PMath 441/641 – Homework 2
Due on Crowdmark at 11:59pm on Wednesday, May 22

1. Let $p \in \mathbb{Z}$ be a prime number with $p \equiv 3 \pmod 4$. Prove that $\mathbb{Z}[i]/(p)$ is a field.

2. Let $K = \mathbb{Q}(\sqrt{33})$. Compute the trace and norm of the following elements of K:

- $\sqrt{33}$
- 1
- $6 + \sqrt{33}$

3. Let α be a root of the polynomial $f(x) = 9x^3 + 2x + 7$. Find an integer n such that $n\alpha$ is an algebraic integer.

4. Let α be an algebraic number such that $N(\alpha)$ and $\mathrm{Tr}(\alpha)$ are both integers in $\mathbb{Z}$. Must α be an algebraic integer? Either prove it, or give a counterexample.

5. Consider the ring $A = \mathbb{Z}\left[\frac{1}{2}\right]$. Is A integrally closed?

6. Let α be a root of the polynomial $x^3 + 3x + 3$. The ring of integers in the field $\mathbb{Q}(\alpha)$ is $\mathbb{Z}[\alpha]$. (You don't have to prove that.)

Find a basis, over $\mathbb{Z}$, of the ideal $I = (3, \alpha)$. That is, we know that I is isomorphic to $\mathbb{Z}^3$ as an additive group. Your job is to find three elements of I that are a basis for I as an additive group.