PMath 441/741 – Homework 3
Due on Crowdmark at 11:59pm on Wednesday, May 29

1. Let $K = \mathbb{Q}(\sqrt{-2})$, with ring of integers $\mathcal{O}_K = \mathbb{Z}[\sqrt{-2}]$. Let $\Psi\colon K \to \mathbb{C}^2$ be the Minkowski map $\Psi = (\Psi_1, \Psi_2)$, where Ψ_1 and Ψ_2 are the embeddings of K in $\mathbb{C}$. List all the elements $a \in \mathcal{O}_K$ such that the length of $\Psi(a)$ is at most 3 ($\|\Psi(a)\| \leq 3$).

2. Let K be a number field with ring of integers $\mathcal{O}_K$, and let $\Psi\colon K \to V_K$ be the Minkowski map:

$$\Psi(\alpha) = (\Psi_1(\alpha), \ldots, \Psi_n(\alpha))$$

where $\Psi_1, \ldots, \Psi_n$ are the $n = [K : \mathbb{Q}]$ embeddings of K into the complex numbers.

Show that if $\alpha \in \mathcal{O}_K$ is an algebraic integer, then the length of $\Psi(\alpha)$ is an algebraic integer.

3. Let d be a squarefree integer, and let $K = \mathbb{Q}(\sqrt{d})$. Assume that the ring of integers of K is $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$. (This is the same as assuming that d is not congruent to 1 modulo 4.) Let $\Psi\colon K \to V_K$ be the Minkowski map.

Prove that the image $\Psi(\mathcal{O}_K)$ is a rectangular lattice. That is, show that there is a basis $\{\alpha, \beta\}$ of $\Psi(\mathcal{O}_K)$ such that $\langle \alpha, \beta \rangle = 0$.

4. Let d be a squarefree integer, and let $K = \mathbb{Q}(\sqrt{d})$. Assume that the ring of integers of K is $\mathcal{O}_K = \mathbb{Z}[\gamma]$, where $\gamma = \frac{1+\sqrt{d}}{2}$. (This is the same as assuming that $d \equiv 1 \pmod 4$.) Let $\Psi\colon K \to V_K$ be the Minkowski map.

Prove that if $\{\alpha, \beta\}$ is a basis for $\mathcal{O}_K$ as a $\mathbb{Z}$-module, then the area of the parallelogram spanned by $\Psi(\alpha)$ and $\Psi(\beta)$ (that is, the parallelogram with vertices 0, $\Psi(\alpha)$, $\Psi(\beta)$, and $\Psi(\alpha + \beta)$) is equal to $\sqrt{|d|}$.

[Hint: Show first that the area of this parallelogram doesn't depend on the choice of basis.]

5. Let d be a squarefree integer congruent to 1 modulo 4, and let $K = \mathbb{Q}(\sqrt{d})$. Let $\Psi\colon K \to V_K$ be the Minkowski map.

Show that $\Psi(\mathcal{O}_K)$ is not a rectangular lattice. That is, show that there is no basis $\{\alpha, \beta\}$ of $\Psi(\mathcal{O}_K)$ as a $\mathbb{Z}$-module such that $\langle \alpha, \beta \rangle = 0$.

6. Let $K = \mathbb{Q}(\sqrt{10})$. Let $\Psi\colon K \to \mathbb{R}^2$ be the Minkowski map; that is, let $\Psi(r + s\sqrt{10}) = (r + s\sqrt{10}, r - s\sqrt{10})$. Find positive real numbers c and C

such that for all $a \in \mathbb{Q}(\sqrt{10})$, we have:

$$c \leq \frac{|a|}{\|\Psi(a)\|} \leq C$$

or prove that no such c and C exist. (The notation $|\cdot|$ means the usual absolute value in $\mathbb{R}$, and $\|\cdot\|$ means the usual length in $\mathbb{R}^2$.)