ is the difference of the arithmetic progression. An upper bound is determined for the planar case by extending the proof of the upper bound of the proper strong parity chromatic number, and then adapted further to apply to toroidal graphs. A lower bound is given by considering a particular graph.

1.1 Preliminaries

A *graph* G consists of a vertex set $V(G)$, an edge set $E(G)$, and a rule that assigns each edge to two (not necessarily distinct) vertices which are called the endpoints of the edge. A graph can be represented as a diagram by drawing each vertex as a point and each edge as a line between its two endpoints. For the graph G, a vertex is said to be *incident* with an edge if the vertex is an endpoint of the edge. Two vertices are said to be *adjacent* if they are incident with a common edge. The *open neighbourhood* of a vertex v, denoted $N_G(v)$, is the set of vertices adjacent to v.

An edge is called a *loop* if its endpoints are the same vertex. All graphs considered will be loopless. A loopless graph is called *simple* if there is at most one edge between each pair of vertices, otherwise it is called a *multigraph*. If two edges e_1 and e_2 share endpoints, they are said to be *parallel*.

The *degree* of a vertex v, denoted $\deg(v)$ or $d(v)$, is the number of edges incident with v. A vertex of degree d is called a *d-vertex*. The minimum degree of a graph G is the minimum degree of the vertices of G and is denoted $\delta(G)$. The maximum degree of a graph G is the maximum degree of the vertices of G and is denoted $\Delta(G)$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ where each edge in $E(H)$ has both endpoints in $V(H)$. If V' is a subset of $V(G)$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set contains all the edges that have both their endpoints in V'.

A graph G is *connected* if there is a sequence of adjacent vertices between every pair of vertices, otherwise it is *disconnected*. A *vertex cut* S is a subset of the vertices such that the graph $G[V(G) - S]$ is disconnected. The *connectivity* of a graph, denoted $\kappa(G)$, is the minimum cardinality of a vertex cut. A graph is *k-connected* if $k \leq \kappa(G)$.

A graph G is *planar* if it can be drawn in the plane without edges crossing; a *plane graph* is a graph together with such an embedding. Similarly, a graph G is *toroidal* if it can be embedded on the torus without edges crossing. The faces of an embedded graph are the regions bounded by the edges of G. The set of faces is denoted $F(G)$. A face is *incident* with an edge if it is bounded by that edge and is *incident* with a vertex if the face and vertex are incident with a common edge. Two faces are *adjacent* if they are incident with a common edge. The set of faces incident with a vertex v is denoted $F_G(v)$. For a face f, the set $E_G(f)$ is the set of edges incident with f, the set $V_G(f)$ is the set of vertices incident with f, and $\deg(f)$ is the number of vertices incident with f. A *facial walk* of f is a shortest walk containing each edge of $E(f)$. A face of degree d is called a d-face. If f is a 2-face, it is called a *digon*. The maximum facial degree of a graph G is the maximum degree of the faces of G and is denoted $\Delta^*(G)$. If each face is a 3-face, the graph is called a *triangulation*. The *dual* of G, denoted G^*, is constructed by a set of vertices $V(G^*)$ corresponding to the faces of G and a set of edges $E(G^*)$ corresponding to the edges of G with an edge between vertices of G^* if their corresponding faces in G are adjacent. If it is clear which graph is being discussed, the subscript G is removed from the above notation.

For all graphs, there is a relation between the number of edges and both the sum of the degrees of the vertices and the sum of the degrees of the faces. Since each edge is incident with two vertices (or the same vertex twice) and with two faces (or the same face twice), it follows that $\sum \deg(v) = \sum \deg(f) = 2|E|$. In addition, if G is embedded in a surface, then Euler's Formula provides a relation between the number of vertices, the number of edges, the number of faces, and the genus, g, of the surface (0 for the plane and 1 for the torus).

Theorem 1.1.1. (Euler's Formula).

$$|V| - |E| + |F| = 2 - 2g$$

These relations are essential and will be used throughout.

Chapter 2

The Four Color Theorem

2.1 Map-Colouring Problem

The first graph colouring problem was developed from the consideration of geographical maps. To better indicate the boundaries between distinct regions, adjacent regions were coloured with different colours. Regions were considered adjacent if they were incident with a common edge, but not if they were only incident at a common point or points.

The problem of determining the maximum number of colours required for all planar maps was first stated by Francis Guthrie in 1852. The problem was posed to Augustus De Morgan, who wrote the following discussion in a letter to William Rowan Hamiltion:

> "*A student of mine asked me today to give him a reason for a fact which I did not know was a fact – and do not yet. He says if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary are differently coloured – four colours may be wanted, but not more.*" [15]

Kempe [37] stated that De Morgan also noted that it was long known to map-makers from experience that four colours will always suffice, but that this experience was likely confined to relatively simple cases. Arthur Cayley brought the problem to the attention of the London Mathematical Society in 1878. The following year, Cayley [18] published a short description of the problem, providing an example requiring four colours, shown in Figure 2.1.1, and noting the difficulty in proving the conjecture is that colouring a new region added to a map may require the complete recolouring of the entire map.

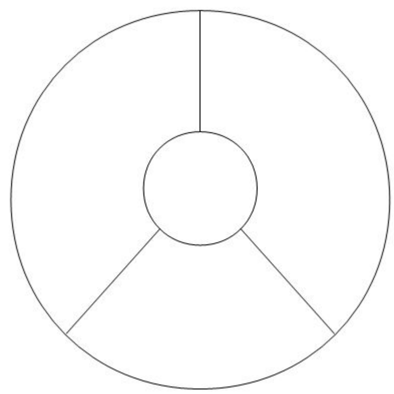


Figure 2.1.1: A planar map which requires four colours

Kempe [37] was thought to have proved the Four Colour Theorem in 1879, but his proof was later shown to be incorrect by Heawood [29]. However, many of the ideas behind Kempe's attempt of a proof were correct and the following ideas were applied in other attempts to prove the theorem. A derivation from Euler's formula showed that every planar map contains regions with at most five neighbours. If a map contains a region with two or three neighbours, it is a simple matter to remove this region, colour the remainder of the map, and assign the region removed a colour not used in an adjacent region. For regions with four or five neighbours, Kempe chains were introduced. Given a colouring of a graph c using at least two colours c_1 and c_2, and a vertex v coloured c_1, the (c_1, c_2)-Kempe chain containing v is the maximal connected subset of regions that are coloured either c_1 or c_2 and include v. If, for a specific (c_1, c_2)-Kempe chain, the colours c_1 and c_2 are interchanged, then c is still a proper colouring, as by definition this cannot cause two regions with the same colour to be adjacent. To deal with a region with four neighbours, the following claim was applied.

Claim 2.1.1. *If a map contains a region with four neighbours, and this region is removed, then there exists a colouring of the resulting map such that the four neighbours of the removed region are coloured with at most three distinct colours.*

Proof. Suppose c is a colouring of the map and each of the neighbours of the removed region is assigned a different colour. Label the regions clockwise R_1, R_2, R_3, and R_4, and their assigned colours c_1, c_2, c_3, and c_4 respectively. If R_1 and R_3 do not belong to the same (c_1, c_3)-Kempe chain, then the colours on the chain containing R_3 can be interchanged, resulting in a colouring using three colours for the four regions. If R_1 and R_3 do belong to the same chain, then R_2 and R_4 cannot belong to the same (c_2, c_4)-Kempe chain, as R_2 and R_4 are on opposite sides of the (c_1, c_3)-Kempe chain. Thus, the colours on the chain containing R_4 can be interchanged, resulting in a colouring using three colours for the four regions. $\square$

A similar method was proposed for a region with five neighbours, but multiple colour interchanges were required and the effects of these multiple interchanges were not fully considered and the result was flawed. Kempe chains did, however, prove two useful facts, as shown by Heawood [29]. The first is that five colours are sufficient to colour any planar map, and the second is that any counterexample that cannot be coloured with four colours cannot contain regions with two, three, or four neighbours, and hence must contain regions with five neighbours. Further results that led to the development of the Discharging Method as an approach to prove the Four Colour Theorem are examined in the next section.

2.2 The Discharging Method

First, the problem of determining a proper colouring of a graph is formally introduced in modern terminology and in the typical form of colouring the vertices of the graph instead of the faces or regions, that is, colouring the planar dual of the graph.

Problem 2.2.1. *A vertex colouring φ is a* proper colouring *of a plane graph G if no two adjacent vertices are assigned the same colour. The* chromatic number, *$\chi(G)$, is the minimum number of colours used in a colouring of G.*

Then the Four Colour Conjecture can be formally presented as follows.

Conjecture 2.2.2. *If G is a plane graph, $\chi(G) \leq 4$.*

One method considered to prove this conjecture was to show that there exists no minimum counterexample to the conjecture that would require more than four colours for a proper colouring. The first tool used in this approach is reducibility. That is, showing that each graph with a certain configuration can be reduced to a graph with fewer vertices whose proper colouring can be extended to the original graph. This shows that the graph cannot be a minimum counterexample. Birkhoff [6] noted the following four reductions in terms of colouring the regions of a map.

Observation 2.2.3. *[6]*

- *If more than three boundary lines meet at any vertex of a map, the coloring of the map may be reduced to the coloring of a map of fewer regions.*

- *If any region of a map is multiply-connected (that is, the region divides the map into multiple components), the coloring of the map may be reduced to the coloring of maps of fewer regions.*

- *If two or three regions of a map form a multiply-connected region, the coloring of the map may be reduced to the coloring of maps of fewer regions.*

- *If the map contains any 1-, 2-, 3- or 4-sided regions, the coloring of the map may be reduced to the coloring of a map with fewer regions.*

The above reductions then reduce the map to either a single region, or a map in which: exactly three boundary lines meet at every vertex, no set of at most three regions form a multiply-connected region, and every region has at least five sides [6]. These conditions are equivalent to that of the dual graph being a triangulation, 4-connected, and having minimum degree of 5. Birkhoff [6] also introduced the concept of a *ring* of regions, that is, a cycle of regions $\mu_1, \mu_2, \ldots, \mu_n$ such that each region is adjacent to the one before it and after it in the cycle, but to no others in the cycle. It was shown that rings of 4 regions are reducible, and rings of 5 regions are reducible if there are at least two regions inside the ring and at least two regions outside the ring.

The second tool used in this approach is unavoidability. If there exists a configuration that must appear in every planar graph, such a configuration is said to be unavoidable, and more generally, if there exists a set of configurations such that at least one of them must appear in every planar graph, then such a set of configurations is said to be unavoidable. As described previously, every planar map without regions of size two, three, or four must contain a region of size five, thus a region of size five is unavoidable. Stronger results were proven by Wernicke [46], who showed that such a map must contain a region of size five adjacent to a region of size five or six, and Franklin [26], who showed that such a map must contain either two adjacent regions of size five or a region of size five adjacent to two regions of size six. Lebesgue [39] described a large collection of configurations, at least one of which must be contained in such a map.

Unavoidability is typically proven through the use of the Discharging Method. To apply the Discharging Method, each vertex and each face are assigned an initial charge dependent on its degree and in such a way that the sum of charges can be easily computed using Euler's formula. Then, through a set of discharging rules, the charges on the vertices and faces are redistributed by transferring charges to adjacent vertices or faces so that the sum of the charges does not change. Depending on the initial charges, if the sum of the charges is positive (respectively negative), then by assuming no configuration in a set occurs, it is shown that each vertex and each face has a nonpositive (respectively nonnegative) charge. This contradiction proves that the set of configurations is unavoidable.

Alternatively, the Discharging Method can be used to find an unavoidable set by letting the sum of the charges be positive (respectively negative) and determining which configurations are required because of their positive (respectively negative) charges. Two examples are shown below; the first does not use the discharging step.

Claim 2.2.4. *Every planar triangulation G without 2-vertices, 3-vertices, or 4-vertices contains a 5-vertex.*

Proof. Each vertex is given $6-\deg(v)$ units of charge and each face is given $6-2\deg(f) = 0$ units of charge. The total charge is:

$$\sum_{v\in V}(6-\deg(v))+\sum_{f\in F}(6-2\deg(f)) = 6|V|-2|E|+6|F|-4|E| = 6(|V|-|E|+|F|) = 12.$$

The only vertices with positive charge are 5-vertices and all faces have charge 0. Thus G contains at least one 5-vertex. □

Claim 2.2.5. *[46] If a planar triangulation G has minimum degree 5, then it contains a 5-vertex adjacent to a 5-vertex or a 6-vertex.*

Proof. Each vertex is given $6-\deg(v)$ units of charge and each face is given $6-2\deg(f) = 0$ units of charge. As above, the sum of the charges is 12. Apply the discharging rule that each 5-vertex gives a charge of $\frac{1}{5}$ to each neighbour. After discharging, the total charge has not changed, so there exists a vertex v with positive charge. Then $\deg(v) \leq 7$, since if $\deg(v) \geq 8$, the charge on v after applying the discharging rule would be at most $6 - \deg(v) + \frac{1}{5}\deg(v) = 6 - \frac{4}{5}\deg(v) \leq 6 - \frac{32}{5} = -\frac{2}{5}$. If v is a 5-vertex, then since v discharged all of its initial charge to its neighbours, it must have received charge from an adjacent 5-vertex. If v is a 6-vertex, then since the initial charge of v was 0, v must have received charge from an adjacent 5-vertex. If v is a 7-vertex, then since the initial charge of v was -1, v must have received charge from at least 6 adjacent 5-vertices. Since the graph is a triangulation, two of these 5-vertices must be adjacent. In all cases the claim is established. □

The Discharging Method is also used to show a minimum counterexample to a theorem does not exist. After applying a set of well designed discharging rules, computing the charge for each vertex and each face based on the assumption that the graph contains no reducible configuration leads to a contradiction of the sum of the charges. Hence, the reducible set is also unavoidable, which proves the theorem. (This technique is used in Chapter 5.)

Heesch [31] was the first to combine the ideas of reducibility and unavoidability. He examined several special cases of triangulations and proved each contained a reducible configuration. Extending this approach, and with the aid of a computer, Appel and Haken [2, 3, 4] proved the Four Colour Theorem in 1976. Their method required about 2000 configurations in the unavoidable reducible set and around 300 discharging rules.

Initially, the approach of Appel and Haken was not fully accepted because a portion of the proof involved a computer, the first major theorem to do so, and could not be verified by hand. In addition, the remaining portion of the proof was tedious and had not been independently verified. Robertson et al. [43] developed an independent proof of the Four Colour Theorem in 1996, leading to a simpler approach than that of Appel and Haken. The authors constructed a set of 633 configurations that was both reducible and unavoidable, through the use of only 32 discharging rules. This independent verification confirmed the Four Colour Theorem. This proof also led to a quadratic time algorithm to find a 4-colouring of a planar graph.

The Discharging Method is also used to prove results regarding other vertex colouring problems, which are considered in the following chapters.

Chapter 3

Cyclic Colourings

3.1 Introduction

The focus of this chapter is the problem of cyclic colourings. Cyclic colourings were introduced by Ore and Plummer [41] in 1969, and, like the problem of proper vertex colourings, the problem was originally formulated in terms of the colouring of the faces, that is, in its planar dual form. Cyclic colouring is defined in terms of colouring the vertices by Definition 3.1.1.

Definition 3.1.1. *A vertex colouring ϕ is a* cyclic colouring *of a connected plane graph G if no two vertices incident with the same face are assigned the same colour. The* cyclic chromatic number, *$\chi_c(G)$, is the minimum number of colours used in a cyclic colouring of G.*

In terms of the dual definition, this is a very natural extension of the original problem of colouring maps. No two regions which share a point on their boundaries are assigned the same colour.

Two main problems regarding the cyclic chromatic number are considered: the upper bound of the cyclic chromatic number of 2-connected (or connected) plane graphs and the upper bound of the cyclic chromatic number of 3-connected plane graphs. As the progress of the results of these problems are at times intertwined, they will be examined together in the next section.

3.2 Upper Bounds

It is clear that the cyclic chromatic number is well-defined by assigning a distinct colour to each vertex of the graph. It is also clear that any cyclic colouring of G requires at least $\Delta^*(G)$ colours, where $\Delta^*(G)$ is the size of the largest face. Ore and Plummer [41] showed that if G is 2-connected, then $\chi_c(G) \leq 2\Delta^*(G)$. Borodin [7, 8] announced the improvement $\chi_c(G) \leq 2\Delta^*(G) - 1$, and implicitly stated the following conjecture, which is given as a formal problem by Jensen and Toft [35].

Conjecture 3.2.1. *If G is a 2-connected plane graph, $\chi_c(G) \leq \frac{3}{2}\Delta^*(G)$.*

Such an upper bound would be the best possible, as shown by the following example of Plummer and Toft [42]. The triangular prism D_3, depicted in Figure 3.2.1, with each of the three edges between the two triangles replaced by the same length path gives an infinite family of graphs with $\chi_c(G) \leq \frac{3}{2}\Delta^*(G)$.

Plummer and Toft [42] introduced the variant considering 3-connected plane graphs, and determined the following general upper bound, as well as improvements on the upper bound if $\Delta^*(G)$ is sufficiently small or sufficiently large.

Theorem 3.2.2. *[42] Let G be a 3-connected plane graph with maximum face size $\Delta^*(G)$. Then $\chi_c(G) \leq \Delta^*(G) + 9$.*

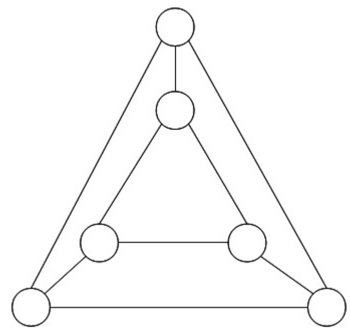


Figure 3.2.1: Graph D_3

Theorem 3.2.3. *[42] Let G be a 3-connected plane graph. Then*

(i) $\Delta^*(G) \leq 10 \rightarrow \chi_c(G) \leq \Delta^*(G) + 8 \leq 18$

(ii) $\Delta^*(G) \leq 9 \rightarrow \chi_c(G) \leq \Delta^*(G) + 7 \leq 16$

(iii) $\Delta^*(G) \leq 8 \rightarrow \chi_c(G) \leq \Delta^*(G) + 6 \leq 14$

(iv) $\Delta^*(G) \leq 7 \rightarrow \chi_c(G) \leq \Delta^*(G) + 6 \leq 13$

Theorem 3.2.4. *[42] Let G be a 3-connected plane graph. Then*

(i) $\Delta^*(G) \geq 14 \rightarrow \chi_c(G) \leq \Delta^*(G) + 8$

(ii) $\Delta^*(G) \geq 15 \rightarrow \chi_c(G) \leq \Delta^*(G) + 7$

(iii) $\Delta^*(G) \geq 18 \rightarrow \chi_c(G) \leq \Delta^*(G) + 6$

(iv) $\Delta^*(G) \geq 24 \rightarrow \chi_c(G) \leq \Delta^*(G) + 5$

(v) $\Delta^*(G) \geq 42 \rightarrow \chi_c(G) \leq \Delta^*(G) + 4$

It was also noted by Plummer and Toft [42] that if $\Delta^*(G) = 3$, then $\chi_c(G) \leq 4 = \Delta^*(G) + 1$ by the Four Colour Theorem. Borodin [7] showed that if $\Delta^*(G) = 4$,

$\chi_c(G) \leq 6 = \Delta^*(G) + 2$. Based on the triangular prism graph, Plummer and Toft [42] illustrated an infinite family of 3-connected planar graphs for which $\chi_c(G) = \Delta^*(G) + 2$, shown in Figure 3.2.2. However, every graph in this family has $\Delta^*(G) = 4$. This observation led to the following conjecture.

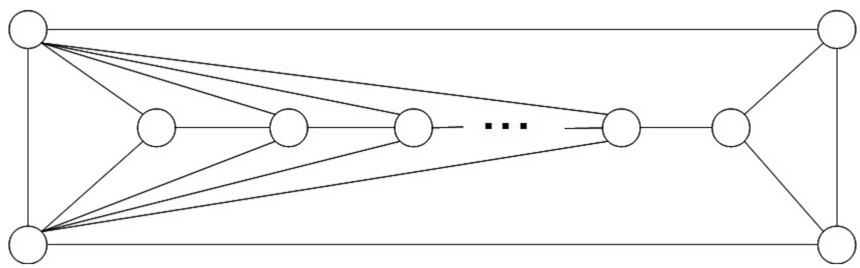


Figure 3.2.2: Family of 3-connected planar graphs such that $\chi_c(G) = \Delta^*(G) + 2$

Conjecture 3.2.5. *If G is a 3-connected plane graph, $\chi_c(G) \leq \Delta^*(G) + 2$.*

Borodin [9] improved the bounds of Theorem 3.2.4, by showing that $\chi_c(G) \leq \Delta^*(G) + 3$ for $\Delta^*(G) \geq 24$. Borodin also proved the following upper bounds for connected plane graphs.

Theorem 3.2.6. *[9] If G is a connected plane graph, then*

$$\chi_c(G) = \begin{cases} 9, & \Delta^*(G) = 5 \\ 11, & \Delta^*(G) = 6 \\ 12, & \Delta^*(G) = 7 \\ 2\Delta^*(G) - 3, & \Delta^*(G) \geq 8. \end{cases}$$

This result was further improved by Borodin, Sanders, and Zhao [12]. This general bound improves the previous results for $\Delta^*(G) = 5$, $\Delta^*(G) = 6$, and $\Delta^*(G) \geq 16$.

Corollary 3.2.7. *[12] If G is a connected plane graph, then $\chi_c(G) \leq \left\lfloor \frac{9}{5}\Delta^*(G) \right\rfloor$, and if $\Delta^*(G) \in \{3, 4, 5, 8, 9, 10\}$, then $\chi_c(G) \leq \left\lfloor \frac{9}{5}\Delta^*(G) \right\rfloor - 1$.*

Improvements to the upper bound of 3-connected plane graphs continued at a rapid pace. Borodin and Woodall [14] proved that $\chi_c(G) \leq \Delta^*(G) + 2$ for $\Delta^*(G) \geq 61$ and $\chi_c(G) \leq \Delta^*(G)+1$ for $\Delta^*(G) \geq 122$. These results were improved by Enomoto, Horňák, and Jendrol' [25] to $\chi_c(G) \leq \Delta^*(G) + 1$ for $\Delta^*(G) \geq 60$. The wheel graphs W_n, formed by joining a vertex to each vertex of a cycle of length $n-1$, demonstrate that $\Delta^*(G)+1$ is the best bound possible. Horňák and Jendrol' [32, 33] showed $\chi_c(G) \leq \Delta^*(G) + 2$ for $\Delta^*(G) \geq 40$, and subsequently for $\Delta^*(G) \geq 24$. Morita [40] improved this bound to $\Delta^*(G) \geq 22$, and also showed that $\chi_c(G) \leq \Delta^*(G) + 3$ for $\Delta^*(G) \geq 20$. Finally, Borodin and Woodall [13] suggested a result based on the minimum cyclic degree of a 3-connected plane graph. The cyclic degree of a vertex v is the number of vertices incident with a face also incident with v. The result was left to the reader as a corollary based on the argument of Plummer and Toft [42] in their proof of Theorem 3.2.2.

Corollary 3.2.8. *[13] If G is a 3-connected plane graph with maximum facial degree $\Delta^*(G)$, then $\chi_c(G) \leq M(\Delta^*(G)) + 1$, where $M(\Delta^*(G))$ is an upper bound on the minimum cyclic degree and is given in the following table.*

$\Delta^*(G)$	3	4	5	6	7	8	9	10	11	$12-16$	$17-20$	≥ 21
$M(\Delta^*(G))$	5	7	9	11	12	13	15	17	18	19	$k+3$	$k+2$

Table 3.1: Upper bound on the minimum cyclic degree in terms of the maximum face degree

Currently, the best general upper bound for connected plane graphs is given by the following result shown by Sanders and Zhao [44].

Theorem 3.2.9. *[44] Every plane graph G has a cyclic $\lceil \frac{5}{3}\Delta^*(G) \rceil$-coloring.*

The previous results verified the above theorem for $\Delta^*(G) \leq 11$, and Sanders and Zhao completed the verification for $\Delta^*(G) \geq 12$. Recently, Hebdige and Král' [30] proved the following result, resolving another case of the conjecture.

Theorem 3.2.10. *[30] Every plane graph with maximum face size at most six has a cyclic coloring using at most nine colours.*

The complete results on connected plane graphs can therefore be summarized as follows.

Corollary 3.2.11. *If G is a connected plane graph, then $\chi_c(G) \leq \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil$, and if $\Delta^*(G) \in \{3, 4, 5, 6, 8\}$, then $\chi_c(G) \leq \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil - 1$. Moreover, if $\Delta^*(G) \in \{3, 4, 6\}$, the upper bound is sharp.*

The best general upper bound for 3-connected plane graphs is the following result shown by Enomoto and Horňák [24].

Theorem 3.2.12. *[24] If G is a 3-connected plane graph, then $\chi_c(G) \leq \Delta^*(G) + 5$.*

Based on the previous results, it only remained to verify the upper bound for $9 \leq \Delta^*(G) \leq 14$.

Finally, additional progress was made in the verification of Conjecture 3.2.5 when Horňák and Zlámalová [34] verified the conjecture for $\Delta^*(G) \geq 18$. Moreover, as the only known examples of graphs such that $\chi_c(G) = \Delta^*(G) + 2$ satisfy $\Delta^*(G) = 4$, Horňák and Zlámalová proposed strengthening the conjecture as follows.

Conjecture 3.2.13. *If G is a 3-connected plane graph G with $\Delta^*(G) \neq 4$, then $\chi_c(G) \leq \Delta^*(G) + 1$.*

As previously shown, this conjecture has been verified for $\Delta^*(G) \geq 60$, however, all three conjectures in this section remain open in general.

This section concludes with the following table summarizing the current best upper bounds on the cyclic chromatic number for 3-connected plane graphs.

$\Delta^*(G)$	3	4	5	6	$7-15$	$16-17$	$18-59$	≥ 60
$\chi_c(G) \leq \Delta^*(G)+$	1	2	3	4	5	4	2	1

Table 3.2: Upper bounds of cyclic chromatic number for 3-connected plane graphs

3.3 Other Perspectives

Alternative perspectives or approaches have been introduced in considering the proposed conjectures of the previous section. The following graph parameter was introduced by Borodin et al. [11].

Definition 3.3.1. *Let G be a plane graph, f be a face of G, and $V_G(f)$ be the set of vertices incident with f. Then $k^*(G)$ (or simply k^*) is the maximum number of vertices that two faces of G can have in common, ie. $k^*(G) = \max\{|V_G(f_1) \cap V_G(f_2)| : f_1, f_2 \in F_G, f_1 \neq f_2\}$.*

The use of this parameter led to the following result.

Theorem 3.3.2. *[11] Every connected plane graph G has $\chi_c(G) \leq \max\{\Delta^*(G) + 3k^*(G) + 2, \Delta^*(G) + 14, 3k^*(G) + 6, 18\}$.*

The above theorem can be simplified in most cases as follows.

Corollary 3.3.3. *[11] If G is a plane graph, and $\Delta^*(G) \geq 4$ and $k^*(G) \geq 4$, then $\chi_c(G) \leq \Delta^*(G) + 3k^*(G) + 2$.*

The above theorem improves the best upper bound when $k^*(G)$ is small, and would verify Conjecture 3.2.1 in the case where $k^*(G) \leq \frac{1}{6}\Delta^*(G)$. The following conjecture was suggested, and would be the best result possible, as shown by the same set of graphs as for Conjecture 3.2.1.

Conjecture 3.3.4. *[11] Every plane graph G with $\Delta^*(G)$ and $k^*(G)$ large enough has a cyclic coloring with $\Delta^*(G) + k^*(G)$ colors.*

If $\Delta^*(G)$ is large enough, the above conjecture implies $\chi_c(G) \leq \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor$.

In another approach, Amini, Esperet, and van den Heuvel [1] generalized the concept of coloring the square of a graph. The *square* G^2 of a graph G is the graph formed with the vertex set $V(G)$ with an edge between vertices if they have at most distance two in G. Thus, the proper colouring of G^2 is equivalent to colouring G such that vertices receive distinct colours if they are adjacent or they share a common neighbour. In the more general case, a subset $\Sigma(v) \subseteq N(v)$ of the neighbourhood of each vertex is given, and a Σ-colouring of G requires that vertices receive distinct colours if they are adjacent or both appear in some set $\Sigma(v)$. So, if $\Sigma(v) = N(v)$ for all $v \in V(G)$, this is precisely the colouring of G^2. A cyclic colouring can be determined as follows. Construct the graph G_F from G by adding a vertex x_f for each face f of G, and add an edge between x_f and each vertex incident with f. Define $\Sigma(v)$ as $\emptyset$ if $v \in V(G)$, and $N(v)$ otherwise. Then applying a Σ-colouring of G_F to the vertices of G gives a cyclic colouring of G, as no two vertices of G incident with the same face f receive the same colour since they all appear in $\Sigma(v_f)$. The following result was determined.

Corollary 3.3.5. *[1] Let S be a fixed surface. Every embedding G^S of a graph G of maximum face order Δ^* has cyclic list chromatic number at most $\frac{3}{2}\Delta^* + o(\Delta^*)$.*

The above result demonstrates that Conjecture 3.2.1 is asymptotically true.

The next approach considers an extension of cyclic colourings called facial colourings, introduced by Král', Madaras, and Škrekovski [38]. Two vertices are said to be *l-facially adjacent* if there exists a path of at most l edges each incident with the same face between the two vertices. A colouring is said to be an *l-facial colouring* if any pair of vertices that are l-facially adjacent are assigned distinct colours. If $\Delta^*(G) \leq 2l+1$, then every cyclic colouring of G is an l-facial colouring. If G is 2-connected, the converse is also true. Facial colourings remove the complication of restricting the results of cyclic colourings by the size of the maximum face. Let $f_c(\Delta^*)$ denote the minimum number of colours needed in a cyclic colouring of every plane graph with maximum face size at most $\Delta^*(G)$. Let $f_f(l)$ denote the minimum number of colours needed in an l-facial colouring of every plane graph. Clearly $f_c(2l+1) \leq f_f(l)$. Thus, the upper bounds determined for facial colourings imply upper bounds for cyclic colourings. Král', Madaras, and Škrekovski [38] matched the bound $f_c(\Delta^*(G)) \leq \left\lfloor \frac{9}{5}\Delta^*(G) \right\rfloor$ of Theorem 3.2.7 by proving $f_f(l) \leq \left\lfloor \frac{18l}{5} \right\rfloor + 2$. The bound $f_c(\Delta^*(G)) \leq \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil$ of Theorem 3.2.9 raises the question of the existance of the bound $f_f(l) \leq \left\lceil \frac{10l+5}{3} \right\rceil$. The following conjecture, analogous to Conjecture 3.2.1, has also been proposed.

Conjecture 3.3.6. *[38] Every plane graph has an l-facial coloring with at most $3l+1$ colors for each $l \geq 0$, ie. $f_f(l) = 3l+1$ for all $l \geq 0$.*

The following improved result for $l = 3$ was shown by Havet, Sereni, and Škrekovski [28].

Theorem 3.3.7. *[28] Every plane graph is 3-facially 11-colourable.*

The above result implies $f_c(7) \leq 11$ which improves the upper bound of the previous section for $\Delta^*(G) = 7$ to $\chi_c(G) \leq \Delta^*(G) + 4$. The following improvement to the general bound was shown by Havet et al. [27]

Theorem 3.3.8. *[27] Every plane graph has an l-facial coloring with at most $\left\lfloor \frac{7l}{2} \right\rfloor + 6$ colors.*

Finally, Azarija et al. [5] considered the upper bound on the cyclic chromatic number for graphs whose faces of size at least four are vertex disjoint, and proved the following result.

Theorem 3.3.9. *[5] Let G be a plane graph with maximum face size Δ^*. If each face of degree at least four is vertex-disjoint, then G has a cyclic coloring with at most $\Delta^* + 1$ colors.*

The approaches discussed in this section have offered new tools and insight to determining the best upper bounds for the cyclic chromatic number of connected plane graphs and the cyclic chromatic number of 3-connected plane graphs, and in one case have improved the previous upper bound. A new strategy may be necessary, however, to confirm the conjectures which have been posed.

Chapter 4

Parity Vertex Colourings

4.1 Introduction

In this chapter, I expand the discussion of vertex colourings which are restricted by the faces of the graph. Compared to cyclic colourings, which were examined in the previous chapter and require that two vertices incident with the same face have different colours, the requirements of parity vertex colourings are relaxed. Problems 4.1.2 and 4.1.3 were proposed by Czap and Jendrol' [20], and require Definition 4.1.1.

Definition 4.1.1. *Let φ be a vertex colouring, not necessarily proper, of a connected plane graph G. We say that a colour c is used k times on a face α of G under the colouring φ if this colour appears k times along the facial walk of α. (The first and the last vertex of the facial walk is counted as one appearance only.)*

Problem 4.1.2. *A vertex colouring φ is a* weak parity vertex colouring *of a connected plane graph G if each face of G uses at least one colour an odd number of times. The problem is to determine the minimum number $\chi_w(G)$ of colours used in a weak parity* *vertex colouring of G. The number $\chi_w(G)$ is called the* weak parity chromatic number*.*

Problem 4.1.3. *A vertex colouring φ is a* strong parity vertex colouring *of a 2-connected plane graph G if for each face α and each colour c the face α uses the colour c an odd number of times or does not use it at all. The problem is to find the minimum number $\chi_s(G)$ of colours used in a strong parity vertex colouring of G. The number $\chi_s(G)$ is called the* strong parity chromatic number*.*

The parity vertex colouring problems were motivated by the study of parity edge colourings and strong parity edge colourings of graphs by Bunde et al. [16, 17]. In a parity edge colouring, each path uses some colour an odd number of times. In a strong parity edge colouring, each open walk uses some colour an odd number of times. In addition, facial parity edge colourings were studied by Czap et al. [22]. In a facial parity edge colouring, no two adjacent edges are assigned the same colour. For each face α and each colour c, the face α uses the colour c an odd number of times or not at all.

A brief overview of weak parity vertex colouring will be given in the next section, but the main focus of this chapter is strong parity vertex colourings. Also considered are *proper weak parity vertex colouring* and *proper strong parity vertex colouring* (which is also abbreviated to proper spv-colouring or called parity vertex colouring). These colourings require that no two adjacent vertices are assigned the same colour, ie. the vertex colouring is proper. The *proper weak parity chromatic number*, $\chi_{pw}(G)$, and *proper strong parity chromatic number*, $\chi_p(G)$ are defined analogously.

4.2 Weak Parity Vertex Colouring

The main goal of the weak parity vertex colouring problem is to determine the best constant upper bound on the weak parity vertex chromatic number of all planar graphs, or determine that no such bound exists. It has been shown by Czap and Jendrol' [20] that the weaker restrictions of weak parity vertex colourings permit a constant upper bound, whereas the bound for cyclic colourings was known to be dependent on the size of the largest face. To show that $\chi_w(G)$ is well defined for every connected plane graph G, the following structural claim is required.

Claim 4.2.1. *If G is a loopless plane graph then every face f is incident with a vertex that occurs only once in the facial walk of f.*

Proof. Suppose that f is a face such that every vertex in the facial walk of f occurs at least twice. Let H be the graph induced by the edges of f. Let e be the total number of occurrences of each vertex in f. Let n be the number of vertices incident with f. If x counts the number of edges that occur twice in the facial walk of f, then $e - x$ is the number of edges in H. Since H is planar, then by Euler's formula, $|F(H)| = |E(H)| - |V(H)| + 2 = e - x - n + 2$. The $e - 2x$ edges used only once in the facial walk of f must bound the $e - x - n + 1$ remaining faces of H. Then the average number of edges used by each of the remaining faces is $\frac{e-2x}{e-x-n+1} = \frac{2e-2x-2n+2-(e-2n+2)}{e-x-n+1} = 2 - \frac{e-2n+2}{e-x-n+1} < 2$. Thus, there exists a face of H bounded by at most 1 edge. It follows there is a face of G bounded by at most 1 edge, contradicting that G is loopless. □

As a result of Claim 4.2.1, $\chi_w(G)$ is well defined for every connected (loopless) plane graph G. Each plane graph G has the weak parity vertex colouring that assigns a distinct colour to each vertex, since some vertex occurs once in the facial walk for each face. On the other extreme, trivially $\chi_w(G) = 1$ if and only if all faces of G are of odd degree.

The following theorem, due to Czap and Jendrol' [20], gives the weak parity vertex chromatic number of the graphs of r-sided prisms, and will be used later in this section. An r-sided prism, D_r, is a plane graph consisting of two r-faces and r 4-faces. The faces are: an internal r-face $\alpha = (u_1, u_2, \ldots, u_r)$, an external r-face $\beta = (v_1, v_2, \ldots, v_r)$, and for $1 \leq i \leq r$, $\alpha_i = (u_i, u_{i+1}, v_{i+1}, v_i)$, where indices are taken modulo r.

Theorem 4.2.2. *[20] Let D_r be an r-sided prism, $r \geq 3$. Then*

$$\chi_w(D_r) = \begin{cases} 2 \text{ if } r \equiv 0 \pmod{4}, \\ 3 \text{ if } r \not\equiv 0 \pmod{4}. \end{cases}$$

It follows from the definitions of the weak parity vertex chromatic number and the proper weak parity vertex chromatic number that $\chi_w(G) \leq \chi_{pw}(G)$. Czap and Jendrol' [20] proved the following result for 2-connected plane graphs.

Theorem 4.2.3. *[20] Let G be a 2-connected plane graph. Then there is a proper weak parity vertex 4-colouring of G, such that each face of G uses some colour exactly once.*

This theorem can be extended as follows to all plane graphs.

Theorem 4.2.4. *Let G be a connected plane graph. Then there is a proper weak parity vertex 4-colouring of G, such that each face of G uses some colour exactly once.*

Proof. Let α be a face of degree at least 4, and let v_i be a vertex of the facial walk of α that appears exactly once in the facial walk, the existence of which is given by Claim 4.2.1. For each vertex v_j in the facial walk of α, add edge v_iv_j if $v_i \nsim v_j$. Adding edges in this manner to each such face, form the planar graph H (which inherits the obvious embedding). Applying the Four Colour Theorem to H, there is a proper colouring φ of H that uses at most four colours.

If φ is then assigned to G, for each face α of degree at least 4, no vertex is assigned the same colour as the vertex v_i (chosen above) since v_i is adjacent in H to each vertex of the facial walk of α in G. Thus, for each face α, the colour of v_i is used only once on α. Thus, φ is a proper weak parity vertex 4-colouring of G having the desired property. $\square$

The previous theorem implies the following upper bound.

Corollary 4.2.5. *Let G be a connected plane graph. Then $\chi_{pw}(G) \leq 4$ and $\chi_w(G) \leq 4$.*

Proof. By the previous theorem, every connected plane graph G has a proper weak parity vertex 4-colouring. The result follows. $\square$

The upper bound of 4 is the best bound possible for proper weak parity vertex colourings, as the graph K_4 requires 4 colours for such a colouring. Also, when considering the problem for plane multigraphs in general, the upper bound of 4 is the best bound possible for weak parity vertex colourings. As an example, consider the graph H formed by doubling each edge of the graph K_4. However, it is unknown if the upper bound of 4 is the best bound possible for weak parity vertex colourings of simple graphs. There is no known example of a graph that requires 4 colours for such a colouring. In a proof by Czap and Jendrol' [20], in which they showed an upper bound of 4, there were only two cases in which the upper bound could not be improved to three. Thus, they made the following conjecture.

Conjecture 4.2.6. *[20] Let G be a connected plane graph of minimum face degree at least 3. Then $\chi_w(G) \leq 3$.*

The following theorem of Czap and Jendrol' [20] gives a class of graphs for which this upper bound holds.

Theorem 4.2.7. *[20] Let G be a 2-connected cubic plane graph. Then $\chi_w(G) \leq 3$. Moreover, the bound 3 is best possible.*

It is clear that the bound 3 is best possible by Theorem 4.2.2, which states that certain r-prisms have a weak parity vertex chromatic number of 3. In general, however, the problem remains open.

4.3 Strong Parity Vertex Colouring

The main goal of the strong parity vertex colouring problem is to determine the best constant upper bound on the strong parity vertex chromatic number of all 2-connected plane graphs, or determine that no such bound exists. Czap and Jendrol' [20] state that $\chi_s(G)$ is well defined for every 2-connected plane graph G, as each 2-connected plane graph G has the strong parity vertex colouring that assigns a distinct colour to each vertex.

By Definition 4.1.1, however, $\chi_s(G)$ is not well defined in general for plane graphs that are not 2-connected. Consider the graph H shown in Figure 4.3.1. If H has a strong parity vertex colouring φ, let $\varphi(u_1) = c$. As c appears twice on the 6-face, it must appear at least three times on the 6-face, thus without loss of generality, $\varphi(u_2) = c$. But then c appears twice on the face $u_1u_2u_3$, so $\varphi(u_3) = c$. Then c appears four times on the 6-face, so by the parity condition it must appear exactly five times on the 6-face. Without loss of generality, $\varphi(u_4) = c$ and $\varphi(u_5) \neq c$. But then c appears twice on the face $u_1u_4u_5$, which is a contradiction. Thus H does not have a strong parity vertex colouring.

Kaiser et al. [36] proposed that Definition 4.1.1 could be modified to say a colour c is used k times on a face α of G under the colouring φ if k vertices are coloured with

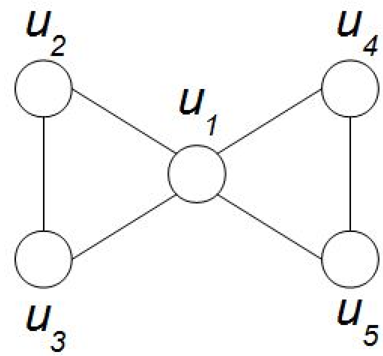


Figure 4.3.1: Graph H

c in the boundary of α. This definition conincides with Definition 4.1.1 in the case of 2-connected plane graphs, and $\chi_s(G)$ is now well defined for every plane graph G, as each plane graph G has the strong parity vertex colouring that assigns a distinct colour to each vertex. However, this extension to all plane graphs was not pursued in their paper and remains an open problem. This section focuses only on 2-connected plane graphs, and hence, will use Definition 4.1.1 exclusively.

The following theorems of Czap and Jendrol' [20] provide examples of two connected plane graphs G for which the exact value of $\chi_s(G)$ is known. As in the case of weak parity vertex colouring, trivially $\chi_s(G) = 1$ if and only if all faces of G are of odd degree.

A set of vertices $S \subseteq V(G)$ of a 2-connected plane graph is said to be *face-independent* if no two vertices of S are incident with a common face, and is said to be *face-dominating* if every face of G is incident with a vertex in S.

Theorem 4.3.1. *[20] Let G be a 2-connected plane graph all faces of which have even degree. If G contains a set S of vertices which is face-independent and face-dominating, then $\chi_s(G) = 2$.*

The wheel graph on n vertices, $n \geq 4$, denoted W_n, is the graph formed by joining a vertex to each vertex of a cycle of length $n-1$.

Theorem 4.3.2. *[20] Let W_n, $n \geq 4$, be an n-vertex wheel. Then*

$$\chi_s(W_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2, \\ 3 & \text{if } n \equiv 3 \pmod 4, \\ 5 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

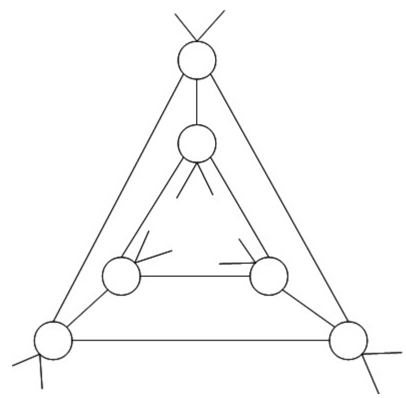


Figure 4.3.2: Configuration K

Lemma 4.3.3. *[20] If a 2-connected plane graph G has an induced subgraph K and the three 4-faces of K are faces of G, as depicted in Figure 4.3.2, then $\chi_s(G) \geq 6$.*

Theorem 4.3.4. *[20] Let D_r be an r-sided prism, $r \geq 3$. Then*

$$\chi_s(D_r) = \begin{cases} 2 \text{ if } r \equiv 0 \pmod 4, \\ 4 \text{ if } r \not\equiv 0 \pmod 4 \text{ and } r \notin \{3,7\}, \\ 5 \text{ if } r = 7, \\ 6 \text{ if } r = 3. \end{cases}$$

While upper bounds for the strong parity vertex chromatic number follow from the upper bounds for the cyclic chromatic number since every cyclic colouring is a strong parity vertex colouring, the following conjecture was made by Czap and Jendrol' [20].

Conjecture 4.3.5. *[20] There is a constant K such that for every 2-connected plane graph G, $\chi_s(G) \leq K$.*

The proper strong parity chromatic number analog for this conjecture has been verified independently by Czap, Jendrol', and Voigt [23] and Kaiser et al. [36]. Since every proper strong parity vertex colouring is a strong parity vertex colouring, the conjecture has therefore been verified for the strong parity chromatic number. Specifically, the following upper bounds were obtained.

Theorem 4.3.6. *[23] Let G be a 2-connected plane graph. Then $\chi_p(G) \leq 118$.*

Theorem 4.3.7. *[36] Every 2-connected plane multigraph has a proper spv-colouring with at most 97 colours.*

The bounds for graphs that do not contain large faces can be improved based on the results of cyclic colourings of Corollary 3.2.11 and Theorem 3.3.7 as follows.

Corollary 4.3.8. *Let G be a 2-connected plane graph. Then*

$$\chi_p(G) = \begin{cases} \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil - 1, & \text{if } \Delta^*(G) \in \{3,4,5,7,8\}, \\ \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil, & \text{if } \Delta^*(G) \leq 57 \text{ and } \Delta^*(G) \notin \{3,4,5,7,8\}, \\ 97, & \text{if } \Delta^*(G) \geq 58. \end{cases}$$

The next section considers the sharp upper bound on the strong parity chromatic number.

4.4 Sharp Upper Bound

Let X_p^* be the sharp upper bound for the proper strong parity chromatic number. That is, X_p^* is the smallest integer such that $\chi_p(G) \leq X_p^*$ for every 2-connected plane graph.

Theorem 4.3.7 states $X_p^* \leq 97$. We now consider lower bounds on X_p^*. A progression of graphs with the largest determined proper strong parity chromatic number is given. If multigraphs are allowed, form G' from G by replacing every edge with a digon, a face bounded by two edges. Kaiser et al. [36] demonstrated that every strong parity vertex colouring of G' is a proper strong parity vertex colouring of G. Thus, for the class of 2-connected plane multigraphs, all bounds on $\chi_s(G)$ are also bounds on $\chi_p(G)$ and vice-versa. In the case of simple graphs, this does not appear to be the case. A large proper strong parity chromatic number does not necessarily imply a large strong parity chromatic number. For instance, for C_5, a cycle on 5 vertices, the proper strong parity chromatic number is 5, but the strong parity chromatic number is 1. The results in both cases will be examined, starting with examples for proper strong parity chromatic numbers. In their introductory paper, Czap and Jendrol' used the r-sided prism to give a lower bound on X_p^*. The specific result is given in Theorem 4.4.1.

Theorem 4.4.1. *[20] Let D_r be an r-sided prism, $r \geq 3$. Then*

$$\chi_p(D_r) = \begin{cases} 4 \text{ if } r \equiv 0 \pmod 2, \\ 5 \text{ if } r \equiv 1 \pmod 2 \text{ and } r \neq 3, \\ 6 \text{ if } r = 3. \end{cases}$$

The lower bound of X_p^* was improved from 6 to 10 by Kaiser et al. [36]. The authors constructed the graph G_{55} on 10 vertices by joining two disjoint 5-cycles by two edges such that the outerface was bounded by 10 vertices, as shown in Figure 4.4.1. For each of the 5-faces, its incident vertices must be assigned distinct colours. (The face cannot contain three nonadjacent vertices of the same colour.) Thus, considering the 10-face, a colour can appear at most twice in its facial walk, and therefore must appear exactly once. It follows that each vertex is assigned a different colour, so a proper strong parity

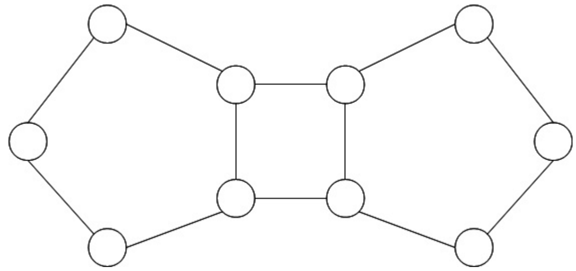


Figure 4.4.1: Graph G_{55}

vertex colouring must use 10 colours.

Presented now is a graph whose proper strong parity chromatic number is 12, and thus demonstrates that $X_p^* \geq 12$.

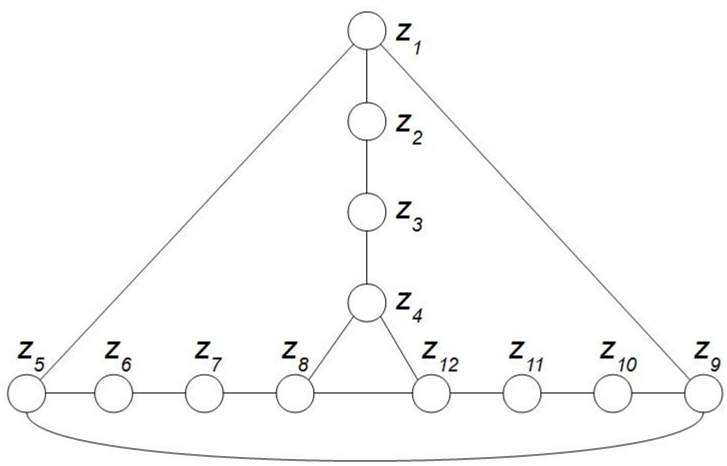
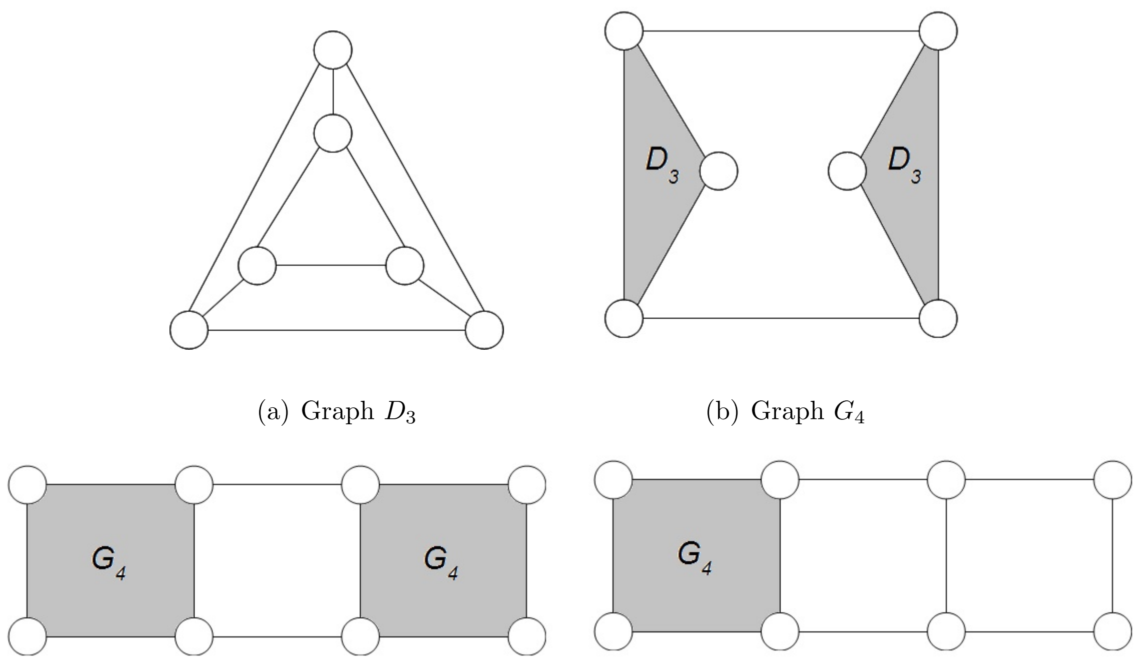


Figure 4.4.2: Graph Z

Theorem 4.4.2. *The graph Z, depicted in Figure 4.4.2, has a proper strong parity chromatic number of 12.*

Proof. Let φ be a proper strong parity vertex colouring of Z. Consider the vertices z_1, z_2, z_3, z_4. No colour can occur three times on these four vertices, as two adjacent vertices would have the same colour. By symmetry, no colour can occur three times on the vertices z_5, z_6, z_7, z_8, and no colour can occur three times on the vertices $z_9, z_{10}, z_{11}, z_{12}$. Now, consider the face $z_1z_2z_3z_4z_8z_7z_6z_5$. No colour can occur five times on this cycle, as two adjacent vertices would have the same colour. Suppose the colour c occurs 3 times on this cycle. Then without loss of generality, it occurs twice on the vertices z_1, z_2, z_3, z_4, and once on the vertices z_5, z_6, z_7, z_8. Considering the face $z_1z_2z_3z_4z_{12}z_{11}z_{10}z_9$, c must occur exactly once on the vertices $z_9, z_{10}, z_{11}, z_{12}$ so that c occurs an odd number of times on this face. But then c appears twice on the face $z_5z_6z_7z_8z_{12}z_{11}z_{10}z_9$, which contradicts that φ is a proper strong parity vertex colouring. By symmetry, no colour can occur three times on any 8-face, thus each vertex must be assigned a different colour. □



(a) Graph D_3 (b) Graph G_4

(c) Graph G_{44} (d) Graph G'_{44}

Figure 4.4.3: Graphs D_3, G_4 G_{44}, and G'_{44}

To conclude this section, the case where G is a simple graph is discussed. The graph with the largest known strong parity chromatic number was constructed by Kaiser et al. [36]. The construction was described as follows. First, take a 3-sided prism D_3, shown in Figure 4.4.3(a), which has a strong parity chromatic number of 6 by Theorem 4.3.4, and note that this result was shown without considering the outer face. Now the graph G_4 is constructed from C_4, a cycle on four vertices, by replacing two opposite edges with D_3, as depicted in Figure 4.4.3(b).[1] As each vertex in a copy of D_3 must be assigned a different colour, then each colour can occur at most twice on the 6-cycle, thus it must occur exactly once. Hence, the colours on the 4-face are distinct. Finally, two disjoint copies of G_4 are joined by two edges such that the outer face is bounded by 8 vertices, forming the graph G_{44}, pictured in Figure 4.4.3(c). Then each colour appears at most once on each 4-cycle, and hence must appear exactly once on the 8-face. Thus, G_{44} requires 8 colours in any strong parity vertex colouring. Thus the upper bound on the strong parity chromatic number of simple graphs is therefore at least 8. Note that it was stated in [36] that the graph G'_{44}, in Figure 4.4.3(d), constructed by replacing a copy of G_4 in G_{44} with a copy of C_4 also has a strong parity chromatic number of 8. This assertion is incorrect. Figure 4.4.4 demonstrates G'_{44} has a strong parity vertex colouring with 6 colours.

There is still a large gap between the best known upper bound and the best known lower bound on X_p^* and its analog for strong parity vertex colourings simple graphs. For strong parity vertex colouring of simple graphs, the best known upper bound is 97 and the best known lower bound is 8. We have shown $12 \leq X_p^* \leq 97$. Thus, the problem of determining the sharp upper bound for the strong parity chromatic number and the

[1] The figure depicting the graph G_4 given as Figure 3.1 (b) by Kaiser et al. [36], does not match their description of G_4. We verified that the graph described does have the property that each strong parity vertex colouring colours the vertices of the outer face with distinct colours, but the graph in the figure does not. Therefore, we assume the graph described was the graph intended by the authors.

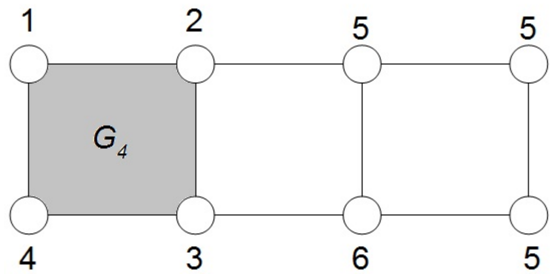


Figure 4.4.4: A strong parity vertex colouring of G'_{44} using 6 colours.

proper strong parity chromatic number remains open.

4.5 Additional Upper Bounds

In this section, upper bounds on the proper strong parity chromatic number for specific classes of graphs are outlined. The first class considered are outerplane graphs. A graph is *outerplane* if it is embedded in the plane such that every vertex is on the outer face. Outerplane graphs were studied by Czap [19], who showed that $\chi_p(G) \leq 12$ for every 2-connected plane graph G. This result was improved by Wang, Finbow, and Wang [45].

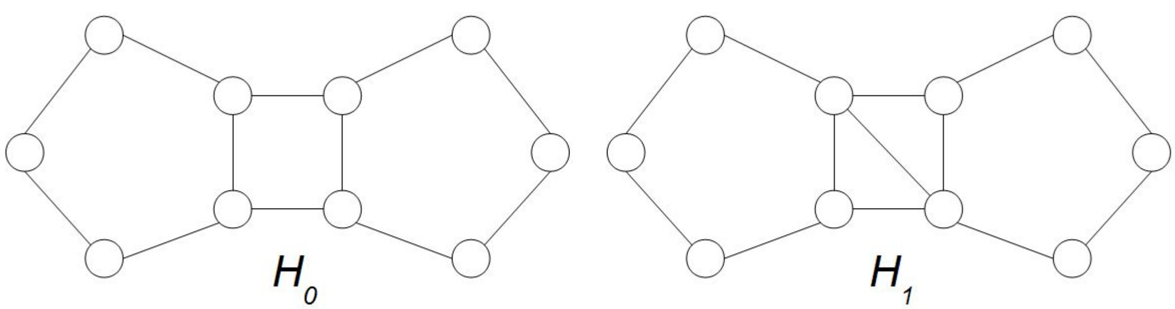


Figure 4.5.1: Graphs H_0 and H_1

Theorem 4.5.1. *[45] If G is a 2-connected outerplane graph, different from H_0 and H_1, then $\chi_p(G) \leq 9$.*

The graph H_0 is precisely the graph G_{55} constructed by Kaiser et al. [36], and has a proper strong parity chromatic number of 10. It is clear that H_1 also has a proper strong parity chromatic number of 10.

In the case of bipartite outerplane graphs, Wang, Finbow, and Wang [45] proved that each has a proper strong parity colouring with at most 8 colours. The authors characterized the class of bipartite outerplane graphs with proper strong parity chromatic number 8 as follows. If G is an outerplane graph, each face that is not the outer face is called an *inner face*, and an inner face f of G is called an *end face* if the boundary of f contains exactly two vertices of degree greater than 2. Let $\mathcal{F}$ denotes the set of 2-connected outerplane graphs that have exactly three inner faces, and the degree of each end face of G is divisible by four and the degree of the face which is not an end face is four. Then $\mathcal{F}$ is precisely the class of graphs with proper strong parity chromatic number 8. The authors also proved that a 2-connected outerplane graph has a proper strong parity vertex colouring using two colours if and only if the degree of each face is divisible by two but not four.

Additionally, Czap, Jendrol', and Kardoš [21] outlined several upper bounds on the proper strong parity vertex chromatic number for graphs with certain facial properties. They depend on the following fundamental lemma, Lemma 4.5.2. This lemma can be used to recolour a face with a proper colouring so that the colouring restricted to this face is a proper strong parity vertex colouring.

Lemma 4.5.2. *[21] Let $C = v_1, \ldots, v_k$ be a cycle on k vertices. Then there is a proper strong parity vertex colouring φ of C using the colours a, b, c, d, e, where the colours a, b, c*

are used at most once.

The first set of upper bounds require that faces exceeding a certain size do not influence each other. Two faces f and g are said to *influence* each other if they share a vertex, or if there is a face h that shares a vertex with f and shares a vertex with g. Theorem 4.5.3 is based on the application of the Four Colour Theorem, and Theorem 4.5.4 and Theorem 4.5.5 are based on the application of upper bounds on cyclic colourings, which are given by Corollary 3.2.11.

Theorem 4.5.3. *[21] Let G be a 3-connected plane graph in which no two non-triangle faces influence each other. Then there is a proper strong parity vertex colouring of G which uses at most six colours $1, \ldots, 6$ such that each vertex which is not incident with any non-triangle face has a colour from the set $\{1, 2, 3, 4\}$. Moreover, this bound is sharp.*

Theorem 4.5.4. *[21] Let G be a 3-connected plane graph in which the faces of size at least 5 do not influence each other. Then there is a proper strong parity vertex colouring of G which uses at most 8 colours.*

Theorem 4.5.5. *[21] Let G be a 3-connected plane graph such that the faces of size at least 6 do not influence each other. There there is a proper strong parity vertex colouring of G which uses at most 10 colours.*

The second set of upper bounds require that faces exceeding a certain size are isolated. Two faces f and g are said to be *isolated* if they do not share a vertex. These bounds are based on the application of results regarding k-planar graphs. A graph is *k-planar* if it can be drawn in the plane such that each edge is crossed by at most k other edges. The following lemma constructs a k-planar graph from a graph whose faces exceeding a certain size are isolated.

Lemma 4.5.6. *[21] Let j be a fixed integer from the set $\{3, 4, 5\}$. Let G be a 2-connected plane graph such that any face of size at least $j+1$ is isolated. Let H be a graph obtained from G in the following way: for each face in G of size at least $j+1$ insert a vertex to H, join two vertices of H by an edge if the corresponding faces influence each other in G. Then*

1. *If $j = 3$ then H is a planar graph.*
2. *If $j = 4$ then H is a 1-planar graph.*
3. *If $j = 5$ then H is a 2-planar graph.*

The following results on the proper colouring of k-planar graphs are used to derive the next set of upper bounds for proper strong parity vertex colourings.

Theorem 4.5.7. *[10] If a graph is 1-planar, then it is vertex 6-colourable.*

Theorem 4.5.8. *[21] If a graph is 2-planar, then it is vertex 10-colourable.*

Then, the following results were derived.

Theorem 4.5.9. *[21] Let G be a 3-connected plane graph such that any face of size at least 4 is isolated. Then there is a proper strong parity vertex colouring of G which uses at most 12 colours.*

Theorem 4.5.10. *[21] Let G be a 3-connected plane graph such that any face of size at least 5 is isolated. Then there is a proper strong parity vertex colouring of G which uses at most 18 colours.*

Theorem 4.5.11. *[21] Let G be a 3-connected plane graph such that any face of size at least 6 is isolated. Then there is a proper strong parity vertex colouring of G which uses at most 28 colours.*

The results outlined in this section can be applied to the problem of determining a sharp upper bound for the proper strong parity chromatic number. For the lower bound, an example whose proper strong parity chromatic number is 12 is given in Theorem 4.4.2. The problem is solved for: outerplane graphs (Theorem 4.5.1), 3-connected plane graphs whose faces of size at least 6 do not influence each other (Theorem 4.5.5), and 3-connected plane graphs whose faces of size at least 4 are isolated (Theorem 4.5.9). For the upper bound, the results in this section may be used to eliminate certain cases.

Chapter 5

Proper ℓ-Residue Vertex Colourings

5.1 Introduction

In this chapter, the concept of parity vertex colourings is extended to vertex colourings with an arbitrary sequence of values $S_\ell = \{i | i \equiv 1 \pmod{\ell}\}$ that a colour may occur on a face. These colourings are defined in the following problem.

Problem 5.1.1. *A vertex colouring φ is a* proper ℓ-residue colouring *of a 2-connected plane graph G if for each face α and each colour c the number of times the colour c is used by the face α is an element of S_ℓ or is not used at all. The problem is to find the minimum number of colours used in a proper ℓ-residue colouring, called the* proper ℓ-residue chromatic number.

Notice that a proper 2-residue vertex colouring is equivalent to a proper strong parity vertex colouring. Hence, it similarly follows that every closed 2-cell embedded graph G has a proper l-residue colouring, and the condition that G be closed 2-cell embedded cannot be dropped. Our main results are the following upper bounds for plane and toroidal graphs. Note that for plane graphs, our bound matches that of Kaiser et al. [36] in the case $l = 2$, but we include it for completeness.

Theorem 5.1.2. *Every 2-connected plane graph has a proper l-residue colouring with at most $20l^2 + A(l)l + 1$ colours, where*

$$A(l) = \begin{cases} 8 & \text{if } 2 \leq l \leq 6, \\ 6 & \text{if } l = 7, \\ 4 & \text{if } l = 8, \\ 2 & \text{if } l = 9, 10, \\ 0 & \text{if } l \geq 11. \end{cases}$$

Theorem 5.1.3. *Every closed 2-cell embedding of a toroidal graph has a proper l-residue colouring with at most $20l^2 + B(l)l + 1$ colours, where*

$$B(l) = \begin{cases} 10 & \text{if } l = 2, \\ A(l) & \text{if } l \geq 3. \end{cases}$$

The proofs of the previous theorems are given in Sections 5.3 and 5.4, after establishing the necessary preliminaries in Section 5.2. In Section 5.5, we give an example graph showing the upper bound for both plane and toroidal graphs must be at least $2l^2 + l$.

5.2 Preliminaries

Let G be a closed 2-cell embedded graph and let f be a face of G. The number of vertices of degree at least 3 on the boundary of f is called the *weight* of f, written as $w(f)$. The face f is a *pseudodigon* if $w(f) = 2$ and f is not a digon. The *modified weight* of f,

$w'(f)$, is defined as 3 if $w(f) = 2$, and $w(f)$ otherwise. Suppose that v is a vertex of G. The *configuration* of v is the tuple obtained by ordering the elements from the multiset $\{w(g) : g \in F(v)\}$ in a nondecreasing manner. The *f-reduced configuration* of v is the tuple obtained by ordering the elements from the multiset $\{w(g) : g \in F(v), g \neq f\}$ in a nondecreasing manner. The *modified configuration* of v and *modified f-reduced configuration* of v are defined analogously, with $w(g)$ replaced by $w'(g)$. The *(open) face-vertex neighbourhood* of v, denoted by $N_G^F(v)$, is defined as $\bigcup_{g \in F(v)} V(g) - \{v\}$, and the *$f$-reduced (open) face-vertex neighbourhood* of v, denoted by $N_G^F(v, f)$ is defined as $\bigcup_{g \in F(v), g \neq f} V(g) - \{v\}$. The sizes of these sets are called the *face degree* and *f-reduced face degree* respectively, and are denoted $d_G^F(v)$ and $d_G^F(v, f)$. As before, if it is clear which graph is being discussed, the subscript G is removed from the above notation.

The annihilation of a vertex v of a plane graph G was defined by Kaiser et al. [36] to establish structural conditions on a minimal counterexample; we extend the definition to closed 2-cell embedded graphs for use in the proof of Lemma 5.3.1. Let G be a closed 2-cell embedded graph and v a vertex of G with degree $d \geq 2$. Suppose the edges incident with v are enumerated in a clockwise order as $e_i := vv_i, i \in \mathbb{Z}_d$. Suppose that the vertices v_i are distinct; that is, v is incident with no pair of parallel edges. The *annihilation* of v is the construction of a 2-cell embedded graph G' from G defined as follows:

(1) add edges $e'_i := v_i v_{i+1}, i \in \mathbb{Z}_d$, embedded in the plane so that for each i, the edges e_i, e_{i+1}, and e'_i, in this order, constitute a facial walk;

(2) delete v together with all the edges e_i.

The following observations are straightforward extensions of the case of 2-connected plane graphs given by Kaiser et al. [36].

Observation 5.2.1.

(1) Every face of G not in $F_G(v)$ is also a face of G';

(2) each face $g \in F_G(v)$ has its counterpart g' in G' such that a facial walk of g' may be obtained from a facial walk W of g by replacing each subsequence of W of the form $e_i v e_{i+1}$ with e'_i, and hence $V(g') = V(g) - \{v\}$;

(3) there is precisely one more face in G', having the sequence $v_0 e'_{d-1} v_{d-1} e'_{d-2} \dots v_1 e'_0 v_0$ as its facial walk.

Lemma 5.2.2. *Let v be a vertex of a (loopless) closed 2-cell embedded graph G, $|V(G)| \geq 4$, such that v is incident with no pair of parallel edges. Then the graph G' obtained from G by annihilating v is closed 2-cell embedded (and loopless).*

The following technical lemma was used by Kaiser et al. [36] and significantly shortens the case analysis of the proof of Claim 5.3.6.

Lemma 5.2.3. *[36] Let (l_i), (l'_i), $i = 0, \dots, k$, be tuples of positive integers such that $l_j \leq l'_j$ for every $j \neq k$, and $l'_k \geq l'_j$ for every $j \neq k$ with $l_j < l'_j$. Then $\sum_{i=0}^{k} l'_i \leq \sum_{i=0}^{k} l_i$ or $\sum_{i=0}^{k} 1/l'_i \leq \sum_{i=0}^{k} 1/l_i$.*

5.3 Plane Graphs

This section gives the proof of Theorem 5.1.2, following the approach of Kaiser et al. [36]. We proceed by contradiction; take a minimal counterexample G with respect to the number of vertices first and to the number of edges next. In Section 5.3.1, we give the structural conditions that G must satisfy as a minimal counterexample. In Section 5.3.2, these structural conditions are used in an application of the discharging method.

5.3.1 Reducibility

In the following subsection, we introduce the following constraints on a minimal counterexample G.

Lemma 5.3.1. *Any minimal counterexample G has the following properties:*

(1) $|G| > 20l^2 + A(l)l + 1$*;*

(2) G *does not contain parallel edges; in particular,* G *is without digons;*

(3) no facial walk of a face of G *contains* $2l$ *consecutive 2-vertices;*

(4) for every vertex v *of* G*, it holds that* $d^F(v) > 20l^2 + A(l)l$*;*

(5) for every l *vertices* $u_1, u_2, \ldots, u_l$ *of* G *such that* $F(u_i) \cap F(u_j) = \{f\}$ *for all* $i \neq j$*, it holds that* $\sum_{i=1}^{l} d^F(u_i, f) > 20l^2 + (A(l) - 1)l + 1$*.*

Proof. By assumption, G is a 2-connected graph. We prove each of the assertions by contradiction.

Property (1) is trivial; assign a different colour to each vertex of G.

To show property (2), let e_1 and e_2 be parallel edges of G. If e_1 and e_2 are the two edges of a digon f, let G' be the graph $G - e_1$. Then G' is 2-connected, and by the minimality of G, G' must have a proper l-residue colouring c with at most $20l^2+A(l)l+1$ colours. Since the vertices of f are different colours, and each other face in G has its counterpart in G', then c is also a proper l-residue colouring of G. If e_1 and e_2 are not the two edges of a digon, let G_1 and G_2 be graphs formed by deleting the interior and exterior of e_1e_2 respectively. As both these graphs are 2-connected and smaller than G with respect to order, they have proper l-residue colourings c_1 and c_2 respectively with at most $20l^2 + A(l)l + 1$ colours, chosen such that the end vertices of e_1 are the same colour in c_1 and c_2. Then $c = c_1 \cup c_2$ is a proper l-residue colouring with at most $20l^2 + A(l)l + 1$ colours.

For property (3), if there exists a chain of 2-vertices $x_1x_2x_3 \ldots x_{2l}$, let v_1 (respectively v_{2l}) be the other neighbour of x_1 (respectively x_{2l}). The vertices v_1 and v_{2l} are distinct from all x_i, $1 \leq i \leq 2l$, otherwise the facial walk would contain at most $2l + 1$ vertices, and as G is 2-connected, the vertices of this facial walk would be the only vertices of G, contradicting property (1). Let $G' = G[V(G) - \{x_i : 1 \leq i \leq 2l\}] + v_1v_{2l}$. Then G' has a proper l-residue colouring c with at most $20l^2 + A(l)l + 1$ colours. If we assign the colour $c(v_1)$ to the even indexed vertices of the chain, and the colour $c(v_{2l})$ to the odd indexed vertices of the chain, the result is a proper l-residue colouring of G with at most $20l^2 + A(l)l + 1$ colours.

To show property (4), let G' be the graph created from G by the annihilation of v. From properties (1) and (2) and Lemma 5.2.2, G' is 2-connected. Thus, by the minimality of G, G' has a proper l-residue colouring c with at most $20l^2 + A(l)l + 1$ colours. Extend c to G by assigning v a colour not used on any vertex in $N_G^F(v)$, which is possible as by assumption, $d^F(v) \leq 20l^2 + A(l)l$. Then c is an l-residue colouring of G with at most $20l^2 + A(l)l + 1$ colours. By Observation 5.2.1, the only faces of G whose boundary vertices differ from the corresponding face in G' are the elements of $F_G(v)$. Since the colour of v is not used by any other vertex incident with one of these faces by construction, c is a proper l-residue colouring of G, a contradiction.

To show property (5), suppose the opposite is true. We construct a graph G_1 by the annihilation of u_1. From properties (1) and (2) and Lemma 5.2.2, G_1 is 2-connected. Also, since for any i and j such that $1 \leq i < j \leq l$, u_i and u_j have exactly one common incident face, and since G is 2-connected, they are not adjacent. Therefore the annihilation of any u_i creates no parallel edges at any other u_j such that $i < j$. Therefore, for each i such that $2 \leq i \leq l$, we construct a graph G_i from the graph G_{i-1} by the annihilation of u_i, and by the previous observation and the fact that any G_i has at least $20l^2 + A(l)l + 1 - i$ vertices, it follows from Lemma 5.2.2 that G_i is 2-connected. Thus, the graph G_l is smaller than G with respect to order. By the minimality of G, G_l has a proper l-residue colouring c' with at most $20l^2 + A(l)l + 1$ colours. We extend the colouring to G as follows. If there exists a colour used by c' on a vertex in $V_G(f) - \bigcup_{i=1}^{l} N_G^F(u_i, f)$ but on no vertex in $\bigcup_{i=1}^{l} N_G^F(u_i, f)$, we assign this colour to each u_i. If the opposite is true, every colour used in $V_G(f) - \bigcup_{i=1}^{l} N_G^F(u_i, f)$ is used in $\bigcup_{i=1}^{l} N_G^F(u_i, f)$ by c', we colour each u_i with a different colour not used by c' on any vertex in $\bigcup_{i=1}^{l} N_G^F(u_i, f)$, but, if possible, appearing in c'. Such a colouring is possible as, by assumption, at most $\sum_{i=1}^{l} d^F(u_i, f) \leq 20l^2 + (A(l) - 1)l$ colours are used by c' on $\bigcup_{i=1}^{l} N_G^F(u_i, f)$ and thus G may be coloured using at most $20l^2 + A(l)l$ colours. Both cases yield a colouring c.

We show that c is a proper l-residue colouring of G. If g is a face in $\bigcup_{i=1}^{l} F(v_i) - \{f\}$, then there is only one vertex u_i incident with g, and since u_i is assigned a colour not used by this face, the face has a proper l-residue colouring. For f, either the same colour already occurring on f is assigned to each u_i, or each u_i is assigned a distinct colour not already used by a vertex of f. This maintains a proper l-residue colouring of f. The remaining faces have the same boundary in both G and G'. Therefore, c is a proper l-residue colouring of G. □

A graph that contradicts Lemma 5.3.1 is defined to be *reducible*. We now prove the following bounds for the (reduced) face degree of a vertex in G in terms of the sum of its (reduced) configuration.

Lemma 5.3.2. *Let v be a vertex of a minimal counterexample G.*

(1) Let the faces incident with v be $f_0, f_1, f_2, \ldots, f_{d(v)-1}$ (and take subscripts modulo $d(v)$ when required). Then:

(a) If $d(v) = 2$, then $d^F(v) \leq 2l(w(f_0) + w(f_1)) - 2l - 2$;

(b) If $d(v) \geq 3$, then $d^F(v) \leq 2l \sum_{i=0}^{d(v)-1} w(f_i) - (2l+1)d(v) - (2l-1)\sigma$, where σ is the number of all i with the property that $w(f_i) + w(f_{i+1}) \leq 10l + \frac{1}{2}A(l) + 1$;

(c) If $d(v) = 3$, $w(f_0) = 3$, $w(f_1) \leq 10l + \frac{1}{2}A(l) - 2$, and $w(f_2) \leq 10l + \frac{1}{2}A(l) - 2$, then $w(f_1) + w(f_2) \geq 10l + \frac{1}{2}A(l) + 4$.

(2) Let f_0 be a face of G incident with v. Let the other faces incident with v be $f_1, f_2, \ldots, f_{d(v)-1}$. Then:

(a) If $d(v) = 2$, then $d^F(v, f_0) \leq 2lw(f_1) - 1$;

(b) If $d(v) \geq 3$, then $d^F(v, f_0) \leq 2l \sum_{i=1}^{d(v)-1} w(f_i) - (2l+1)d(v) + 4l + 1 - (2l-1)\sigma$, where σ is the number of all i $(1 \leq i \leq d(v) - 2)$ with the property that $w(f_i) + w(f_{i+1}) \leq 10l + \frac{1}{2}A(l) + 1$.

Proof. Assume the faces incident with v are enumerated in clockwise order. Suppose $d(v) = 2$, and let v_0 and v_1 be the neighbours of v. As every vertex incident with the same faces has the same face degree, we can assume without loss of generality that $d(v_0) \geq 3$. Let P be the walk from (but not including the ends of) v to v_1 obtained by concatenating the path from v to v_1 incident with both f_0 and f_1 and the portions of the facial walks of f_0 and f_1 from v_0 to v_1. Then P consists of $w(f_0) + w(f_1) - 2$ vertices of degree at least three, and $w(f_0) + w(f_1) - 1$ segments of at most $2l - 1$ 2-vertices, the first of which includes v. Hence, $d^F(v) \leq 2l(w(f_0)+w(f_1))-2l-2$, which verifies statement (1a).

Now suppose $d(v) \geq 3$ and let v_i be the other vertex of degree greater than 2 incident with both f_i and f_{i+1}. Let $P(v,f_i)$ be the portion of the facial walk of f_i between (but not including) v and v_{i-1} that includes v_i. Then $d^F(v) \leq \sum_{i=0}^{d(v)-1} |P(v,f_i)|$. Suppose $w(f_i)+w(f_{i+1}) \leq 10l+\frac{1}{2}A(l)+1$ and let u be the vertex of $P(v,f_i)$ adjacent to v. Then u is incident with f_i and f_{i+1}. If $d(u)=2$, then by statement (1a), $d^F(v) \leq 20l^2+A(l)l-2$, which contracts Lemma 5.3.1 (4). Hence, if $w(f_i)+w(f_{i+1}) \leq 10l+\frac{1}{2}A(l)+1$, $d(u) \geq 3$, and the segment between v and v_i contains no 2-vertices. Let $\sigma_i = 1$ if $w(f_i)+w(f_{i+1}) \leq 10l+\frac{1}{2}A(l)+1$ and $\sigma_i = 0$ otherwise. Notice that $\sigma = \sum_{i=1}^{d(v)} \sigma_i$. As $P(v,f_i) = V(f_i) - P(v,v_i)$, $|P(v,f_i)| \leq 2lw(f_i)-2l-1-(2l-1)\sigma_i$. Since $P(v,f_i)$ covers all the vertices of $N^F(v)$, we have

$$d^F(v) \leq \sum_{i=0}^{d(v)-1} |P(v,f_i)| \leq 2l \sum_{i=0}^{d(v)-1} w(f_i) - (2l+1)d(v) - (2l-1)\sigma,$$

which verifies statement (1b).

Suppose $d(v)=3$, $w(f_0)=3$, $w(f_1) \leq 10l+\frac{1}{2}A(l)-2$, $w(f_2) \leq 10l+\frac{1}{2}A(l)-2$, and $w(f_1)+w(f_2) \leq 10l+\frac{1}{2}A(l)+3$. Let v_i be the other vertex of degree greater than 2 incident with both f_i and f_{i+1} and let u_i be the vertex adjacent to v incident with both f_i and f_{i+1}. We prove for all i, $u_i = v_i$, that is, v is adjacent to no 2-vertex. If $u_i \neq v_i$ for some i, then $d(u_i)=2$. If $d(u_0)=2$ or $d(u_2)=2$, then by statement (1a), $d^F(u_2) \leq 20l^2+A(l)l-2$, which contradicts Lemma 5.3.1 (4). Suppose $d(u_1)=2$. As v and v_1 are incident with both f_1 and f_2 and $w(f_1)+w(f_2) \leq 10l+\frac{1}{2}A(l)+3$, there are at most $10l+\frac{1}{2}A(l)+1$ vertices of degree at least three in the closed face-vertex neighbourhood of u_1, but by the previous argument, v is adjacent to v_0 and v_2. Therefore the closed face-vertex neighbourhood of u_1 contains at most $10l+\frac{1}{2}A(l)+1$ segments of 2-vertices between vertices of degree at least three. Hence, there are $10l + \frac{1}{2}A(l) + 3$ vertices of degree at least three in the face-vertex neighbourhood of u_1, but at most $10l + \frac{1}{2}A(l)$ segments of two vertices, one of which includes u_1, so $d^F(u_1) \leq 20l^2 + A(l)l$, which contradicts Lemma 5.3.1 (4). Therefore, we correct our bound obtained in the proof of statement (1b) by removing the 2-vertices between v and v_1, and obtain $d^F(v) \leq 20l^2 + A(l)l$, which contradicts Lemma 5.3.1 (4) and verifies statement (1c).

Statements (2a) and (2b) follow immediately from statements (1a) and (1b) respectively. Assuming f_0 is between $f_{d(v)-1}$ and f_1 with u_0 the other vertex of degree greater than 2 incident with both f_0 and f_1, we need simply remove the vertices counted in the facial walk of f_0 between $v_{d(v)-1}$ and u_0. If $d(v) = 2$, there are $w(f_0) - 2$ vertices of degree at least three between $v_{d(v)-1}$ and u and $w(f_0) - 1$ segments of at most $2l - 1$ 2-vertices, so the number of such vertices counted is $2lw(f_0) - 2l - 1$. If $d(v) \geq 3$, there are $w(f_0) - 3$ vertices of degree at least three between $v_{d(v)-1}$ and u_0 and $w(f_0) - 2$ segments of at most $2l - 1$ 2-vertices, so the number of such vertices counted is $2lw(f) - 4l - 1$. □

5.3.2 Discharging

Now that we have determined some of the properties of the minimal counterexample G, we derive a contradiction through the use of the discharging method.

To start, the following charges are assigned:

- Each vertex v is given $d(v) - 6$ units of charge;
- Each face f is given $2|V(f)| - 6$ units of charge.

The following observation is easily derived through use of Euler's formula.

Observation 5.3.3. *The sum of the charges of the graph G is -12.*

Next, the defined rules 1 through 3 are applied sequentially. These rules are based on those of [36]. In our case, a face is *large* if

$$w(f) \geq \begin{cases} 20 & \text{if } l = 2, \\ \lceil 8.5l + 4.7 \rceil & \text{if } l = 3, 4, \\ \lceil 9.25l + 5.75 \rceil & \text{if } l \geq 5, \end{cases}$$

otherwise, it is *small*.

Rule 1: Every face not being a pseduodigon sends two units of charge to each incident 2-vertex. Each pseudodigon does the same, except that one of the respective 2-vertices receives no charge.

Rule 2: Every small face distributes its remaining charge evenly to all incident vertices of degree at least 3. Every large face sends 1.5 units of charge to all incident vertices of degree at least 3.

Rule 3: Every large face sends each incident vertex v_i with a negative charge c_i a charge of $-c_i$.

This discharging method is designed to obtain nonnegative charges for all vertices and all faces, contradicting Observation 5.3.3. Let $c_0(v)$ and $c_0(f)$ be the initial charge assigned to vertex v and face f respectively, and let $c_i(v)$ and $c_i(f)$ be the charge on vertex v and face f after Rule i is applied. The charges are examined after the application of each rule.

Observation 5.3.4. *After the application of Rule 1, each face has a charge of* $c_1(f) = 2w'(f) - 6$.

Proof. If f is not a pseudodigon, $c_1(f) = c_0(f) - 2(|V(f)| - w(f)) = 2|V(f)| - 6 - 2(|V(f)| - w(f)) = 2w(f) - 6 = 2w'(f) - 6$. If f is a pseudodigon, $c_1(f) = c_0(f) - 2(|V(f)| - w(f)) + 2 = 2|V(f)| - 6 - 2(|V(f)| - w(f)) + 2 = 2w(f) - 4 = 2w'(f) - 6$. □

Since $w'(f) \geq 3$, $2w'(f) - 6 \geq 0$, thus the charge of each face is nonnegative by Observation 5.3.4. If f is a large face, $w'(f) \geq 20$, so $c_2(f) = c_1(f) - 1.5w'(f) = 0.5w'(f) - 6 \geq 4$, thus each large face has a positive charge after the application of Rule 2.

Observation 5.3.5. *After the application of Rule 2, each vertex v has the following charge:*

$$c_2(v) = 3d(v) - 6 - \left(6 \sum_{f_i \ small} \left(\frac{1}{w'(f_i)}\right)\right) - \left(\sum_{f_i \ large} \frac{1}{2}\right).$$

Proof. By Rule 2, each small face f sends $\frac{c_1(f)}{w(f)}$ to each of its incident vertices of degree at least three, and each large face f sends $\frac{3}{2}$ to each of its incident vertices of degree at least three. Thus, for each vertex v of degree at least three:

$$\begin{aligned}
c_2(v) &= c_1(v) + \sum_{f_i \ small} \left(\frac{c_1(f_i)}{w(f_i)}\right) + \sum_{f_i \ large} \frac{3}{2} \\
&= d(v) - 6 + \sum_{f_i \ small} \left(\frac{2w'(f_i) - 6}{w(f_i)}\right) + \sum_{f_i \ large} \frac{3}{2} \\
&= d(v) - 6 + \sum_{f_i \ small} \left(2 - \frac{6}{w'(f_i)}\right) + \sum_{f_i \ large} \left(2 - \frac{1}{2}\right) \\
&= 3d(v) - 6 - \left(6 \sum_{f_i \ small} \left(\frac{1}{w'(f_i)}\right)\right) - \left(\sum_{f_i \ large} \frac{1}{2}\right),
\end{aligned}$$

as required. □

We can now consider the application of Rule 3. If $c_2(v) < 0$, then v is referred to as a *special vertex*. Let $d'(v)$ be the number of large faces incident with v. The following two claims are established.

Claim 5.3.6. *Every special vertex has degree at most 5 and is incident with a large face.*

Proof. If $d(v) \geq 6$, then $c_2(v) \geq 3d(v) - 6 - 2(d(v) - d'(v)) - d'(v)/2 \geq 3d(v) - 6 - 2d(v) \geq d(v) - 6 \geq 0$. Thus every special vertex has degree at most 5.

We now proceed by contradiction. Let v be a special vertex. If $d(v) = 2$, at least one of the faces, say f_0, is a pseudodigon, and since f_1 is small, $w(f_1) \leq 9.25l + 5.75$. Hence, by Lemma 5.3.2 (1a), $d^F(v) \leq 18.5l^2 - 9.5l - 2$, which contradicts Lemma 5.3.1 (4).

Therefore, v is a special vertex of degree $d \geq 3$. Let the faces incident with v be $f_0, f_1, f_2, \ldots, f_{d(v)-1}$, ordered so that $w'(f_0) \leq w'(f_1) \leq w'(f_2) \leq \cdots \leq w'(f_{d(v)-1})$. By Observation 5.3.5, $c_2(v) = 3d(v) - 6 - 6\sum_i 1/w'(f_i) < 0$, which implies the following inequality:

$$\sum_i 1/w'(f_i) > d/2 - 1 \tag{5.1}$$

Assume v is of degree 3. By (5.1), $w'(f_0) \leq 5$. Suppose $w'(f_0) = 3$. We show $w'(f_0) + w'(f_2) < 10l + \frac{1}{2}A(l) + 1$, by using the fact that f_2 is a small face:

$$\begin{array}{ll}
l = 2 & w'(f_0) + w'(f_2) < 23 \leq 25 \\
l = 3, 4 & w'(f_0) + w'(f_2) < 8.5l + 7.7 \leq 10l + 5 \\
l = 5, 6 & w'(f_0) + w'(f_2) < 9.25l + 8.75 \leq 10l + 5 \\
l = 7 & w'(f_0) + w'(f_2) < 9.25l + 8.75 \leq 10l + 4 \\
l = 8 & w'(f_0) + w'(f_2) < 9.25l + 8.75 \leq 10l + 3 \\
l = 9, 10 & w'(f_0) + w'(f_2) < 9.25l + 8.75 \leq 10l + 2 \\
l \geq 11 & w'(f_0) + w'(f_2) < 9.25l + 8.75 \leq 10l + 1
\end{array}$$

Then by Lemma 5.3.2 (1b), $d^F(v) \leq 2l\sum_i w'(f_i) - 10l - 1$, which, together with

Lemma 5.3.1 (4), implies the following inequality:

$$\sum_i w'(f_i) \geq 10l + \frac{1}{2}A(l) + 6. \tag{5.2}$$

For $l \geq 5$, if $w'(f_1) \leq 6$, then by (5.2), the modified configuration of v is $(3, 6, 10l+\frac{1}{2}A(l)-3)$, which contradicts Lemma 5.3.2 (1c) and if $w'(f_1) \geq 7$ then, by Lemma 5.2.3 applied to the tuples $(3, 7, 42)$ and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1/2$ or $\sum_i w'(f_i) \leq 52 < 10l + \frac{1}{2}A(l) + 6$, which contradicts either (5.1) or (5.2). For $l = 3, 4$, if $w'(f_1) \leq 7$, then $l = 3$, and the modified configuration of v is $(3, 7, 30)$, which contradicts Lemma 5.3.2 (1c), and if $w'(f_1) \geq 8$ then, by Lemma 5.2.3 applied to the tuples $(3, 8, 24)$ and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1/2$ or $\sum_i w'(f_i) \leq 35 < 10l + \frac{1}{2}A(l) + 6$, which contradicts either (5.1) or (5.2). For $l = 2$, if $w'(f_1) \leq 9$, then the modified configuration of v is either $(3, 8, 19)$, which contradicts Lemma 5.3.2 (1c), or $(3, 9, 18)$ or $(3, 9, 19)$, which contradict (5.1), and if $w'(f_1) \geq 10$ then, by Lemma 5.2.3 applied to the tuples $(3, 10, 15)$ and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1/2$ or $\sum_i w'(f_i) \leq 28 < 10l + \frac{1}{2}A(l) + 6$, which contradicts either (5.1) or (5.2).

Suppose $w'(f_0) = 4$. Then $w'(f_1) \leq 7$, so by Lemma 5.3.2 (1b), $d^F(v) \leq 2l \sum_i w'(f_i) - 8l - 2$, which, together with Lemma 5.3.1 (4), implies the following inequality:

$$\sum_i w'(f_i) \geq 10l + \frac{1}{2}A(l) + 5. \tag{5.3}$$

If $w'(f_1) = 4$, then the modified configuration of v is $(4, 4, 10l + \frac{1}{2}A(l) - 3)$, which contradicts Lemma 5.3.2 (1b). If $l \geq 3$ and $w'(f_1) \geq 5$ then, by Lemma 5.2.3 applied to the tuples $(4, 5, 20)$ and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1/2$ or $\sum_i w'(f_i) \leq 29 < 10l + \frac{1}{2}A(l) + 6$, which contradicts either (5.1) or (5.3). If $l = 2$ and $w'(f_2) = 5$, then by (5.3), v is incident with a large face. If $l = 2$, $w'(f_1) \geq 6$ and by

Lemma 5.2.3 applied to the tuples $(4, 6, 12)$ and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1/2$ or $\sum_i w'(f_i) \leq 22 < 10l + \frac{1}{2}A(l) + 6$, which contradicts either (5.1) or (5.3).

Suppose $w'(f_0) = 5$. Then $w'(f_1) \leq 6$, so by Lemma 5.3.2 (1b), $d^F(v) \leq 2l \sum_i w'(f_i) - 8l - 2$, which, together with Lemma 5.3.1 (4), again implies (5.3). By Lemma 5.2.3 applied to the tuples $(5, 5, 10)$ and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1/2$ or $\sum_i w'(f_i) \leq 20 < 10l + \frac{1}{2}A(l) + 5$, which contradicts either (5.1) or (5.3).

If v is of degree 4, then by Lemma 5.3.2 (1b) and Lemma 5.3.1, we again have (5.3). If $w'(f_0) = w'(f_1) = w'(f_2) = 3$, then by (5.3), the modified configuration of v is $(3, 3, 3, 10l + \frac{1}{2}A(l) - 4)$, which, by Lemma 5.3.2 (1b), contradicts Lemma 5.3.1 (4). Thus, by Lemma 5.2.3 applied to the tuples (3, 3, 4, 12) and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 1$ or $\sum_i w'(f_i) \leq 22 < 10l + \frac{1}{2}A(l) + 5$, which contradicts either (5.1) or (5.3).

If v is of degree 5, then by Lemma 5.3.2 (1b) and Lemma 5.3.1, we have (5.2). By Lemma 5.2.3 applied to the tuples (3,3,3,3,6) and the modified configuration of v, it must hold that $\sum_i 1/w'(f_i) \leq 3/2$ or $\sum_i w'(f_i) \leq 18 < 10l + \frac{1}{2}A(l) + 6$, which contradicts either (5.1) or (5.2). $\square$

Claim 5.3.7. *Every large face has a nonnegative charge after the application of Rule 3.*

Proof. Let f be an arbitrary large face of G. For each special vertex v incident with f, we establish that it is incident with exactly one large face and list its the possible f-reduced or modified f-reduced configurations. For each case, we find an upper bound on the charge of these vertices. The results are summarized in Table 5.1.

If v is a 2-vertex, then, by Rule 1, its f-reduced configuration is the 1-tuple (2) and its charge is -2.

Suppose now that v has degree $d \geq 3$. Let the other faces incident with v be $f_1, f_2, \ldots, f_{d(v)-1}$, ordered so that $w'(f_1) \leq w'(f_2) \leq \cdots \leq w'(f_{d(v)-1})$, and let d' denote the number of large faces incident with v. By Claim 5.3.6, $d \leq 5$. By Observation 5.3.5,

$$c_2(v) = 3d(v) - 6 - \left(6 \sum_{f_i \ small} \left(\frac{1}{w'(f_i)}\right)\right) - \left(\sum_{f_i \ large} \frac{1}{2}\right).$$

Since v is a special vertex, its charge is negative, so we obtain that

$$3d - 6 - \left(6 \sum_{f_i \ small} \left(\frac{1}{w'(f_i)}\right)\right) - d'/2 < 0 \tag{5.4}$$

Furthermore, $w'(f_i) \geq 3$, and hence

$$d - 6 + (3/2)\, d' = 3d - 6 - 2(d - d') - d'/2 < 0.$$

From this, it immediately follows that $d \leq 4$ and $d' = 1$, i.e, f is the only large face incident with v.

Assume first that $d = 3$. Then from (5.4), we have $5/2 - 6(1/w'(f_1) + 1/w'(f_2)) < 0$, so either $w'(f_1) = 3$ and $w'(f_2) \leq 11$, or $w'(f_1) = 4$ and $w'(f_2) \leq 5$. Now, if $d = 4$, then we have $11/2 - 6\sum_i 1/w'(f_i) < 0$, and if some $w'(f_i)$ were greater than or equal to 4, this inequality would not hold, so the modified f-reduced configuration of v is $(3, 3, 3)$. In each case a bound on the charge on v after the application of Rule 1 and Rule 2 can be found by applying (5.4). The results are summarized in Table 5.1.

We now consider the sum of the charges of the special vertices incident with f. For $l \geq 5$, let S' denote the set of special vertices incident with f with the f-reduced configuration $(3, 10)$ or $(3, 11)$, for $l = 3, 4$, let S' denote the set of special vertices incident with f with the f-reduced configuration $(3, 11)$, and for $l = 2$, let $S' = \emptyset$. Let S denote the set of all the special vertices incident with f not in S'. We observe the following:

(modified) f-reduced configuration	charge
(2)	-2
$(3, x), x \leq 11$	$1/2 - 6/x \geq -3/2$
$(4, x), x \leq 5$	$1 - 6/x \geq -1/2$
$(3, 3, 3)$	$-1/2$

Table 5.1: The possible f-reduced or modified f-reduced configuration of special vertices incident with f, and their charges.

Observation 5.3.8. *In every subset $\{u_1, u_2, \ldots, u_l\}$ of vertices of S, there exist two vertices incident with at least two common faces.*

Proof. Suppose the contrary, that is, there exists a subset $U' = \{u'_1, u'_2, \ldots, u'_l\}$ such that no two vertices are incident with two common faces, i.e., the only common face of each pair of vertices is f. Consider each of the possible f-reduced or modified f-reduced configurations of each u_i given by Table 5.1. If the f-reduced configuration of u_i is (2), then by Lemma 5.3.2 (2a), $d^F(u_i, f) \leq 4l - 1$. If the modified f-reduced configuration of u_i is $(3, x)$, then by Lemma 5.3.2 (2b), $d^F(u_i, f) \leq 2l(3 + x) - 4l - 1$. If $l \geq 5$, $x \leq 9$, so $d^F(u_i, f) \leq 20l - 1$; if $l = 3, 4$, $x \leq 10$, so $d^F(u_i, f) \leq 22l - 1$; and if $l = 2$, $x \leq 11$, so $d^F(u_i, f) \leq 24l - 1$. If the modified f-reduced configuration of u_i is $(4, x), x \leq 5$, then by Lemma 5.3.2 (2b), $d^F(u_i, f) \leq 2l(4 + x) - 4l - 1 \leq 12l - 1$. If the modified f-reduced configuration of u_i is $(3, 3, 3)$, then by Lemma 5.3.2 (2b), $d^F(u_i, f) \leq 14l - 3$. Therefore, if $l \geq 5$, $d^F(u_i, f) \leq 20l - 1$ and $\sum_{i=1}^{l} d^F(u_i, f) \leq 20l^2 - l$, if $l = 3, 4$, $d^F(u_i, f) \leq 22l - 1$ and $\sum_{i=1}^{l} d^F(u_i, f) \leq 22l^2 - l \leq 20l^2 + 7l + 1$, and if $l = 2$, $d^F(u_i, f) \leq 47$ and $\sum_{i=1}^{l} d^F(u_i, f) \leq 94$. A contradiction follows from Lemma 5.3.1

(5). $\square$

Let G^* be formed from G by sequentially deleting each 2-vertex of G and adding an edge between its neighbours. Let T be a subset of S defined as follows. For $i \geq 1$, let t_i be a vertex chosen from $S \setminus \bigcup_{j=1}^{i-1} S_j$ (where S_i is the set of special vertices in the closed f-reduced face-vertex neighbourhood of t_i minus the set $\bigcup_{j=1}^{i-1} S_j$) with the following priority:

1. a 2-vertex,

2. a 4-vertex,

3. a 3-vertex for which all special face-vertex neighbours in $S \setminus \bigcup_{j=1}^{i-1} S_j$ are 3-vertices and incident with a common small face,

4. a 3-vertex with at most one neighbour in G^* which is also in the set $S \setminus \bigcup_{j=1}^{i-1} S_j$ and not all special face-vertex neighbours in $S \setminus \bigcup_{j=1}^{i-1} S_j$ are incident with a common small face

5. a 3-vertex with more than one neighbour in G^*, each of which is also in the set $S \setminus \bigcup_{j=1}^{i-1} S_j$ and not all special face-vertex neighbours in $S \setminus \bigcup_{j=1}^{i-1} S_j$ are incident with a common small face.

Then $T = \{t_i\}$ and every pair of vertices t_i and t_j are incident with only one common face, namely f. Hence, by Observation 5.3.8, $|T| \leq l - 1$. We establish the following claim.

Claim 5.3.9. *Let R_i denote the total charge of the vertices in S_i after Rule 1 and Rule 2 have been applied. Then R_i is at least -4.*

Proof. Suppose $R_i < -4$. We consider the following cases.

Case 1: $d(t_i) = 2$**.** Then all vertices in S_i are incident with the same pseudodigon, and hence $S_i \subseteq \{t_i, v_1, v_2\}$, where v_1 and v_2 are the two vertices of degree at least 3 on the pseudodigon. By Table 5.1, $S = \{t_i, v_1, v_2\}$, and $d(v_1) = d(v_2) = 3$. Hence $F(v_1) = F(v_2)$, and so v_1 and v_2 each have the modified f-reduced configuration $(3, 3)$. Then v_1 and v_2 form a small face of weight 3 with v_3, but v_3 is a cut-vertex. Hence $R_i \geq -4$.

Case 2: $d(t_i) = 4$**.** Then the f-reduced configuration of t_i is $(3, 3, 3)$. By the priority we have chosen the vertices, $S_i \subseteq N_{G^*}[t_i]$. Suppose $N_{G^*}[t_i] = \{a, b, c, d\}$ and b and c each share two faces of weight 3 with t_i. By Table 5.1, S_i contains at least four vertices. If all five vertices are incident with f, then $d(b) = d(c) = 4$. Hence, a and d have the f-reduced configuration $(3, 3)$ and b and c have the f-reduced configuration $(3, 3, 3)$. But then if e is the third vertex of degree at least three of the weight 3 face with c and d but not t_i, then e is a cut vertex. Hence, only four vertices are incident with f, then without loss of generality, b is not incident with f. But then $d(c) \geq 4$, so $R_i \geq -4$.

Case 3: $d(t_i) = 3$ **and all face-vertex neighbours in** $S \setminus \bigcup_{j=1}^{i-1} S_j$ **are incident with a common small face** f'**.** If $w(f') = 3$, then each vertex in S_i has the f-reduced configuration $(3, 3)$. As with Case 1, this configuration creates a cut-vertex. Hence, $w(f') \geq 4$, so $R_i \geq w(f')(\frac{1}{2} - \frac{6}{w(f')}) \geq -4$.

Case 4: $d(t_i) = 3$ **and** t_i **has at most one neighbour in** G^* **which is also in the set** $S \setminus \bigcup_{j=1}^{i-1} S_j$**.** Let f_1 and f_2 be the two small faces incident with t_i such that $w(f_1) \leq w(f_2)$ and $w(f_1) \leq 4$. Let S_{i_2} be the set of vertices in S_i incident with f_2 and let $S_{i_1} = S_i \setminus S_{i_2}$. If R_{i_j} is the total charge of the vertices of S_{i_j}, then $R_i = R_{i_1} + R_{i_2}$. If t_i has a neighbour in G^* which is also in the set $S \setminus \bigcup_{j=1}^{i-1} S_j$ and incident with f_2, then $R_{i_2} \geq (w(f_2) - 1)(\frac{1}{2} - \frac{6}{(w(f_2))}) \geq -3$, and $R_{i_1} = (w(f_1) - 3)(\frac{1}{2} - \frac{6}{w(f_1)}) \geq -1$. Hence $R_i \geq -4$. Otherwise t_i has no neighbour in G^* which is also in the set $S \setminus \bigcup_{j=1}^{i-1} S_j$ and incident with f_2. If $w(f_2) \neq 5$, then $R_{i_2} \geq (w(f_2) - 2)(\frac{1}{2} - \frac{6}{(w(f_2))}) \geq -2$, and $R_{i_1} = (w(f_1) - 2)(\frac{1}{2} - \frac{6}{w(f_1)}) \geq -2$ and the result follows. If $w(f_2) = 5$ and $w(f_1) = 3$, a similar calculation shows $R_{i_2} \geq -2.1$ and $R_{i_1} \geq -1.5$, and the result follows. If $w(f_2) = 5$ and $w(f_1) = 4$, then $R_{i_2} \geq c(v) + (w(f_2) - 3)(\frac{1}{2} - \frac{6}{(w(f_2))}) = -0.2 + (5 - 3)(\frac{1}{2} - \frac{6}{5}) \geq -1.6$ and $R_{i_1} \geq -2$ and the result follows.

Case 5: $d(t_i) = 3$ and t_i has multiple neighbours in G^*, each of which is also in the set $S \setminus \bigcup_{j=1}^{i-1} S_j$. Then every vertex in $S \setminus \bigcup_{j=1}^{i-1} S_j$ has this property. Hence, there exists a cycle of these vertices, the exterior of which is f, and since $|T| \leq l - 1$, there are at most $2l - 1$ vertices on the cycle. But then $w(f) \leq 2l - 1$, which contradicts that f is a large face. Hence, there are no such vertices in T. □

For $|T| < i < l$, $S_i = \emptyset$. Let S_i^* be formed by adding as many vertices of S' to S_i as possible such that the charge of the vertices in S_i^* is at least -4 so that the collection of sets S_i^* are mutually disjoint. We now consider the number of vertices in $S' \setminus \bigcup_{i=1}^{l-1} S_i^*$.

Claim 5.3.10. $|S' \setminus \bigcup_{i=1}^{l-1} S_i^*| \leq w(f) - 3(l - 1)$.

Proof. If $S' \setminus \bigcup_{j=1}^{l-1} S_i^*$ is empty, then the result is obvious. Suppose some S_i^* contains fewer than three vertices of degree at least three. Note that any vertex of $S_i^* \setminus S_i$ has (negative) charge at least $-1/10$ and hence the sum of the charges of the vertices in S_i^* is less than -3.9. By Table 5.1, S_i^* contains a 2-vertex, and the two vertices of degree at least three each have degree three. Hence, they must have the same modified f-reduced configuration, which must be $(3, 4)$ (otherwise G has a cut vertex). Suppose some vertex of degree at least three on the 4-face but not the pseudodigon is special. Then this vertex is degree 3, and has modified f-reduced configuration $(4, x)$, $4 \leq x \leq 5$. Then this vertex shares no face with a special 2-vertex, so it is in some S_j^*, and by Table 5.1, S_j^* contains at least four vertices. Otherwise, every such S_i^* without a special vertex of degree at least 3 on the 4-face but not the pseudodigon can be assigned a unique non-special vertex from its 4-face, which is clearly not in S'. The result follows. $\square$

We conclude by computing $c_3(f)$. For $l \geq 5$, each vertex in S' has charge at least $-1/10$, so the charge required for $S' \setminus \bigcup_{i=1}^{l-1} S_i^*$ is at most $\frac{1}{10}(w'(f) - 3(l-1))$, and

$$
\begin{aligned}
c_3(f) &= \frac{1}{2}w'(f) - 6 - 4(l-1) - \frac{1}{10}(w'(f) - 3(l-1)) \\
c_3(f) &= \frac{2}{5}w'(f) - \frac{37}{10}l - \frac{17}{10} \\
c_3(f) &\geq \frac{37}{10}l + \frac{23}{10} - \frac{37}{10}l - \frac{17}{10} \\
c_3(f) &\geq \frac{3}{5}.
\end{aligned}
$$

For $l = 3, 4$, each vertex in S' has charge at least $-1/22$, so the charge required for $S' \setminus \bigcup_{i=1}^{l-1} S_i^*$ is at most $\frac{1}{22}(w'(f) - 3(l-1))$, and

$$
\begin{aligned}
c_3(f) &= \frac{1}{2}w'(f) - 6 - 4(l-1) - \frac{1}{22}(w'(f) - 3(l-1)) \\
c_3(f) &= \frac{5}{11}w'(f) - \frac{85}{22}l - \frac{41}{22} \\
c_3(f) &\geq \frac{85}{22}l + \frac{47}{22} - \frac{85}{22}l - \frac{41}{22} \\
c_3(f) &\geq \frac{3}{11}.
\end{aligned}
$$

For $l = 2$, there are no vertices in S', so

$$\begin{aligned}
c_3(f) &= \frac{1}{2}w'(f) - 6 - 4(l-1)\\
c_3(f) &= \frac{1}{2}w'(f) - 10\\
c_3(f) &\geq 10 - 10\\
c_3(f) &\geq 0.
\end{aligned}$$

Thus every large face has a nonnegative charge after the application of Rule 3, completing the proof of Claim 5.3.7. $\square$

We now complete the proof of Theorem 5.1.2.

Theorem 5.1.2. *Every 2-connected plane graph has a proper l-residue colouring with at most $20l^2 + A(l)l + 1$ colours, where*

$$A(l) = \begin{cases} 8 & \text{if } 2 \leq l \leq 6,\\ 6 & \text{if } l = 7,\\ 4 & \text{if } l = 8,\\ 2 & \text{if } l = 9, 10,\\ 0 & \text{if } l \geq 11. \end{cases}$$

Proof. By Claim 5.3.6, every special vertex of a minimum counterexample G finishes with a nonnegative charge after the application of the discharging rules, and hence every vertex finishes with a nonnegative charge; and by Claim 5.3.7, the charge of every face is also nonnegative, which is a contradiction to Observation 5.3.3. The result follows as no minimal counterexample exists. $\square$

5.4 Toroidal Graphs

The proof of Theorem 5.1.3 follows almost immediately from the proof of Theorem 5.1.2. We proceed by contradiction; take a minimal counterexample G with respect to the number of vertices first and to the number of edges next. The structural conditions that G must satisfy are identical to those of Section 5.3.1 with $A(l)$ replaced by $B(l)$. We derive a contradiction through the use of the discharging method.

To start, the following charges are assigned:

- Each vertex v is given $d(v) - 6$ units of charge;
- Each face f is given $2|V(f)| - 6$ units of charge.

The following observation is easily derived through use of Euler's formula.

Observation 5.4.1. *The sum of the charges of the graph G is* 0*.*

Next, the defined rules 1 through 3 are applied sequentially and are the same as the proof for plane graphs. Here, a face is *large* if

$$w(f) \geq \begin{cases} 21 & \text{if } l = 2, \\ \lceil 8.5l + 4.7 \rceil & \text{if } l = 3, 4, \\ \lceil 9.25l + 5.75 \rceil & \text{if } l \geq 5, \end{cases}$$

otherwise, it is *small*.

This discharging method is designed to obtain nonnegative charges for all vertices and all faces, and a positive charge for at least one face, contradicting Observation 5.4.1. Let $c_0(v)$ and $c_0(f)$ be the initial charge assigned to vertex v and face f respectively, and let $c_i(v)$ and $c_i(f)$ be the charge on vertex v and face f after Rule i is applied. The charges are examined after the application of each rule. As the rules were not changed, Observations 5.3.4 and 5.3.5 still apply.

We can now consider the application of Rule 3. If $c_2(v) < 0$, then v is referred to as a *special vertex*. Let $d'(v)$ be the number of large faces incident with v. The following three claims are established.

Claim 5.4.2. *There exists a special vertex.*

Proof. Suppose there are no special vertices. Then every vertex has nonnegative charge, and by Rule 2, every face has nonnegative charge. Hence, by Observation 5.4.1, every vertex and every face has zero charge, so every vertex has degree at most six and every face is small.

Let v be a vertex of degree $d \geq 3$. Let the faces incident with v be $f_0, f_1, f_2, \ldots, f_{d-1}$, ordered so that $w'(f_0) \leq w'(f_i) \leq w'(f_2) \leq \cdots \leq w'(f_{d-1})$. As $c_2(v) = 0$, Observation 5.3.5 implies the following inequality

$$\sum_i 1/w'(f_i) = d/2 - 1.$$

Hence, the modified configuration of v is one of $(3,7,42)$, $(3,8,24)$, $(3,9,18)$, $(3,10,15)$, $(3,12,12)$, $(4,5,20)$, $(4,6,12)$, $(4,8,8)$, $(5,5,10)$, $(6,6,6)$, $(3,3,4,12)$, $(4,4,4,4)$, $(3,3,3,3,6)$, $(3,3,3,3,3,3)$. Each of these configurations contradict Lemma 5.3.1 (4) by applying Lemma 5.3.2 (1b); note that 42 is a small face only if $l \geq 4$ and 24 is a small face only if $l \geq 3$. Hence, every vertex in G is a 2-vertex, so G is a cycle of at most $2l-1$ vertices by Lemma 5.3.1 (3), contradicting Lemma 5.3.1 (1). Therefore, G contains a special vertex as required. □

Claim 5.4.3. *Every special vertex has degree at most 5 and is incident with a large face.*

Proof. The result follows from Claim 5.3.6 except when $l = 2$, a face of weight 20 is now a small face. If v is of degree 3, and $w'(f_0) = 3$, then $w'(f_0) + w'(f_2) \leq 23 < 26$. If $w'(f_1) \leq 9$, then the modified configuration of v is either $(3, 8, 20)$, which contradicts Lemma 5.3.2 (1c), or $(3, 9, 19)$ or $(3, 9, 20)$, which contradicts (5.1). Otherwise, the result follows from Claim 5.3.6. $\square$

Claim 5.4.4. *Every large face has a positive charge after the application of Rule 3.*

Proof. The result follows from Claim 5.3.7 except when $l = 2$ and $w(f) = 20$. But then f is not a large face, which completes the proof. $\square$

We now complete the proof of Theorem 5.1.3.

Theorem 5.1.3. *Every closed 2-cell embedding of a toroidal graph has a proper l-residue colouring with at most $20l^2 + B(l)l + 1$ colours, where*

$$B(l) = \begin{cases} 10 & \textit{if } l = 2, \\ \\ A(l) & \textit{if } l \geq 3. \end{cases}$$

Proof. By Claim 5.4.3, every special vertex of a minimum counterexample G finishes with a nonegative charge after the application of the discharging rules, and hence every vertex finishes with a nonnegative charge. By Rule 2, the charge of every small face is nonnegative. By Claim 5.4.2, there exists a special vertex, so by Claim 5.4.3, there exists a large face, and by Claim 5.4.4, the charge of every large face is positive. Hence, the sum of the charges is positive, which is a contradiction to Observation 5.4.1. The result follows as no minimal counterexample exists. $\square$

5.5 Lower Bound

In this section, we give an example showing that the sharp upper bound on the number of colours required for a proper l-residue colouring is at least $2l^2+l$. Let Q_l be the graph on $2l^2+l$ vertices constructed as follows. Take l disjoint cycles of length $2l+1$, and label each vertex $z_{i,j}$, where i corresponds to the copy of C_{2l+1} and j to the position of the vertex in that cycle. For each i, add an edge between $z_{i,2l+1}$ and $z_{i+1,1}$, and embed the graph such that it is outerplane, that is, the outerface contains every vertex. The graph is depicted in Figure 5.5.1.

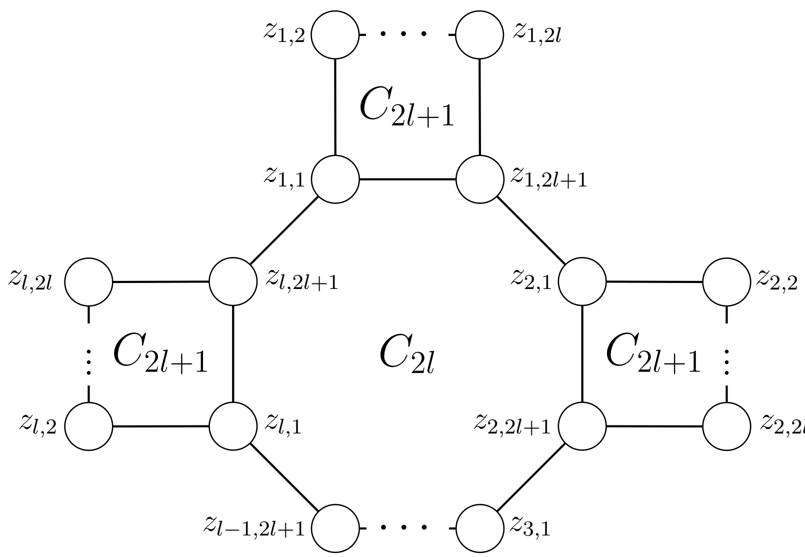


Figure 5.5.1: Graph Q_l

Theorem 5.5.1. *A proper l-residue colouring of Q_l requires $2l^2+l$ colours.*

Proof. We show that each vertex of Q_l must be a different colour. No colour can appear $l+1$ times on any of the cycles of length $2l+1$, thus it can occur at most once on each cycle. Now, considering the outer face, any colour can occur a maximum of l times, thus it can only occur once on the graph. Therefore, each vertex must be assigned a different colour. □

Since we do not make use of the $2l$-face in the proof of the previous theorem, we can take any closed 2-cell embedded graph with a $2l$-face and replace such a face with a copy of Q_l to produce a graph, the proper l-residue colouring of which requires at least $2l^2 + l$ colours.

For proper strong parity vertex colouring, the graph Q_2 is precisely the graph H_0 depicted in Figure 4.5.1. We now consider the order of the sharp upper bound on the proper l-residue chromatic number.

Theorem 5.5.2. *For closed 2-cell embedded plane or toroidal graphs, the sharp upper bound on the proper l-residue chromatic number is $\Theta(l^2)$.*

Proof. By Theorems 5.1.2 and 5.1.3, the sharp upper bound is at most $20l^2 + A(l)l + 1$ and $20l^2 + B(l)l + 1$ respectively, so the sharp upper bound is $O(l^2)$. By Theorem 5.5.1, the sharp upper bound is at least $2l^2 + l$, so the sharp upper bound is $\Omega(l^2)$. The result follows. □

Thus the sharp upper bound is on the order of l^2, and has been determined to within a factor of 10.

Chapter 6

Conclusion

The Four Colour Conjecture was first posed in 1852 and its validity was finally confirmed in 1976. Work on this problem of the upper bound of proper vertex colourings inspired further variations of colouring the vertices of planar graphs, which have been central to the development of graph theory. The solution also provided a useful approach, the Discharging Method, used to prove many of the upper bounds of vertex colouring problems.

In comparison to the proof of the Four Colour Theorem, which required 633 reducible configurations and 32 discharging rules, the proofs of the upper bounds of the restricted colouring problems are less complex. The proof of Theorem 3.2.9, the upper bound of the cyclic chromatic number, involved 5 reducibility lemmas and 7 discharging rules. Similarly, the proof of Theorem 3.2.12, the upper bound for 3-connected plane graphs, involved 5 reducibility rules and 13 discharging rules. In the case of strong parity vertex colourings, the proof of Theorem 4.3.7 involved 5 reducibility claims and 3 discharging rules. It is likely that more complex proofs involving more reducibility rules and more discharging rules will be required to determine the sharp upper bound for these restricted colouring problems. Future consideration of expanded sets of reducible configurations and discharging rules should lead to tighter bounds on the proper ℓ-residue vertex chromatic number; the lower and upper bounds are currently separated by a factor of 10.

Further extending the idea of proper ℓ-residue vertex colourings, it would be interesting to consider the problem for non-arithmetic sequences of values a colour is allowed to occur on a face. It is clear that the number 1 must be present in any such sequence, otherwise any graph with a 3-face or a 5-face cannot be assigned such a colouring. Natural choices of sequences would also be recursive, including geometric sequences or the Fibonacci sequence. Some of the arguments are lost for non-arithmetic sequences, however, by being unable to add a constant number of vertices with a colour to a face on which that colour is already used. A new approach would be required.

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