Lecture 3

Today: Quantum phase estimation cont'd

Amplitude Estimation

Clean computation

QFT over Abelian groups

References:

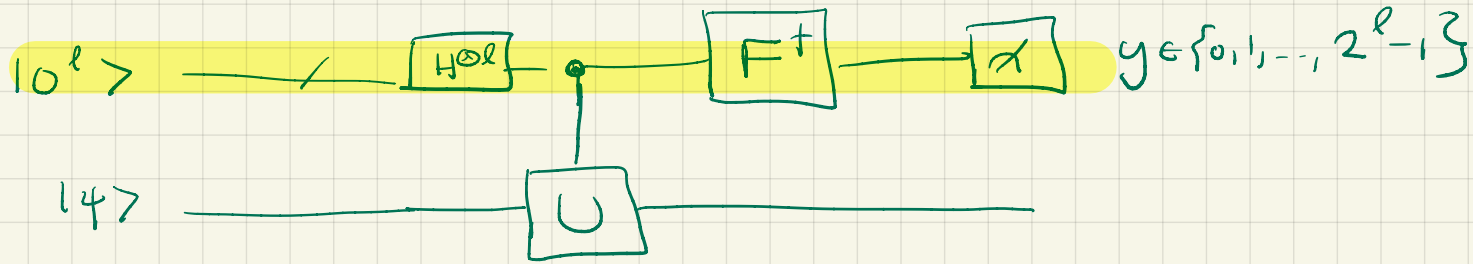
N+C CH 5

BHMT 2000

AMC lecture notes CH 4

Kitaev '95, Section 5

Recall: Phase estimation circuit



$$U|\psi\rangle = e^{2\pi i \theta}|\psi\rangle$$

estimate $\tilde{\theta} = y/2^\ell$ of θ

How big should ℓ be to make $|\tilde{\theta} - \theta|$ small

Analysis

$$|0^\ell\rangle|\psi\rangle \xrightarrow{H^{\otimes \ell}\otimes I} \frac{1}{\sqrt{2^\ell}}\sum_{y=0}^{2^\ell-1}|\bar{y}\rangle|\psi\rangle$$

$$\xrightarrow{\boxed{U}} \frac{1}{\sqrt{2^\ell}}\sum_{y=0}^{2^\ell-1}|\bar{y}\rangle U^y|\psi\rangle$$

$$= \underbrace{\left(\frac{1}{\sqrt{2^\ell}}\sum_{y=0}^{2^\ell-1}|\bar{y}\rangle e^{2\pi i\theta y}\right)}_{\text{State of 1st register before } F^\dagger \text{ and measurement}}|\psi\rangle$$

How big does ℓ need to be?

<u>Easy case</u>: Suppose $\theta = x/2^r$ $\quad x, r \geq 1$

Choose $\ell = r$ $\quad F^\dagger\left(\frac{1}{\sqrt{2^\ell}}\sum_{y=0}^{2^\ell-1}|\bar{y}\rangle e^{\frac{2\pi i xy}{2^\ell}}\right) = |\bar{x}\rangle$

Measurement gives x with prob. 1, so $\tilde{\theta} = x/2^r = \theta$

General case

Choose $\ell = t + \lceil \log(2 + 1/2p) \rceil$

to ensure $|\theta - \tilde{\theta}| < 1/2^t$ with prob. $\geq 1-p$

[see N+C CH 5]

Quantum Amplitude Estimation

BHMT 2000

Given: $(n+1)$-qubit unitary U s.t

$$U|0^{n+1}\rangle = \sqrt{p}\,|\varphi_1\rangle|1\rangle + \sqrt{1-p}\,|\varphi_0\rangle|0\rangle$$

where $\|\varphi_0\| = \|\varphi_1\| = 1$

Compute an estimate of p

Recall from discussion of AA

$$R_B = 2(I \otimes |0\rangle\langle 0|) - I$$

$$R_A = 2\,U|0^{n+1}\rangle\langle 0^{n+1}|U^\dagger - I$$

Lemma (recall): $(R_A R_B)^j \, U|0^{n+1}\rangle = \sin\left((2j+1)\theta_p\right)|\varphi_1\rangle|1\rangle$

$$+ \cos\left((2j+1)\theta_p\right)|\varphi_0\rangle|0\rangle$$

$$p = \sin^2\theta_p$$

$$V = \text{span}\{|\varphi_1\rangle|1\rangle, |\varphi_0\rangle|0\rangle\}$$

$R_A R_B$ preserves V

[TYPO CORRECTED HERE]

$$(R_A R_B)|_V = \begin{array}{cc} \begin{pmatrix} \cos(2\theta_p) & \sin(2\theta_p) \\ -\sin(2\theta_p) & \cos(2\theta_p) \end{pmatrix} \\ \begin{array}{cc} \underbrace{\qquad\qquad}_{|\varphi_1\rangle|1\rangle} & \underbrace{\qquad\qquad}_{|\varphi_0\rangle|0\rangle} \end{array} \end{array}$$

Eigenvals of this mtx: $e^{\pm 2i\theta_p}$

Note: $p = \sin^2(\theta_p) = \sin^2(-\theta_p)$

CAN USE EITHER EIGENSTATE + PHASE ESTIMATION.

Can use p.e with input state $U|0^{n+1}\rangle$

$$= \alpha\,|+\theta_p\rangle + \beta\,|-\theta_p\rangle$$

Clean computation

Suppose U_f is a q. circuit s.t

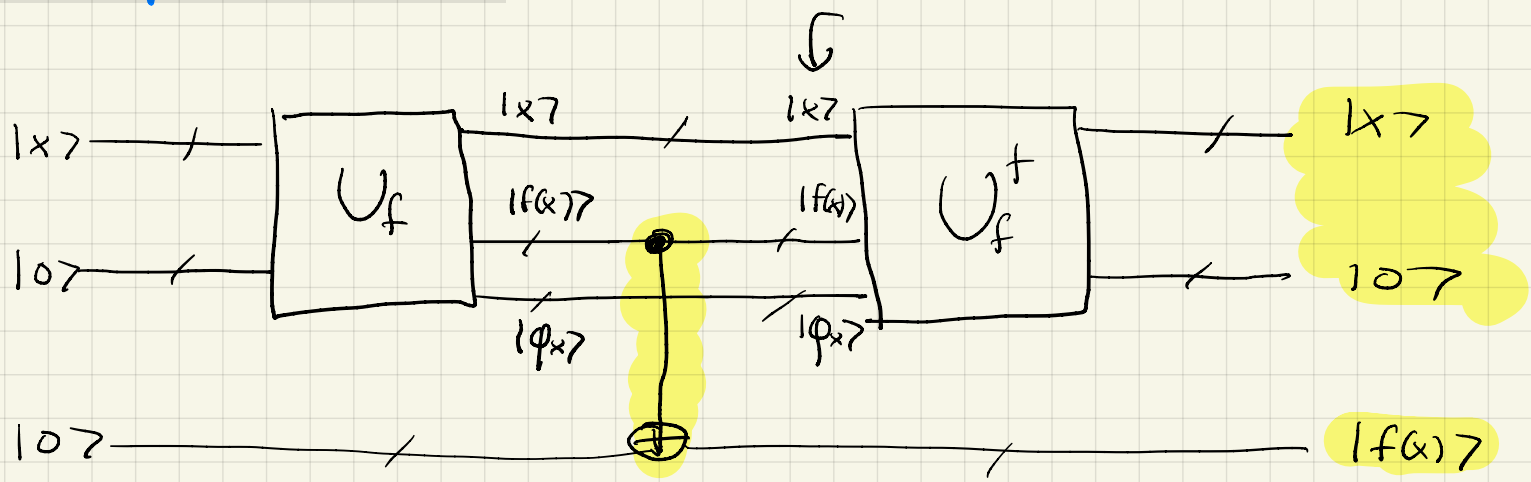
unwanted garbage state

$$U_f |x\rangle|0\rangle = |x\rangle|f(x)\rangle|\phi_x\rangle$$

where $f: \{0,1\}^n \to \{0,1\}^m$

Let C_f be the "clean version" $C_f |x\rangle|0\rangle = |x\rangle|f(x)\rangle$

How to implement C_f:



Example Phase estimation on a unitary U

Let $\{|\psi_j\rangle\}_j$ be the eigenstates of U

s.t $U|\psi_j\rangle = e^{2\pi i \theta_j}|\psi_j\rangle$

Phase estimation w/ $\ell = t + O(\log \frac{1}{\rho})$ ancillas

gives unitary $\tilde{Q}$ s.t

$$\|\tilde{Q} - Q\| \leq O(\sqrt{\rho})$$

where $Q|\psi_j\rangle|0\rangle = |\psi_j\rangle \underbrace{|\theta_j^1 \theta_j^2 \ldots \theta_j^t\rangle}_{t \text{ bits correct}} \underbrace{|\phi_j\rangle}_{\text{garbage}}$

Using clean computation, get $O(\sqrt{\rho})$ approx. to "clean version"

$$|\psi_j\rangle|0\rangle \longrightarrow |\psi_j\rangle|\theta_j^1 \ldots \theta_j^t\rangle$$

Fourier Transform over Abelian groups

G = a finite abelian group, with group operation "+"

<u>Dfn</u>: A character of G is a function

$$\chi : G \longrightarrow \mathbb{C} \setminus \{0\}$$

with the property $\chi(g_1 + g_2) = \chi(g_1) \cdot \chi(g_2)$

Properties

① $\chi(0) = 1$

<u>Pf</u>: $\chi(0+g) = \chi(0) \cdot \chi(g) \Longrightarrow \chi(0) = 1$

② $|\chi(g)| = 1 \quad \forall g \in G$

<u>Pf</u>: recall $\underbrace{g + g + \cdots + g}_{|G| \text{ times}} = 0 \quad \forall g \in G$ (order of a grp element divides $|G|$)

$$1 = \chi(0) = \chi(g + g + \cdots + g) = (\chi(g))^{|G|}$$

③ The set of characters of G forms a group "$\hat{G}$" with multiplication as the group operation

i.e, Let χ, χ' be characters of G

Define $\chi \cdot \chi'(g) \overset{\text{def}}{=} \underline{\chi(g)\,\chi'(g)}$

To prove this, check

$\chi \cdot \chi'$ is a character

multiplication of characters is associative $(\chi \cdot \chi')\chi'' = (\chi) \cdot (\chi' \cdot \chi'')$

identity element is $I(g) = 1 \quad \forall g \in G$

every character has an inverse

$$\chi^{-1}(g) = \frac{1}{\chi(g)} = \chi^{*}(g)$$

④ The character group $\hat{G}$ satisfies $\hat{G} \simeq G$

We won't prove this

It means we can label characters by elements of G.

e.g cyclic group $\mathbb{Z}_N$ (integers mod N)

$$\chi_y(x) = e^{\frac{2\pi i x y}{N}} \qquad x, y \in \mathbb{Z}_N$$

This is (almost) the whole story b/c any Abelian group can be decomposed as a direct sum

$$G = G_1 \oplus G_2 \oplus \dots \oplus G_L$$

$$G_i \approx \mathbb{Z}_{N_i}$$

⑤ Orthogonality of characters

$$\frac{1}{|G|} \sum_{g \in G} \chi_\alpha^*(g)\, \chi_\beta(g) = \begin{cases} 1 & , \ \chi_\alpha = \chi_\beta \\ 0 & , \ \text{o/w} \end{cases}$$