Today:

1. Facts from complexity theory
2. Hardness of exactly simulating quantum computation
3. Hardness of approximately simulating QC

The polynomial hierarchy

$$\mathsf{PH} = \bigcup_{k=0,1,2,\ldots} \Sigma_k$$

where

- $$\Sigma_0 = \mathsf{P}$$
- $$\Sigma_1 = \mathsf{NP}$$
- $$\Sigma_2 = \mathsf{NP}^{\mathsf{NP}}$$
- $$\vdots$$
- $$\Sigma_k = \mathsf{NP}^{\Sigma_{k-1}}$$

If $$\Sigma_i = \Sigma_{i+1}$$ for some $$i$$,

then $$\mathsf{PH} = \Sigma_i$$ and this "polynomial hierarchy collapse" is thought to be extremely unlikely.

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Refs:
- [Bremner Jozsa Shepherd 10]
- [Terhal Divincenzo 02]
- [Fenner Green Homer Zhang 03]
- [Aaronson 04]
Post BQP

A decision problem \( L \in \text{postBQP} \) if \( \exists \) a uniformly generated polynomial-size circuit family \( \{ C_n \}_{n \in \mathbb{N}} \) s.t.

\[
\begin{align*}
  x \in L & \implies \Pr \left[ z = 1 \mid y = 0 \right] \geq \frac{2}{3} \\
  x \notin L & \implies \Pr \left[ z = 1 \mid y = 0 \right] \leq \frac{1}{3}
\end{align*}
\]

Can likewise define \( \text{postBPP} \) by replacing the BQP circuit with a BPP circuit.

Can also define \( \text{post A} \) for any restricted family \( A \) of BQP circuits.
Fact: Quantum postselection $\Rightarrow$ classical postselection

$p_{postBPP} \subseteq \Sigma_3$ \cite{Han, Hemaspaandra, Thierauf 97}

$p_{postBQP} \subseteq PH$ \cite{Aaronson 2005} + Today's Thm

$\Rightarrow$ If postBPP contains postBQP then PH collapses!

2) Classical hardness of exact simulation \cite{see Brenner Jozsa Shepherd 10}

Claim: Suppose $A$ is a restricted circuit family such that

$$post A = postBQP$$

Suppose that is an efficient classical alg. that, given a circuit in $A$, samples from the distribution

$$p(x, y, r)$$

(see (8))

Then PH collapses.
Pf: Take the efficient (BPP) circuit that samples \( P_{(x,y,r)} \) and then postselect on \( y \). This gives a postBPP circuit that exactly simulates the post-selected \( q \) circuit.

\[
\text{post-BQP} = \text{post-A} \leq \text{post-BPP} \Rightarrow \text{PH} \subseteq \Sigma_3
\]

Restricted circuit families \( A \) s.t. \( \text{post-A} = \text{post-BQP} \) can be much less powerful than BQP.

Examples

\( A = \text{depth 3 circuits} \) [Terhal Divincenzo 2002]
\( A = \text{IQP circuits} \) [Bremner Jozsa Shepherd 2010]
\( A = \text{QAOA circuits} \) [Farhi Harrow 2016]
Want to show \( \text{postselected} (\text{depth 3}) = \text{post BQP} \)

An \( n \)-qubit state can be teleported using \( n \) parallel copies of the above:
Suppose we have a q. circuit measurement + postselected

We can insert postselected teleportation after the 1st gate to get an equivalent circuit:

Now insert postselected teleportation after 2nd, 3rd, ..., m-1st gates:
This is a depth-3 postselected circuit that simulates the given postselected circuit with $m$ gates.

No efficient classical algorithm can exactly sample from output of depth 3 poly-size quantum circuits, unless $PH$ collapses.

Depth 2 circuits can be efficiently simulated, see homework.
IQP circuits

"Instantaneous Quantum polynomial time" [Brenner Jozsa Shepherd 2010]

\[
D = \left( \prod_{j<k} \exp \frac{i \pi}{8} \omega_{jk} z_j z_k \right) \left( \prod_{l=1}^n \exp \frac{i \pi}{8} \nu_l z_l \right)
\]

2-qubit gates

\[\omega_{jk}, \nu_l \in \mathbb{R}, \ell \geq 3\]