PMath 441/641 – Homework 1
Due 11:59pm on Wednesday, May 15 on Crowdmark

1. Let $M = \mathbb{Z}^3$ and $N = \mathbb{Z}^2$, both $\mathbb{Z}$-modules. Let $\varphi\colon M \to N$ be the $\mathbb{Z}$-module homomorphism $\varphi(a, b, c) = (2a + b, 3b + 2c)$. Find generators for the $\mathbb{Z}$-module $\ker \varphi$.

2. Prove that $\frac{1}{3}(1 + 10^{1/3} + 10^{2/3})$ is an algebraic integer.

3. Prove that $a = \frac{1+\sqrt{-5}}{2}$ is not integral over $\mathbb{Z}$.

4. Let $\alpha \in \mathbb{C}$ be a root of $x^3 - x + 1$. Compute the cardinality of the ring $\mathbb{Z}[\alpha]/(\alpha + 2)$. Is the ideal $(\alpha + 2)$ prime?

5. Let K be a number field of degree d over $\mathbb{Q}$, and let P be a nonzero prime ideal of the ring of integers $\mathcal{O}_K$ of K. Prove that P contains a prime integer p (that is, $p \in \mathbb{Z} \cap P$), and that $\mathcal{O}_K/P$ contains at most p^d elements.

6. Prove that the ring of integers of $\mathbb{Q}(\sqrt{33})$ is $\mathbb{Z}\left[\frac{1+\sqrt{33}}{2}\right]$. This is a special case of the general fact that the ring of integers of $\mathbb{Q}(\sqrt{d})$ is (assuming d is squarefree):

$$\begin{array}{ll} \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \pmod 4 \\ \mathbb{Z}[\sqrt{d}] & \text{otherwise} \end{array}$$

You don't have to do the general case. Just $d = 33$.

In fact, *do not* do the general case. I will take off marks if you do.