Correctness of Feymann-Kiteav

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Correctness of Kiteav's Hamiltonian

Need to show: $\exists\ 0 < a < b < n$ $a-b = O\left(\frac{1}{poly(n)}\right)$

s.t

$x \in L \Rightarrow \lambda_{min}(H_x) < a$ $\quad H_x = H_{in} + H_{evo} + H_{end}$

$x \notin L \Rightarrow \lambda_{min}(H_x) > b$

"Yes case" $x \in L$, we can construct

$$|\psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes |\psi_t\rangle$$

Based on the computation step, s-t

$$\langle\psi|H_x|\psi\rangle = \underbrace{\langle\psi|H_{in}|\psi\rangle}_{0} + \underbrace{\langle\psi|H_{evo}|\psi\rangle}_{0} + \underbrace{\langle\psi|H_{out}|\psi\rangle}_{\leq \varepsilon} \leq \frac{\varepsilon}{(T+1)}$$

"No case" $x \notin L$,

Need to show $\exists$ some $b > \frac{\varepsilon}{T+1}$ s-t $\lambda_{min}(H_x) > b$

or $\langle\psi|H_x|\psi\rangle > b$ $\forall |\psi\rangle$

2 case of $|\psi\rangle$:

subspace

1. $|\psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes U_t \cdots U_1 |x\rangle_I |\psi\rangle_p |0\rangle_o$ (Vaild History state)
2. $|\psi\rangle$ is in the orthogonal space to case 1

Case 1:

$|\psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle \otimes |\psi_t\rangle$ is a vaild history state

$\langle\psi|H_x|\psi\rangle = \langle\psi|H_{out}|\psi\rangle \geq Pr(\text{run the circuit and obtain } 0) \geq \frac{(1-\varepsilon)}{(T+1)}$

2. $|\psi\rangle$ is not, suppose further that $\langle\psi|H_{out}|\psi\rangle$ is relatively "small"

In this case, how "big" is $\langle\psi|H_{in} + H_{evo}|\psi\rangle$

Want to argue:

a) $Nul(H_{in} + H_{out}) \Rightarrow$ Case 1 $\Rightarrow \langle\psi|H_{in} + H_{out}|\psi\rangle > 0$

b) The minimal non-zero eigenvalue $\geq O\left(\frac{1}{T^3}\right)$

a) Null space of H_{in}:

$\rightarrow$ projector on register I and T

a) Null space of H_{in}:

$H_i^{(I)} = |0\rangle\langle 0|_C \otimes |\bar{x}_i\rangle\langle \bar{x}_i|_i$ → projector on register I and T

$Nul(H_i^{(I)}) = \{|i\rangle_C \otimes |\psi\rangle_{IPA}\ i \neq 0\} \cup \{\underbrace{|0\rangle|\psi'\rangle \otimes |x_i\rangle \otimes |\psi''\rangle}_{\text{Valid starting state}}\} = L_1$

$H_i^{(L)} H_j^{(R)} = 0 \quad L, R \in \{I, 0\}$

$\Rightarrow Nul(H_{in}) = \bigcap_{H \in H_{in}} Nul(H) = \{|i\rangle_C \otimes |\psi\rangle_{IPC}\ i \neq 0\} \cup \{|0\rangle_C |x\rangle_I |\psi\rangle_p |0\rangle_A\}$

(Exercise)

$Nul(H^{(t \to t+1)}) = \{|i\rangle_T \otimes |\psi\rangle, i \neq t, t+1\} \cup \{|t\rangle_T \otimes |\psi\rangle \otimes |t+1\rangle_T U_t |\psi\rangle\}$

$\Rightarrow Nul(H_{Evo}) = \{\underbrace{\sum_{i=1}^{T} |i\rangle \otimes U_i \cdots U_1 |\psi\rangle}_{\text{Valid compute step}}\} = L_2$

$\Rightarrow Nul(H_{in} + H_{Evo}) = \{\underbrace{\frac{1}{\sqrt{T+1}} \sum_{i=1}^{T} |i\rangle \otimes U_i \cdots U_1 |x\rangle_I |\psi\rangle_p |0\rangle_A}_{\text{Valid computation state + step}}\} = L$

What is the smallest non zero eigenvalue of $H_{in} + H_{out}$

Lemma 14.4 [KSV02]: Let A_1, A_2 be nonnegative operator s.t

$Nul(A_1) \cap Nul(A_2) = \{0\}$

and no nonzero eigenvalues of A_1, A_2 smaller than V. Then

$A_1 + A_2 \geq V \cdot 2(1 - \cos^2(\theta))$

θ → angle between L_1 and L_2

$\cos^2(\theta) = \max_{|\psi_1\rangle \in L_1, |\psi_2\rangle \in L_2} |\langle\psi_1|\psi_2\rangle|^2$

Want to apply this to $A_1|_{L^\perp}$ and $A_2|_{L^\perp}$

Angle (scroll up)

$\cos^2(\theta) = \max_{|\psi_1\rangle \in L_1, |\psi_2\rangle \in L_2} |\langle\psi_1|\psi_2\rangle|^2 \leq \frac{T}{T+1}$

Eigenvalue

Apply a change of basis (no change in eigenvalue)

$W = \sum_{j=0}^{T} |j\rangle\langle j|_T \otimes U_j \cdots U_1$ ← Unitary $\Rightarrow \begin{pmatrix} I & & \\ & U_1 & \\ & & U_2 U_1 \\ & & & \ddots \end{pmatrix}$

Change of Basis of H_{in}

Change of Basis of H_{in}

$W^\dagger H_{in} W = H_{in}$ (W does not change the basis of $|0\rangle\langle 0|$)

$H_{in} \Rightarrow$ sum of commuting projector (eigenvalue 0,1)

$\Rightarrow$ smallest nonzero eigenvalue is 1

On W_{evo}. consider $|t+1\rangle\langle t| \otimes U_t$

$$W^\dagger (|t+1\rangle\langle t| \otimes U_t) W$$
$$= \sum_{i,j} \big(\underbrace{|i\rangle\langle i|}_{\text{force to be } t+1} \otimes U_i \cdots U_1\big)^\dagger (|t+1\rangle\langle t| \otimes U_t)(\underbrace{|j\rangle\langle j|}_{\text{forces to be } t} \otimes U_j \cdots U_1)$$
$$= (|t+1\rangle\langle t+1| \otimes U_{t+1} \cdots U_1)^\dagger (|t+1\rangle\langle t| \otimes U)(|t\rangle\langle t| \otimes U_t \cdots U_1)$$
$$= |t+1\rangle\langle t| \otimes I$$

Hence $W^\dagger H^{(t\to t+1)} W = \frac{1}{2}(|t\rangle\langle t| + |t+1\rangle\langle t+1| - |t\rangle\langle t+1| - |t+1\rangle\langle t|) \otimes I = E_t \otimes I$

$$\Rightarrow W^\dagger H_{Evo} W = \sum_{j=1}^{T} E_j = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

Eigenvalue $\lambda_k = 1 - \cos\left(\frac{\pi k}{(T+1)}\right) \quad (k = 0, \ldots, T)$

$$\Rightarrow v \geq 1 - \cos\left(\frac{\pi}{(T+1)}\right) \geq O(T^{-2})$$

$\Rightarrow$ eigenvalue at most $1/c'' T^2$

$\Rightarrow$ By Lemma eigenvalue at least $\frac{1}{c' T^2}\left(1 - \frac{T}{T+1}\right) \geq O\left(\frac{1}{T^3}\right)$ $\square$

What do we have:

(Yes case): $\frac{\varepsilon}{(T+1)}$

(No case): $\min\left\{\frac{c'}{T^2}, \frac{(1-\varepsilon)}{(T+1)}\right\}$

Pick appropriate ε from amplification

$\square$

Application: Adiabatic computing!

[KSV02]: Classical and Quantum Computation.

[AN02]: Quantum NP - a survey

[Yue 20]: The complexity of Entanglement. Lectures 2, 3, 4