PMath 441/741 – Homework 4

Due on Crowdmark at 11:59pm on Wednesday, June 5

1. Let d and e be different squarefree integers greater than 1. Find the discriminant of the ring $\mathbb{Z}[\sqrt{d}, \sqrt{e}]$.

2. Compute the discriminant of the ideal $I = (5, \sqrt{10})$ in $\mathbb{Z}[\sqrt{10}]$.

3. Let $K = \mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3 + x + 1$. Prove that the ring of integers of K is $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

4. Let α be a root of the polynomial $x^3 + 2x + 5$. (You may assume that this polynomial is irreducible.) How many elements does the ring $\mathbb{Z}[\alpha]/(\alpha^2 + 1)$ have?

5. Let $K = \mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3 - x - 1$.
Let $L = \mathbb{Q}(\beta)$, where β is a root of the polynomial $x^3 + 2x - 1$.
Let $M = \mathbb{Q}(\gamma)$, where γ is a root of the polynomial $x^3 - 3x + 25$.
Two of these fields are secretly identical. (You may assume that for the purposes of solving this question.) Which two?
[Bonus points if you can prove that the two identical fields are actually identical!]

6. Let α be a root of $x^3 - 4x + 2$, and let $K = \mathbb{Q}(\alpha)$. Prove that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. [Hint: First prove that either $\mathcal{O}_K = \mathbb{Z}[\alpha]$, or else $\mathbb{Z}[\alpha]$ has index 2 in $\mathcal{O}_K$. Then let $T = 2\mathcal{O}_K$, so that $T \subset \mathbb{Z}[\alpha]$ (because it has index dividing 2), and show that T has index at least 5 in $\mathbb{Z}[\alpha]$.]