Recall: Phase estimation circuit

$$U |147\rangle = e^{2\pi i \theta} |147\rangle$$

Estimate $$\tilde{\theta} = \frac{1}{2}\varepsilon$$ of $$\theta$$

How big should $$\varepsilon$$ be to make $$|\tilde{\theta} - \theta|$$ small?

References:
- N+C CH 5
- BHMT 2000
- AMC lecture notes CH 4
- Kitaev ’95, Section 5
How big does $l$ need to be?

Easy case: Suppose $\theta = \frac{x}{2^r}$, $x, r \geq 1$.

Choose $l = r$

$$\begin{align*}
F^+ \left( \frac{1}{\sqrt{2^e}} \sum_{y=0}^{2^e-1} |y\rangle e^{-\frac{2\pi i xy}{2^e}} \right) = 1\chi_7
\end{align*}$$

Measurement gives $x$ with prob. 1, so $\tilde{\theta} = \frac{x}{2^r} = \theta$. 

State of 1st register before $F^+$ and measurement.
General case

Choose \( l = t + \lceil \log_2 (2 + \frac{1}{2p}) \rceil \)

to ensure \( |18 - \xi| < \frac{1}{2^t} \) with prob. \( > 1 - p \)

[see N+C CH5]

---

Quantum Amplitude Estimation

Given: \((n+1)\)-qubit unitary \( U \) s.t.

\[
U |0^{n+1}\rangle = \sqrt{p} |\psi_0\rangle |1^n\rangle + \sqrt{1-p} |\psi_1\rangle |0^{n-1}\rangle
\]

where \( \langle \psi_0 \rangle = \langle \psi_1 \rangle = 1 \)

Compute an estimate of \( p \)

Recall from discussion of AA

\[
R_0 = 2 (I \otimes |0\rangle \langle 0|) - I
\]

\[
R_A = 2 U |0^{n+1}\rangle \langle 0^{n+1}| U^\dagger - I
\]
Lemma: \((R_A R_B)^j V|\phi_{117}\rangle = \sin \left((2j+1)\theta_p\right)|\phi_{117}\rangle + \cos \left((2j+1)\theta_p\right)|\phi_{107}\rangle\)

\(p = \sin^2 \theta_p\)

\(V = \text{Span}\{|\phi_{117}\rangle, |\phi_{107}\rangle\}\)

\((R_A R_B)|_V = \begin{pmatrix} \cos(2\theta_p) & \sin(2\theta_p) \\ -\sin(2\theta_p) & \cos(2\theta_p) \end{pmatrix}\)

Eigenvalues of this matrix: \(e^{\pm 2i\theta_p}\)

Note: \(p = \sin^2 (\theta_p) = \sin^2 (-\theta_p)\)

Can use \(p\) e with input state \(U|\phi_{117}\rangle\)

= \(a|1+\theta_p\rangle + b|-\theta_p\rangle\)
Suppose $U_f$ is a q. circuit s.t. unwanted garbage state $U_f \ket{1x7} \rightarrow \ket{\phi_x}$

where $f : \{0,1\}^{m} \rightarrow \{0,1\}^{m}$

Let $C_f$ be the "clean version" $C_f \ket{1x7} = \ket{1\times7f(x)}$

How to implement $C_f$:

\[ \begin{array}{c}
1x7 \\
\uparrow \\
U_f \\
\downarrow \\
1\phi_x \\
\uparrow \\
1x7 \\
\rightarrow \\
\hline \\
107 \\
\uparrow \\
f(x) \\
\downarrow \\
1\phi_x \\
\uparrow \\
1x7 \\
\rightarrow \\
\hline \\
107 \\
\uparrow \\
f(x) \\
\downarrow \\
1\phi_x \\
\uparrow \\
1x7 \\
\rightarrow \\
\hline \\
107 \\
\uparrow \\
f(x) \\
\downarrow \\
1\phi_x \\
\uparrow \\
1x7 \\
\rightarrow \\
\hline \\
107 \\
\end{array} \]
**Example**

Phase estimation on a unitary $U$

Let $|\psi_j\rangle$ be the eigensates of $U$ such that

$$U|\psi_j\rangle = e^{2\pi i \theta_j} |\psi_j\rangle$$

Phase estimation with $L = +1 + O(\log \frac{1}{\delta})$ ancillas gives unitary $	ilde{Q}$ such that

$$||\tilde{Q} - Q|| \leq O(\sqrt{\delta})$$

where

$$Q|\psi_j\rangle = 1|\psi_j\rangle |\psi_1\rangle |\psi_2\rangle ... |\psi_L\rangle$$

Using clean computation, get $O(\sqrt{\delta})$ approx. to "clean version"

$$|\psi_j\rangle 10^L \rightarrow |\psi_j\rangle |\psi_1\rangle ... |\psi_L\rangle$$
Fourier Transform over Abelian groups

\( G = \) a finite abelian group, with group operation "+",

**Dfn:** A character of \( G \) is a function

\[ \chi : G \rightarrow \mathbb{C} \setminus \{0\} \]

with the property

\[ \chi(g_1 + g_2) = \chi(g_1) \cdot \chi(g_2) \]

**Properties**

1. \( \chi(0) = 1 \)
   - **pf:** \( \chi(0 + g) = \chi(0) \cdot \chi(g) \Rightarrow \chi(0) = 1 \)

2. \( |\chi(g)| = 1 \) \( \forall g \in G \)
   - **pf:** recall \( g + g + \ldots + g = 0 \) \( \forall g \in G \) (order of \( g \) is \( |G| \))

   \[ 1 = \chi(0) = \chi(g + g + \ldots + g) = (\chi(g))^{|G|} \]
(3) The set of characters of $G$ forms a group $\hat{G}$ with multiplication as the group operation.

i.e., let $\chi, \chi'$ be characters of $G$

define $\chi \cdot \chi' (g) \overset{\text{def}}{=} \chi(g) \chi'(g)$

To prove this, check

- $\chi \cdot \chi'$ is a character
- multiplication of characters is associative: $(\chi \cdot \chi') \chi'' = \chi \cdot (\chi' \chi'')$
- identity element is $I(g) = 1 \quad \forall g \in G$
- every character has an inverse

$$\chi^{-1}(g) = \frac{1}{\chi(g)} = \chi^{\#}(g)$$

(4) The character group $\hat{G}$ satisfies $\hat{G} \cong G$

We won't prove this.

It means we can label characters by elements of $G$. 
e.g. cyclic group \( \mathbb{Z}_N \) (integers mod \( N \))

\[
\chi_y(x) = e^{\frac{2\pi i xy}{N}} \quad x,y \in \mathbb{Z}_N
\]

This is (almost) the whole story b/c any Abelian group can be decomposed as a direct sum

\[
G = G_1 \oplus G_2 \oplus \ldots \oplus G_k
\]

\[G_i \cong \mathbb{Z}_{N_i}\]

5. Orthogonality of characters

\[
\frac{1}{|G|} \sum_{g \in G} \chi_x^*(g) \chi_y(g) = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases}
\]