Lecture 11

Today:

1. detecting marked item with discrete-time QW
2. Examples
3. Ambainis alg. for element distinctness

References

[Szegedy 04]

[MNRS]

[AMC] CH 17

[Ambainis 03]

Recall: X = finite set, $|X| = n$

P $n \times n$ symmetric stochastic mtx, δ = spectral gap of P

$M \subseteq X$ set of marked items $\epsilon \leq \frac{|M|}{n}$

Last time: classical absorbing RW finds a marked item in time

$$t = O\left(\frac{1}{\delta \epsilon}\right)$$

② Discrete time QW [This version: Szegedy 04]

idea: want to make transitions in superpositions.

$x \in X$ $\quad |x\rangle \xrightarrow{?} \sum_{y \in X} \sqrt{P_{yx}}\, |y\rangle$ NOT UNITARY

Make Hilbert space bigger

$$\mathcal{H} = \mathbb{C}^n \times \mathbb{C}^n = \{ |x\rangle|y\rangle : x, y \in X \}$$

define $\quad |P_y\rangle \stackrel{def}{=} \sum_{x \in X} \sqrt{P_{xy}}\, |x\rangle$

projector $\quad \Pi_A \stackrel{def}{=} \sum_{y \in X} |y\rangle\langle y| \otimes |P_y\rangle\langle P_y|$

$$\Pi_B \stackrel{def}{=} \sum_{y \in X} |P_y\rangle\langle P_y| \otimes |y\rangle\langle y|$$

QW from P: $\quad W \stackrel{def}{=} (2\Pi_B - I)(2\Pi_A - I)$

Aside : Jordan's lemma (1875)

Let A, B be subspaces of a finite dim. Hilbert space $\mathcal{H}$.

let Π_A, Π_B project onto A, B.

let $W = (2\Pi_B - I)(2\Pi_A - I)$

$$\Delta \overset{\text{def}}{=} \Pi_B \Pi_A$$

Let SVD of Δ be $\Delta = \sum_{i=1}^{d} \sigma_i |b_i\rangle\langle a_i|$

- $d = \text{rank}(\Delta)$
- $\sigma_i > 0$
- $\{|a_i\rangle\}$ orthonormal
- $\{|b_i\rangle\}$ orthonormal

Lemma Let $\mathcal{H}_i = \text{span}\{|a_i\rangle, |b_i\rangle\}$ $\quad 1 \le i \le d$

(i) $\mathcal{H}_i \perp \mathcal{H}_j$ whenever $i \neq j$

Π_A, Π_B, and W preserve each subspace $\mathcal{H}_i$ $\quad 1 \le i \le d$

(ii) $W|_{\mathcal{H}_i}$ is a rotation by angle $2\theta_i$ where

$$\cos\theta_i = \sigma_i \qquad \theta_i \in [0, \pi/2)$$

(iii) Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_d \oplus R$

$W|_R$ has eigenvals ± 1

If $W|v\rangle = |v\rangle$ for $v \in R$ then $v \in A^\perp \cap B^\perp$

Pf:

(i) $\Delta = \Pi_B \Pi_A = \sum_{i=1}^{d} \sigma_i |b_i\rangle\langle a_i|$ $\Rightarrow$ $a_i \in A$ $\forall i$

$b_i \in B$

$$\langle b_j|a_i\rangle = \langle b_j|\Pi_B \Pi_A|a_i\rangle = \langle b_j|\Delta|a_i\rangle$$
$$= \sigma_i \delta_{ij} \Rightarrow \mathcal{H}_i \perp \mathcal{H}_j \quad i \neq j$$

$$\Pi_B|a_i\rangle = \Pi_B \Pi_A|a_i\rangle = \Delta|a_i\rangle = \sigma_i|b_i\rangle$$

likewise $\Pi_A|b_i\rangle = \sigma_i|a_i\rangle$

$\Pi_B|b_i\rangle = |b_i\rangle$

$\Pi_A|a_i\rangle = |a_i\rangle$

Π_B preserves $\mathcal{H}_i$

Π_A " "

W " "

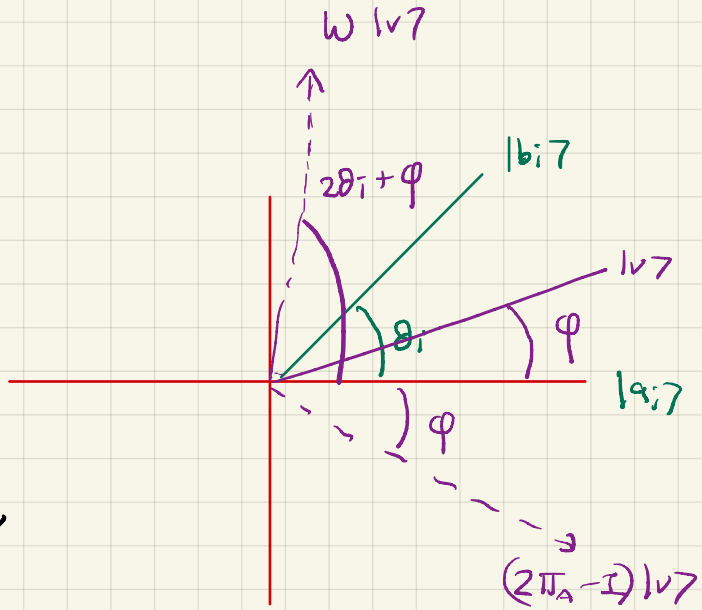
(ii) $\Pi_B|_{\mathcal{H}_i} = |b_i\rangle\langle b_i|$

$\Pi_A|_{\mathcal{H}_i} = |a_i\rangle\langle a_i|$

$W|_{\mathcal{H}_i} = (2\Pi_B - I)|_{\mathcal{H}_i}\,(2\Pi_A - I)|_{\mathcal{H}_i}$

$= \underbrace{(2|b_i\rangle\langle b_i| - I)(2|a_i\rangle\langle a_i| - I)}$

rotation by angle $2\theta_i$ ✓



(iii) R = nullspace of $\Delta^\dagger\Delta$ = $\Pi_A\Pi_B\Pi_A$

$= A^\perp \oplus (A \cap B^\perp)$

$W = (2\Pi_B - I)(2\Pi_A - I)$

$= \underbrace{(A^\perp \cap B)}_{w=-1} \oplus \underbrace{(A^\perp \cap B^\perp)}_{w=+1} \oplus \underbrace{(A \cap B^\perp)}_{w=-1}$

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③ Detecting presence of marked items by QW

Use QW with

$$P' = \begin{pmatrix} P_{X/M} & 0 \\ * & I \end{pmatrix}$$

Look at $\Delta = \Pi_B \Pi_A$:

Goal: relate singular v's of Δ to eigenv's of $P_{X/M}$

Define unitaries U, V

$$U|i\rangle|y\rangle = |y\rangle|p''_y\rangle$$

$$V|i\rangle|y\rangle = |p'_y\rangle|y\rangle$$

$$V^\dagger \Pi_B = \sum_{y \in X} (|i\rangle|y\rangle)(\langle p'_y | \langle y|)$$

$$\Pi_A U = \sum_{z \in X} (|z\rangle|p'_z\rangle)(\langle i| \langle z|)$$

Consider $V^\dagger \Delta U = (V^\dagger \Pi_B)(\Pi_A U) = \sum_{y,z \in X} |i\rangle\langle i| \otimes |y\rangle\langle z| \sqrt{p'_{zy} p'_{yz}}$

$$V^\dagger \Delta U = |1\rangle\langle 1| \otimes T \qquad \text{where} \qquad T = \begin{pmatrix} P_{X/M} & 0 \\ 0 & I \end{pmatrix}$$

NONZERO SING. VALS OF Δ are eigenvals of $P_{X/M}$ (and $+1$)

$\Rightarrow$ largest sing. vale of Δ is $+1$

If $M = \emptyset$ $\quad U|1\rangle|v\rangle$ where $|v\rangle$ is uniform superpos. over X

If $M \neq \emptyset$ $\quad U|1\rangle|y\rangle$ for any $y \in M$.

2nd largest sing. vale of Δ is $\leq \| P_{X/M} \| \leq 1 - \delta\epsilon$

Algorithm

Start in the initial state

$\Pi_M, I - \Pi_M$

$\Pi_M = \sum_{j \in M} |j\rangle\langle j|$

$$|\psi_0\rangle = \frac{1}{\sqrt{n - |M|}} \sum_{j \in X/M} |j\rangle |\beta_j'\rangle$$

perform P.E on $W = (2\Pi_B - I)(2\Pi_A - I)$

what precision do we want?

USES RUNTIME $O\left(\frac{1}{\sqrt{\delta\epsilon}}\right)$ to resolve $\sim \sqrt{\delta\epsilon}$ phase

Case ①: $M = \emptyset$

Then $|\psi_0\rangle$ is the left sing. vec of Δ w $\sigma_j = +1$

$$\boxed{W|\psi_0\rangle = |\psi_0\rangle}$$

eigenvalue $= e^{i \cdot 0}$

Case ②: $M \neq \emptyset$

$|\psi_0\rangle$ is $\perp$ to $+1$ eigenvalues of W

$\Rightarrow$ supported on e-vals $e^{\pm 2i\theta_j}$

$$1 - \theta_j^2/2 \leq \sigma_j = \cos\theta_j \leq 1 - \delta\epsilon$$

$$\Rightarrow \theta_j \geq \left(2\left(1 - (1 - \delta\epsilon)\right)\right)^{1/2} = \Omega\left(\sqrt{\delta\epsilon}\right)$$

eigval $e^{i\Omega(\sqrt{\delta\epsilon})}$

runtime of QW alg $O\left(\frac{1}{\sqrt{\delta\epsilon}}\right)$

runtime of absorbing RW $O\left(\frac{1}{\delta\epsilon}\right)$

Remarks:

- only detects presence of marked items

[MNRS] can find marked items $O\left(\frac{1}{\sqrt{\delta\epsilon}}\right)$

[Ambainis, Gilyén, Jeffery, Kokainis 2019]

find marked item $\tilde{O}\left(\sqrt{HT}\right)$

broad family of M.C