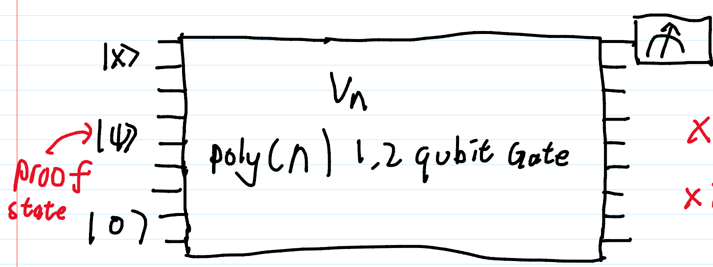
Feymann - kiteav pt 1:

QMA:



$\begin{cases} 1 \Rightarrow \text{True} \\ 0 \Rightarrow \text{False} \end{cases}$

$x \in L \Rightarrow \exists |\psi\rangle$ s.t $V_n |x\rangle|\psi\rangle|0\rangle$ output 1 w.p $\geq 1-\varepsilon$

$x \notin L \Rightarrow \forall |\psi\rangle$, $V_n |x\rangle|\psi\rangle|0\rangle$ Output 1 w.p $\leq \varepsilon$

$(\varepsilon = 2/3$, amplification $e^{-n^c})$

k-local Hamiltonian problem:

Given $(H_1, \cdots, H_m)$ k-local Hamiltonian acting on n-qubit such that

$0 \leq \lambda_1 \leq \cdots \leq \lambda_N \leq 1$ for all H_i $1 \leq i \leq m$

Given $0 \leq a \leq b \leq 1$ s.t $b - a \leq O(m^{-\alpha})$ (α constant)

Let $H = \sum_j H_j$

decide whether.

(Yes case): $\lambda_{min}(H) \leq a$

(No case): $\lambda_{min} \geq b$

Feymann-kitaev: 5-local Hamiltonian is QMA-complete

proof:

5-local Ham $\subseteq$ QMA Continue walk on expander graph (No don't do that)

What, do you really think I am going to give you the solution to the assignment here.

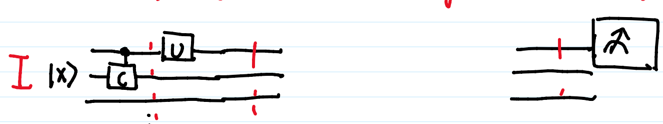
QMA $\subseteq$ 5-local Ham

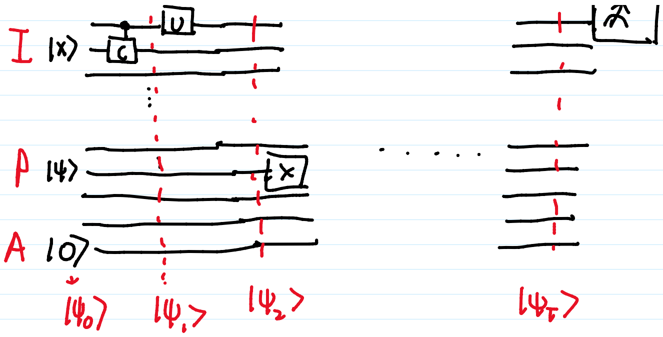
Classical	Quantum
Variables	qubits
Literal	Local Hamiltonians
Clause	$H = \sum_i H_i$
Assignment for the variable	n-qubit state $\lvert\psi\rangle$
(Solution Quality / # of clause satisfy)	$\langle\psi\lvert H\rvert\psi\rangle = \sum_i \langle\psi\lvert H_i\rvert\psi\rangle$
Optimal assignment	Eigenstate of $\lambda_{min}(H)$

Fix $x \in \{0,1\}^*$, $|x| = n$, $\exists V_n, |\psi\rangle$

Let $T = poly(n)$ be the number of gates V_n

Similar to C-L, we want some ways to "encode" the checking circuit.





$|\psi_{i+1}\rangle = U_i|\psi_i\rangle$

Need to check:

1. Start OK: $|\psi_0\rangle \Rightarrow$ first register $\to |x\rangle$
   Last register $\to |0\rangle$
2. Evolve OK: for $i \in [T]$, $|\psi_i\rangle = U_i|\psi_{i-1}\rangle$
3. End OK: $\boxed{A}$ obtains 1 w.p. $\geq 2/3$

"History state"

$$|\psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes |\psi_t\rangle$$

Clock register ($\log T$ qubit)

Want: Encode $H = H_{in} + H_{Evo} + H_{end}$

s.t. $\langle\psi|H_i|\psi\rangle$ is low if condition i are not violated too much

"penaltie score" for violating the 3 conditions.

1. Start OK:

check $|0\rangle$ is on register A on $|\psi_0\rangle$

Consider $|\psi_0\rangle$,

$P_i^{(A)} = (|1\rangle\langle 1|_i)_A \Rightarrow$ design to ensure qubit i on the I^{th} register to be 0.

Note $$\langle\psi_0|P_i^{(A)}|\psi_0\rangle = \underbrace{\langle 0|_A \langle\phi|_P \langle x|_I}^{\langle\psi_0|} \left(I_I \otimes I_P \otimes P_i^{(A)}\right) |x\rangle_I |\phi_P\rangle |0\rangle_A$$
$$= \langle 0|_A (|1\rangle\langle 1|_i)_A |0\rangle_A$$
$$= 0$$
✓

Define $H_i^{(A)} = (\underbrace{|0\rangle\langle 0|_C}_{} \otimes (|1\rangle\langle 1|_i)_A)$

↳ extract $|\psi_0\rangle$ from $|\psi\rangle$.

$$H_i^{(A)}|\psi\rangle = (|0\rangle\langle 0|_C \otimes (|1\rangle\langle 1|_i)_A)\left(\frac{1}{\sqrt{T+1}} \sum_i |i\rangle_C |\psi_i\rangle_A\right)$$
$$= \frac{1}{\sqrt{T+1}} \underbrace{(|1\rangle\langle 1|_i)_A |\psi_i\rangle_A}_{=0}$$

$\Rightarrow \langle\psi|H_i^{(A)}|\psi\rangle = 0$

check $|x\rangle$ is on register I. Same idea

$H_i^{(A)} = |0\rangle\langle 0|_C \otimes |\bar{x}_i\rangle\langle\bar{x}_i|$

let

$$H_{in} = \sum_i H_i^{(I)} + \sum_i H_i^{(I)}$$

poly # of local Hamiltonian !

3. End OK : Circuit output 1 (measure on $|\psi_T\rangle$)

$$H_{out} = |T\rangle\langle T|_c \otimes |0\rangle\langle 0|_1$$

$$\langle\psi|H_{out}|\psi\rangle = \frac{1}{T+1} \langle\psi_T| \, |0\rangle\langle 0|_1 \otimes I \, |\psi_T\rangle$$

$$= \frac{1}{T+1} P(\text{run the circuit and obtain } 0) \leq \frac{1}{\delta(T+1)}$$

2. Evolves OK:

$\forall t \in [T]$, we need to encode:

$U_i|\psi_i\rangle = |\psi_{i+1}\rangle$

Define

$$H^{(t\to t+1)} = \frac{1}{2}\Big(\underline{|t\rangle\langle t|_c \otimes I} + \underline{|t+1\rangle\langle t+1|_c \otimes I}$$
$$- \underline{|t+1\rangle\langle t|_c \otimes U_t} - \underline{|t\rangle\langle t+1|_c \otimes U_t^\dagger} \Big)$$

Note $$\left(|t+1\rangle\langle t|_c \otimes U_t\right)\left(\sum_i |i\rangle_c |\psi_i\rangle\right)$$
$$= |t+1\rangle \otimes U_t|\psi_t\rangle$$
$$= |t+1\rangle \otimes |\psi_{t+1}\rangle$$

$$\left(\sum_i \langle i|_c \langle\psi_i|\right)\left(|t\rangle\langle t+1|_c \otimes U_t^\dagger\right) = \langle t+1|_c \otimes \langle\psi_{t+1}|$$

$$\langle\psi|H^{(t\to t+1)}|\psi\rangle = \langle\psi| \left(|t\rangle\langle t|_c \otimes I\right)|\psi\rangle \quad \Rightarrow \quad \langle\psi_t|\psi_t\rangle = 1$$

$$+ \langle\psi|\left(|t+1\rangle\langle t+1|_c \otimes I\right)|\psi\rangle \quad \Rightarrow \quad \langle\psi_t|\psi_t\rangle = 1$$

$$- \langle\psi|\left(|t+1\rangle\langle t|_c \otimes U_t\right)|\psi\rangle \quad \Rightarrow \quad \langle\psi_{t+1}|U_t|\psi_t\rangle = 1$$

$$- \langle\psi|\left(|t\rangle\langle t+1|_c \otimes U_t^\dagger\right)|\psi\rangle \quad \Rightarrow \quad \langle\psi_t|U_t^\dagger|\psi_{t+1}\rangle = 1$$

$$= 0$$

:)

$H_{Evo} = \sum_t H^{(t\to t+1)}$ poly(|x|) local Hamiltonian

Excercise. each of $H_i^{(I)}$, $H_i^{(A)}$, $H^{(t\to t+1)}$, H_{out} is PSD with $\|H_i\| \leq 1$

(i.e., real eigenvalue with $0 \leq \lambda_i \leq 1$)

Excercise. each of $H_i^{[illegible]}$, $H_i^{[illegible]}$, $H^{[illegible]}$. H_{out} is PSD with $\|H_i\| \le 1$

(i.e, real eigenvalue with $0 \le \lambda_i \le 1$

($\log(n) + 2$ local Hamiltonian) (Problem with clock register)

[KSV02]: Classical and Quantum Computation.

[Yue 20]: The Complexity of Entanglement. Lecture 2, 3, 4