PMath 441/741 – Homework 8
Due on Crowdmark at 11:59pm on Wednesday, July 3

1. Let $\mathbb{Q} \subset L \subset M$ be number fields, with rings of integers $\mathbb{Z}$, $\mathcal{O}_L$, and $\mathcal{O}_M$, respectively. Let p be a prime number, P a prime factor of the ideal $p\mathcal{O}_L$, and Q a prime factor of the ideal $P\mathcal{O}_M$.

Prove that $f(P) \mid f(Q)$.

2. Let $\mathbb{Q} \subset L \subset M$ be number fields, with rings of integers $\mathbb{Z}$, $\mathcal{O}_L$, and $\mathcal{O}_M$, respectively. Let p be a prime number, P a prime factor of the ideal $p\mathcal{O}_L$, and Q a prime factor of the ideal $P\mathcal{O}_M$.

Prove that $e(P) \mid e(Q)$.

3. Let $K \subset L$ be number fields with rings of integers $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively, and let $I \subsetneq \mathcal{O}_K$ be a proper ideal of $\mathcal{O}_K$. Prove that $I\mathcal{O}_L \neq \mathcal{O}_L$ is not the unit ideal of $\mathcal{O}_L$.

4. Let $\mathbb{Q} \subset L \subset M$ be number fields, with rings of integers $\mathbb{Z}$, $\mathcal{O}_L$, and $\mathcal{O}_M$, respectively. A prime number p is said to be totally split in L if and only if the ideal $p\mathcal{O}_L$ factors into a product of $[L : \mathbb{Q}]$ different prime ideals.

Prove that if p is totally split in M, then it is totally split in L.

5. Let A be a domain, K its fraction field. Prove that $A = \cap_P A_P$, where P ranges over all prime ideals of A.

6. Let K be a number field with ring of integers $\mathcal{O}_K$. Let D be a domain satisfying $\mathcal{O}_K \subset D \subset K$. Prove that there is a set S of prime ideals of $\mathcal{O}_K$ such that

$$D = \bigcap_{Q \in S} (\mathcal{O}_K)_Q$$