PMath 441/741 – Homework 7
Due on Crowdmark at 11:59pm on Wednesday, June 26

1. Factor the ideals (2) and (3) in the ring of integers $\mathcal{O}_K$ of the field $K = \mathbb{Q}(\alpha)$, where α is a root of $x^3 + 16x + 24$. It may be useful to know that the ring of integers of $\mathcal{O}_K$ is equal to $\mathbb{Z}[\alpha, \beta]$, where β is a root of $x^3 + 36x + 81$.

[You don't have to prove that $\mathcal{O}_K = \mathbb{Z}[\alpha, \beta]$ – you can take that for granted.]

2. Let L be a number field. Let $Q \subset \mathcal{O}_L$ be a prime ideal, and let $(p) = Q \cap \mathbb{Z}$. Show that p is a generator for the ideal $Q^{e(Q)}(\mathcal{O}_L)_Q$ of the local ring $(\mathcal{O}_L)_Q$.

3. Let $I = (1, \frac{1+\sqrt{6}}{5})$ be the fractional ideal of $D = \mathbb{Z}[\sqrt{6}]$ generated by 1 and $\frac{1+\sqrt{6}}{5}$. Let P be the ideal of D generated by 7. (You may assume that P is a prime ideal.) Find an element $\alpha \in \mathbb{Q}(\sqrt{6})$ such that $ID_P = \alpha D_P$.

4. Let K be the number field $\mathbb{Q}(\alpha)$, where α is a root of the polynomial $x^3 + 9x + 3$. Find an element $a \in K$ such that $\mathcal{O}_K = \mathbb{Z}[a]$.

5. Let $p \in \mathbb{Z}$ be a prime number, α be the root of a p-Eisenstein polynomial – that is, a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_i \in \mathbb{Z}$, $p \mid a_i$ for all i, and $p^2 \nmid a_0$. Let R be the ring $\mathbb{Z}[\alpha]$. Prove that there is exactly one prime ideal P of R that contains p, prove that R_P is a DVR, and prove that α is a uniformizer for R_P.

6. Let D be the ring $\mathbb{Z}[\sqrt{5}]$, and let $P = (2, 1 + \sqrt{5})$ be the ideal generated by 2 and $1 + \sqrt{5}$. Prove that D_P is not a DVR.