Free Functions and Operator Algebras

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CMS Winter Meeting
December 10th, 2017

NC Sets and Functions

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Let V be a vector space, then we set $\mathbb{M}(V) = \sqcup_{n=1}^{\infty} M_n(\mathbb{C}) \otimes V$. For simplicity we set $\mathbb{M}_d = \mathbb{M}(\mathbb{C}^d)$. Note that there is a natural action of GL_n on $M_n(\mathbb{C}) \otimes V$, by $S \cdot (A \otimes v) = (S^{-1}AS) \otimes v$.

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A set $\Omega \subset \mathbb{M}(V)$ is called an nc set, if for every $X, Y \in \Omega$, $X \oplus Y \in \Omega$. We will write $\Omega(n) = M_n(\mathbb{C}) \otimes V \bigcap \Omega$.

A function $f \colon \Omega \to \mathbb{M}_1$ is called non-commutative (nc) or free if:

- $f(\Omega(n)) \subset M_n(\mathbb{C})$;
- If $X \in \Omega(n)$, $Y \in \Omega(m)$ and $T \in M_{m,n}$, such that $TX = YT$, then $Tf(X) = f(Y)T$.

The second condition can be replaced with the requirement that f respects direct sums and similarities.

NC Examples

We are in particular interested in the free ball $\mathfrak{B}_d \subset \mathbb{M}_d$, i.e., the set of all d-tuples $X = (X_1, \ldots, X_d)$, such that $\|X_1 X_1^* + \cdots + X_d X_d^*\| < 1$.

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A first example of an nc function is a free polynomial, i.e., an element of the free algebra $\mathbb{F}_d$. More generally a free rational function.
Another example is a power series in free variables with that has radius of convergence at least $r > 0$ on each level. All the examples so far are analytic. However, a surprising fact proved by Kaliuzhnyi-Verbovetskyi and Vinnikov is that local boundedness is sufficient to imply analyticity for nc functions.

NC Kernels

Recently, Ball, Marx and Vinnikov have introduced a notion of nc kernels that generalize the classical notion of reproducing kernels. Given an nc set $\Omega \subset \mathbb{M}(V)$, a cp nc kernel K on Ω is a function satisfying:

- For $Z \in \Omega(n)$ and $W \in \Omega(m)$, $K(Z, W) \in \mathcal{L}(M_{n,m}, M_{n,m})$.
- If $Z, \tilde{Z}, W, \tilde{W} \in \Omega$ and α, β are scalar matrices, such that $\alpha Z = \tilde{Z}\alpha$ and $\beta Z = \tilde{Z}\beta$, then

$$\alpha K(Z, W)(A)\beta^* = K(\tilde{Z}, \tilde{W})(\alpha A \beta^*).$$

- $K(Z, Z)\colon M_n \to M_n$ is a positive map for every $Z \in \Omega$.

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- $K(Z, Z) \colon M_n \to M_n$ is a positive map for every $Z \in \Omega$.

As in the classical case, for each cp nc kernel K one associates an nc-RKHS, consisting of nc functions on Ω. One can also consider the multiplier algebra of a nc-RKHS.

More NC Examples

Let $Z \in \mathbb{M}_n(\mathbb{C}) \otimes V$ and set

$$\Omega(k) = \begin{cases} \{Z^{\oplus \frac{k}{n}}\} & n \mid k \\ \emptyset, & n \nmid k \end{cases}.$$

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If K is a kernel on Ω, then for every $m \in \mathbb{N}$ by the properties of the kernel $K(Z^{\oplus m}, Z^{\oplus m}) = K(Z, Z) \otimes I_{M_m}$. Therefore, $K(Z, Z)$ is simply a completely positive map.

NC Szego Kernel

We will denote by $\mathcal{H}_d$ the full Fock space on $\mathbb{C}^d$. This space is an nc-RKHS, with the nc reproducing Szego kernel given by:

$$K(Z,W)(T) = \sum_{\alpha \in \mathbb{W}_d} Z^\alpha T W^{*\alpha}$$

Here $\mathbb{W}_d$ stands for the free monoid on d generators. This is a complete Pick space. The multiplier algebra of $\mathcal{H}_d$ is $H^\infty(\mathfrak{B}_d)$, the algebra of all uniformly bounded nc functions on $\mathfrak{B}_d$. In fact this operator algebra is unitarily equivalent to the WOT-closed subalgebra of $B(\mathcal{H}_d)$ generated by the non-commutative shifts, considered by Arias, Popescu, Davidson and Pitts.

Multiplier Ideals and Varieties

- For an ideal $\mathcal{I} \subset H^\infty(\mathfrak{B}_d)$ we define the nc variety $Z(\mathcal{I}) \subset \mathfrak{B}_d$, to be all the d-tuples $X \in \mathfrak{B}_d$, such that $f(X) = 0$, for every $f \in \mathcal{I}$.

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- Similarly for a subset $V \subset \mathfrak{B}_d$, we set $I(V) \subset \mathfrak{B}_d$, to be the collection of all multipliers that vanish on V.
- One notes that every variety is in particular an nc set, that is additionally closed under "relevant" similarities, restriction to invariant subspaces and compression to coinvariant subspaces. An nc isomorphisms of subvarieties $V \subset \mathfrak{B}_d$ and $W \subset \mathfrak{B}_e$ is an nc function $f \colon V \to W$, that has an nc inverse.

Goal

Davidson, Ramsey and Shalit studied the multiplier algebras on subvarieties of the unit ball of $\mathbb{C}^d$.

Theorem (Davidson,Ramsey,Shalit)

Let $V, W \subset \mathbb{B}_d$ be two varieties, then TFAE:

- *There exists an automorphism F of $\mathbb{B}_d$, such that $F(W) = V$.*
- *The multiplier algebras of V and of W are completely isometrically isomorphic.*
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Question: Does this work in the free case?

Automorphisms of the nc-Ball

It was proved by Popescu and Davidson and Pitts that every analytic automorphism of the operator d-ball is in fact induced by an automorphism of $\mathbb{B}_d$. In the case of the matrix ball $\mathfrak{B}_d(n)$ we can show using elementary methods that:

Lemma

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To prove it note that the analytic automorphisms of $\mathfrak{B}_d(n)$ are $SU(n, dn)/Z$, where Z is the center. So if $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ corresponds to such an automorphism and then for every $V \in U_n$ we have $V^{\oplus(d+1)} T V^{*\oplus(d+1)} = T$ and now we use the special structure of T.

More on Varieties

Let now $V \subset \mathfrak{B}_d$ be an nc variety. Let $\mathcal{I} \subset H^\infty(\mathfrak{B}_d)$ be the ideal of functions vanishing on V. We let $\mathcal{H}_V$ be the orthogonal complement of the closure of $\mathcal{I}$ in $\mathcal{H}_d$. Let us denote by $\mathcal{M}_V$ the algebra of multipliers on V.

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Theorem (Salomon, Shalit, S.)

The map $f \to f|_V$ is a completely contractive and surjective homomorphism $\mathcal{M}_d \to \mathcal{M}_V$ and it induces a completely isometric isomorphism $\mathcal{M}_d/\mathcal{I} \cong \mathcal{M}_V$.

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There are two possible proofs of this result. One uses nc kernel theory, the other uses Davidson-Pitts distance formula.

Weak-$*$ Continuous f.d. Representations

Let us write $\mathrm{Hom}(H^\infty(\mathfrak{B}_d), M_n(\mathbb{C}))$ for the space of all unital completely contractive representations of $H^\infty(\mathfrak{B}_d)$ on $\mathbb{C}^n$.

Theorem (Davidson,Pitts)

For all $d \in \mathbb{N}$ and $n \in \mathbb{N}$, we have there is a natural continuous projection $\pi_{d,k}\colon \mathrm{Hom}(\mathcal{M}_d, M_n(\mathbb{C})) \to \overline{\mathfrak{B}(n)}$, given by $\pi_{d,k}(\varphi) = (\varphi(z_1), \dots, \varphi(z_d))$. Furthermore, for every $T \in \mathfrak{B}(n)$, there is a unique WOT continuous representation φ_T of $\mathcal{M}_d$ corresponding to T, given by the Popescu functional calculus.

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The behavior of $\pi_{d,k}$ on the boundary is more complex. If $T \in \partial\mathfrak{B}_d(n)$, then we say that T is pure if $\sum_{|\alpha|=n} T^\alpha T^{*\alpha} \xrightarrow[n\to\infty]{0}$. Using Popescu's functional calculus one can extend the map $T \mapsto \varphi_T$ from $\mathfrak{B}_d(n)$ to the set of all pure T in the boundary, that we will denote by $\overline{\mathfrak{B}_d(n)}^p$.

Weak-$*$ Continuous f.d. Representations

One can also look at the following much simpler setting. Let $T \in \partial\mathfrak{B}_d(n)$, be such that there exists $S \in \mathrm{GL}_n$, such that $S^{-1}TS \in \mathfrak{B}_d(n)$. In this case there is a unique extension of every *nc*-function in $\mathcal{M}_d$ to T.

$$f(T) = Sf(S^{-1}TS)S^{-1}.$$

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Lemma (Salomon, Shalit, S.)

The set of pure points is precisely the set of points similar to a strict contraction.

CCU Homomorphisms

Let $V \subset \mathfrak{B}_d$ and $W \subset \mathfrak{B}_e$ be nc varieties and let $\alpha \colon \mathcal{M}_V \to \mathcal{M}_W$ be a completely contractive unital homomorphism. Then obviously for every $n \in \mathbb{N}$ we obtain a map:

$$\alpha_n^* \colon \operatorname{Hom}(\mathcal{M}_W, M_n(\mathbb{C})) \to \operatorname{Hom}(\mathcal{M}_V, M_n(\mathbb{C})).$$

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The map α^* clearly respects direct sums and similarities. Hence we get a map $G \colon W \to \overline{\mathfrak{B}_d}$ and we can extend G to a map on $\mathfrak{B}_d$ using the complete Pick property. If α is weak-$*$ continuous, then $G(W) \subset \overline{V}^p$.

CI Isomorphisms

Theorem (Salomon, Shalit, S.)

Let $V \subset \mathfrak{B}_d$ and $W \subset \mathfrak{B}_e$ be nc varieties and assume that $\alpha \colon \mathcal{M}_V \to \mathcal{M}_W$ is a completely isometric isomorphism. Then α gives rise to maps $F, G \colon \mathfrak{B}_d \to \mathfrak{B}_d$, such that $F \circ G|_V = \mathrm{id}_V$ and $G \circ F|_W = \mathrm{id}_W$. In other words V and W are biholomorphic.

Homogeneous nc-Varieties in the Ball

As in the commutative case we can consider the nice case of homogeneous varieties in $\mathfrak{B}_d$. An ideal $\mathcal{I} \subset \mathcal{M}_d$ is said to be homogeneous if for every $f \in \mathcal{I}$, every homogeneous component f_n of f is in $\mathcal{I}$. Therefore we can think of it as an ideal generated by homogeneous polynomials in $\mathbb{F}_d$.

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A subvariety $V \subset \mathfrak{B}_d$ is said to be homogeneous if it is invariant under multiplication by $t \in \overline{\mathbb{D}}$, i.e., $X \in V$ implies $tX \in V$, for every $t \in \overline{\mathbb{D}}$. As in the commutative case, a subvariety of $\mathfrak{B}_d$ is homogeneous if and only if the ideal $\mathcal{I} = I(V)$ of functions that vanish on V is homogeneous.

The Homogeneous Case

Given a subset $\mathcal{X} \subset \mathbb{M}_d$, we define the matrix span of $\mathcal{X}$ to be:

$$\text{mat-span}(\mathcal{X})(n) = \text{Span}\left\{(I_d \otimes L)(A) \mid A \in \mathcal{X},\ L \in \mathcal{L}(M_n(\mathbb{C}))\right\}.$$

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Theorem (Salomon, Shalit, S.)

Let $V, W \subset \mathfrak{B}_d$ be homogeneous varieties, such that $\text{mat-span}(V) = \text{mat-span}(W) = \mathbb{M}_d$. *Then TFAE*

- $\mathcal{M}_V$ *and* $\mathcal{M}_W$ *are completely isometrically isomorphic;*
- *There exists an automorphism F of $\mathfrak{B}_d$, such that $f(V) = W$;*
- *There a unitary $U \in U_d$, such that $UV = W$.*

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The (More) General Case

Lemma (S.)

If $0 \neq X \in \mathfrak{B}_d(n)$ is an irreducible point, then there exists $S \in \mathrm{GL}_n(\mathbb{C})$, such that if $Y = S^{-1}XS$, then $YY^* = rI$, for $0 < r < 1$.

The key ingredient is the Perron-Frobenius theory of completely positive maps on finite-dimensional C^*-algebras.

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Theorem (S.)

If $V, W \subset \mathfrak{B}_d$ are nc varieties, such that $V(1), W(1) \neq \emptyset$, then TFAE:

- *$\mathcal{M}_V$ and $\mathcal{M}_W$ are completely isometrically isomorphic;*
- *There exists an automorphism F of $\mathfrak{B}_d$, such that $f(V) = W$.*

The proof relies on combining the above lemma with classical results on hyperbolic geometry of balls in Banach spaces.

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Thank You!