FREE ANALYSIS — LEARNING SEMINAR

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CONTENTS

1. BASIC DEFINITIONS AND SOME HISTORY

The goal of this seminar is to introduce the basic ideas and some of the applications of free analysis. Free analysis deals with so-called noncommutative (nc for short) functions. One should think of free analysis as a quantized version of complex analysis in the same line as operator spaces are quantized Banach spaces. The template that I will try to fill with content is "analytic function of several noncommuting variables".

Since I have a penchant for history, I will start from afar. The first lecture is dealing with original motivation and the origins of this field. The idea of an nc function goes back to Taylor. In 1970 in [33] Taylor proposed a notion of joint spectrum for several commuting operators based on the idea of the Koszul complex (a complex uniformly loved by geometers and algebraists, alike). He then proceeded to construct functional calculus for several commuting operators in [32]. However, his next two papers [34, 35] were far more ambitious. He has set out to construct a joint spectrum and functional calculus for several arbitrary operators. The ideas that guided Taylor were again algebraic in their core. He recast functional calculus and spectrum in the language of modules. Given $T_1, \ldots T_d \in B(\mathcal{H})$ commuting operators on some Hilbert space we automatically get that $\mathcal{H}$ is a module over the polynomial ring $A = \mathbb{C}[z_1, \ldots, z_d]$. Given a topological algebra B over the polynomial ring, the question of B-function calculus for our operators is "can one extend the module structure on $\mathcal{H}$ to a B-module structure?". To be more precise, one has a natural map $\mathcal{H} \to \mathcal{H} \otimes_A B$, given by $\xi \mapsto \xi \otimes 1$ and the question is whether this is an isomorphism? The natural choice for B is the algebra $\mathcal{O}(U)$, the algebra of analytic functions on $U \subset \mathbb{C}^d$ with the Frechet topology. Where does the spectrum come in? It gives us a tool to determine whether this works for a given B or not. Classically, one has a $\mathcal{O}(U)$-functional calculus if and only if the spectrum is in U.

What happens if the operators do not commute? Just replace $\mathbb{C}[z_1, \ldots, z_d]$ with the free algebra $\mathbb{C}\langle z_1, \ldots, z_d \rangle$. What do you replace $\mathcal{O}(U)$ with? Taylor suggested several options, for example, rings of convergent power series in several non-commuting variables. His most interesting idea was the nc functions. Here are the first definitions that we will carry through the seminar (the notations are not Taylor's).

Definition 1.1. Given a vector space V, we define the nc space over V to me $\mathbb{M}(V) = \sqcup_{n=1}^{\infty} M_n(V)$, where $M_n(V) = M_n(\mathbb{C}) \otimes V$. We will write simply $\mathbb{M}_d$ for $\mathbb{M}(\mathbb{C}^d)$. For every subset $\Omega \subset \mathbb{M}(V)$, we will write $\Omega(n) = \Omega \cap M_n(V)$, the n-th level of Ω.

Definition 1.2. An nc subset of $\mathbb{M}(V)$ is $\Omega \subset \mathbb{M}(V)$, such that for every $X, Y \in \Omega$, we have that $X \oplus Y = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right) \in \Omega$.

Note that $M_n(V) = M_n(\mathbb{C}) \otimes V$ comes equipped with a natural $\mathbf{GL}_n(\mathbb{C})$ (or rather $\mathbf{P}\mathrm{GL}_n(\mathbb{C})$) action, where $S \in \mathbf{GL}_n(\mathbb{C})$ acts on $v \otimes T$ by $v \otimes (S^{-1}TS)$. For $X \in \mathbb{M}(V)$, we will simply write $S^{-1}XS$. This is the last ingredient we need to define nc functions.

Definition 1.3. Let V and W be vector spaces and $\Omega \subset \mathbb{M}(V)$ an nc set. A function $f \colon \Omega \to \mathbb{M}(W)$ is called nc if it satisfies the following properties:

- f is graded, i.e, $f(\Omega(n)) \subset M_n(W)$ for every $n \in \mathbb{N}$.
- f respects direct sums, i.e, $f(X \oplus Y) = f(X) \oplus f(Y)$ for every $X, Y \in \Omega$.
- f respects similarities, i.e, for every $n \in \mathbb{N}$, $X \in \Omega(n)$ and $S \in \mathbf{GL}_n(\mathbb{C})$, such that $S^{-1}XS \in \Omega$, we have that $f(S^{-1}XS) = S^{-1}f(X)S$.

Lemma 1.4 ([18, Proposition 2.1])**.** *The latter two conditions can be replaced by one condition: if $X \in \Omega(n)$, $Y \in \Omega(m)$ and $T \in M_{m,n}(\mathbb{C})$ is such that $TX = YT$, then $Tf(X) = f(Y)T$. One says that f respects intertwinners in this case.*

Proof. If f respects intertwinners, than clearly f respects similarities. Now for direct sums one writes

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} I_n \\ 0 \end{pmatrix} = \begin{pmatrix} I_n \\ 0 \end{pmatrix} X.$$

Let $f(X \oplus Y) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Since f respects intertwinners $A = f(X)$ and $C = 0$. Similarly, one gets that $B = 0$ and $D = f(Y)$.

Now let f be an nc function, $X \in \Omega(n)$ and $Y \in \Omega(m)$. If $T \in M_{m,n}$ is such that $TX = YT$, then consider the matrix $S = \left(\begin{smallmatrix} I_n & 0 \\ T & I_m \end{smallmatrix}\right)$. Then we have

$$S^{-1} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} S = S^{-1} \begin{pmatrix} X & 0 \\ YT & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Since f is nc we have that

$$\begin{pmatrix} f(X) & 0 \\ f(Y)T - Tf(X) & f(Y) \end{pmatrix} = S^{-1} \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix} S = \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix}.$$

Thus f respects intertwinners. □

Example 1.5. Every element of the free algebra $\mathbb{C}\langle z_1, \dots, z_d \rangle$ is naturally a free function on $\mathbb{M}_d$. In fact, one can view $\mathbb{M}_d$ as the parameter space for all finite-dimensional representations of the free algebra on d-generators. This is the analog of $\mathbb{C}^d$ that Taylor was looking for in the case of the free algebra.

Let us write from now on $\mathbb{W}_d$ for the free monoid on d-letters.

Example 1.6. Convergent power series in several noncommuting variables. Let $r > 0$ and consider the series $\sum_{\alpha \in \mathbb{W}_d} a_\alpha z^\alpha$, where $a_\alpha \in \mathbb{C}$ and z^α is just the word α in the free variables $z_1, \ldots, z_d$. Let us write $|\alpha|$ for the length of the word α. Assume that $\limsup_{n \to \infty} \sqrt{\sum_{|\alpha|=n} a_\alpha r^n} \leq 1$. Then the power series defines an nc function on the nc set

$$\mathfrak{D}_r = \{X \in \mathbb{M}_d \mid \forall 1 \leq j \leq d, \|X_j\| < r\}.$$

This set is the noncommutative polydisc of radius r. One can, of course, use a multi-radius for this example.

This example brings up the question of topologies, but I will defer the discussion of this question to the next lecture.

The second father of free analysis is Voiculescu. In [37, 38] he considered nc functions under the name of "fully matricial functions". The "fully matricial sets" of Voiculescu are nc sets we have defined with the additional assumption that they are closed under similiarities on all levels. Let $\Omega \subset \mathbb{M}(V)$ be an nc set, we will denote by $\widetilde{\Omega}$ the similarity envelope of Ω. Namely, for every $n \in \mathbb{N}$ we set

$$\widetilde{\Omega}(n) = \left\{S^{-1} X S \mid X \in \Omega(n),\, S \in \mathbf{GL}_n(\mathbb{C})\right\}.$$

In [18, Proposition A.3] (alternatively see [1, Proposition 3.10]) it is proved that if $f \colon \Omega \to \mathbb{M}(W)$ is an nc function, then there exists a unique nc function $\tilde{f} \colon \widetilde{\Omega} \to M(W)$, such that $\tilde{f}|_\Omega = f$. The functions that interested Voiculescu in particular are the resolvents.

Example 1.7. Let $B \subset A$ be two Banach algebras. Let $a \in A$ and we defien the B-resolvent set of a

$$\rho(a, B)(n) = \{b \in M_n(B) \mid a \otimes I_n - b \text{ is invertible }\}.$$

This is clearly an nc set and it is even similarity invariant, so it is a fully matricial set in the sense of Voiculescu. We define the function $R(a, B) \colon \rho(a, B) \to \mathbb{M}(A)$ by $R(a, B)(b) = (a \otimes I_n - b)^{-1}$. This is an nc function. In fact, it is an example (not quite) of a rational nc function and I will discuss those a bit in the upcoming lectures.

The last example is special to me since this is how I got into the world of free analysis. The commutative generalization of $H^2(\mathbb{D})$ and its multiplier algebra $H^\infty(\mathbb{D})$ is the Drury-Arveson space with its multiplier algebra. If the reproducing kernel of H^2 is $\frac{1}{1 - z\bar{w}}$, then the reproducing kernel of the Drury-Arveson space is $\frac{1}{1 - \langle z, w \rangle}$. The Drury-Arveson space is the symmetric Fock space and thus is a quotient of the full Fock space $\mathcal{F}_d^2 = \oplus_{n=0}^\infty \left(\mathbb{C}^d\right)^{\otimes n}$, where we set $\left(\mathbb{C}^d\right)^{\otimes 0} = \mathbb{C}$. The analog of the multiplier algebra is the free semigroup algebra — the weak-* closed subalgebra of $B(\mathcal{F}_d^2)$ generated by the left creation operators. Davidson and Pitts studied this algebra in [11–13] and proved, in particular, that it satisfies a distance formula analogous to the one introduced by Sarason for Nevanlinna-Pick interpolation. Similar results were obtained by Arias and Popescu in [5]. Popescu in [23, 24] has already studied this algebra in term of functions on the unit ball of $B(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$.

It turns out that the Fock space consists of nc functions on the free ball

$$\mathfrak{B}_d = \left\{X \in \mathbb{M}_d \mid \sum_{j=1}^d X_j X_j^* < I\right\}.$$

This is a nc reproducing kernel Hilbert space, with the kerenl $K(Z,W)(T) = \sum_{\alpha \in \mathbb{W}_d} Z^\alpha T W^{\alpha *}$ (I will explain what are nc functions of several variables, when we discuss Taylor-Taylor series). The multiplier algebra of this space is the space of nc functions uniformly bounded on $\mathfrak{B}_d$ and this algebra is unitarily equivalent to the free semigroup algebra.

Remark 1.8. One may ask why not include $n = \infty$ in the definition of $\mathbb{M}(V)$, where $M_\infty(V) = B(\mathcal{H}) \otimes V$? Of course, the tensor here becomes ambiguous unless V is finite-dimensional. The answer is: one can do that, see for example the work of Agler and McCarthy [2] and the works of Muhly and Solel [21]. Sometimes, even one should do that. Eventually, looking at only finite-dimensional representations implies a form of residual finite-dimensionality and many things are not such. That being said, if one goes to infinite dimensions, one introduces the problems related to analytic functions on Banach spaces. Though such problems will be implicit in our discussion, most analytic tools will be nice and cozy several complex variables stuff.

2. Topologies, analyticity, and the difference-differential formula

Since we are doing analysis, one needs to put a topology on V. Even if V is a topological vector space, then one can still choose various topologies on $\mathbb{M}(V)$. The most immediate one is the disjoint union (du) topology. This is the topology considered by Taylor in his original work. The problem with this topology is that it sometimes allows for various anomalies, that I will present when we discuss analyticity. The first, rather surprising result is that local boundedness for an nc function is enough to guarantee analyticity (at least as a function on each level).

Except for the du topology, I am going to introduce three more; the fine topology, the uniform topology (This is a name from [18], in [3] it is called "the fat topology") and the free topology. From now on V is a topological vector space (more assumptions will be added as necessary).

Definition 2.1. An nc set $\Omega \subset \mathbb{M}(V)$ is called an nc domain if Ω is open in the du topology and invariant under unitary similarities. The topology generated by all nc domains is called the fine topology.

The fine topology is a sort of limit topology. This topology won't be used much in the seminar, but it appears, for instance, in the work of Pascoe on the free Jacobian conjecture [22].

Definition 2.2. Assume that V is an operator space. Let $Y \in \mathbb{M}(V)(n)$ and $r > 0$, we define the ball around Y to be the set with levels

$$\mathfrak{B}_r(kn) = \left\{ X \in \mathbb{M}(V)(kn) \mid \|X - Y^{\oplus k}\| < r \right\}, \, k \in \mathbb{N}.$$

The uniform topology is the topology generated by these balls.

I will be mostly talking about $\mathbb{M}_d$, so we will throw in this third quite interesting topology.

Definition 2.3. This topology is defined on $\mathbb{M}_d$. Let $\delta \in M_k(\mathbb{C}\langle z_1, \ldots, z_d \rangle)$, for some $k \in \mathbb{N}$. We define the following "free polynomial polyhedron"

$$U_\delta = \{ X \in \mathbb{M}_d \mid \|\delta(X)\| < 1 \}.$$

Where the ealuation is a block matrix (tensor product). The free topology is the topology whose basis is the collection of sets U_δ, where δ runs over all possible square polynomial matrices over the free algebra in d variables.

Proposition 2.4 ([3, Proposition 3.10]).

$$\textit{du topology} \geq \textit{fine topology} \geq \textit{uniform topology} \geq \textit{free topology}.$$

Here $\tau \geq \tau'$ means that τ is finer than τ'.

Let us assume from now on that V and W are operator spaces. The reason for this definition will be explained below.

Definition 2.5. Let τ be one of our topologies. Let $\Omega \subset \mathbb{M}(V)$ be an nc domain that is also opne in the topology τ and $f\colon \Omega \to \mathbb{M}(W)$ an nc function. We will say that f is τ-analytic (e.g. uniformly analytic) if f is τ-locally bounded.

Remark 2.6. These topologies differ quite a bit. Agler and McCarthy in [3, Theorem 1.4, 7.7] prove that there are uniformly analytic functions that are not pointwise approximable by polynomials, whereas this is true for the free topology (see [1]). However, they also prove in [3, Theorem 1.5, 6.1] that the uniform topology admits an implicit function theorem, but according to [3, Theorem 1.6] the free topology does not.

Lemma 2.7 ([18, Proposition 2.2]). *Let $\Omega \subset \mathbb{M}(V)$ be an nc set and let $f\colon \Omega \to \mathbb{M}(W)$ be an nc function. If $X \in \Omega(n)$, $Y \in \Omega(m)$ and $Z \in M_{n,m}(V)$ are such that $\left(\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right) \in \Omega$, then*

$$f\left(\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}\right) = \begin{pmatrix} f(X) & \Delta f(X,Y)(Z) \\ 0 & f(Y) \end{pmatrix}.$$

We have that $\Delta f(X,Y)(tZ) = t\Delta f(X,Y)(Z)$, for every $t \in \mathbb{C}$, such that $\left(\begin{smallmatrix} X & tZ \\ 0 & Y \end{smallmatrix}\right) \in \Omega$.

Proof. Write $T = \left(\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right)$ and set $f(T) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Note that

$$T\begin{pmatrix} I_n \\ 0 \end{pmatrix} = \begin{pmatrix} I_n \\ 0 \end{pmatrix} X.$$

By Lemma 1.4 we have that $A = f(X)$ and $C = 0$. Now we also have that

$$\begin{pmatrix} 0 & I_m \end{pmatrix} T = Y \begin{pmatrix} 0 & I_m \end{pmatrix}.$$

Conclude that $D = f(Y)$ and $f(T) = \left(\begin{smallmatrix} f(X) & \Delta f(X,Y)(Z) \\ 0 & f(Y) \end{smallmatrix}\right)$.

Let $t \in \mathbb{C}$ and write $T_t = \left(\begin{smallmatrix} X & tZ \\ 0 & Y \end{smallmatrix}\right)$. If $T, T_t \in \Omega$, then $(tI_n \oplus I_m)T = T_t(tI_n \oplus I_m)$. Thus we get the last statement.

□

Remark 2.8. Due to the generality of [18], they define right (and left) admissible nc sets. Every domain is both, so I will restrict to the case of nc domains.

Remark 2.9. In [18] the operator $\Delta f(X,Y)$ is called the right difference-differential and there is a left one corresponding to lower-triangular matrices. By [18, Proposition 2.8] for nc domains $\Delta_R f(X,Y)(Z) = \Delta_L f(Y,X)(Z)$. So we will focus on the right one.

Why do I care about upper-triangular matrices, anyway? The intuition is rather simple. Let $f\colon \mathbb{D} \to \mathbb{C}$ be an analytic function. Let $A = \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)$, where $a, b \in \mathbb{D}$. What is $f(A)$? Assume first that $a \neq b$. Set $S = \left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right)$ and note that

$$S^{-1}AS = \begin{pmatrix} a & c-db \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c+d(a-b) \\ 0 & b \end{pmatrix}.$$

So fix $d = c/(b-a)$, then $A^n = S\left(\begin{smallmatrix} a^n & 0 \\ 0 & b^n \end{smallmatrix}\right)$. If $f(z) = \sum_{n=0}^{\infty} \alpha - nz^n$ is the Taylor series, then

$$f(A) = S\begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} S^{-1} = \begin{pmatrix} f(a) & c\frac{f(b)-f(a)}{b-a} \\ 0 & f(b) \end{pmatrix}.$$

Note that the upper right corner is the divided difference of f at a, b multiplied by c. If $a = b$, then we either take the limit or write $A = aI + cJ$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. In the second case, it is immediate that $A^n = a^n I + na^{n-1}cJ$, thus the power series yields $f(A) = f(a)I + f'(a)cJ$. Either way the upper right corner contains the derivative of f at a.

Remark 2.10. A (much more) highbrow approach to this, that is in the spirit of Taylor's works is to consider a commutative algebra A and its modules. An extension of an A module M by an A module N is an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0\,.$$

One can define a notion of isomorphism of extensions and a sum of two extensions. This gives us a group $\mathrm{Ext}^1_A(M, N)$. Tee key here is to consider the vector space $M \oplus N$ and endow it with a module structure, so that N is a submodule and M is a quotient. Let us assume that the action of A on M is given by a homomorphism $\pi_M\colon A \to \mathrm{End}_{\mathbb{C}}(M)$ and similarly π_N for N. Then to specify a module action on $M \oplus N$, we need to choose a linear map from $\Delta\colon A \to \mathrm{Hom}_{\mathbb{C}}(M, N)$, such that the map $a \mapsto \left(\begin{smallmatrix} \pi_N(a) & \Delta(a) \\ 0 & \pi_M(a) \end{smallmatrix}\right)$ is a homomorphism. Now assume that A comes from a geometric object, for example a smooth affine algebraic variety X. Let $\mathfrak{m} \subset A$ be a maximal ideal and let x be the point corresponding to $\mathfrak{m}$. The Zariski cotangent space to X at x is $\mathfrak{m}/\mathfrak{m}^2$. One views this as a vector space over the residue field $A/\mathfrak{m}$. Now consider the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow A/\mathfrak{m} \longrightarrow 0\,.$$

Apply $\mathrm{Hom}_A(-, A\mathfrak{m})$ to the sequence to get a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(A/\mathfrak{m}, A/\mathfrak{m}) \longrightarrow A/\mathfrak{m} \longrightarrow \mathrm{Hom}_A(\mathfrak{m}, A/\mathfrak{m}) \longrightarrow \mathrm{Ext}^1(A/\mathfrak{m}, A/\mathfrak{m}) \longrightarrow 0\,.$$

The last zero is since A is a free module over itself so one can only extend A trivially. Now $A?\mathfrak{m}$ is a field, so the first term is just $A.\mathfrak{m}$. Therefore, the first non-trivial arrow is an isomorphism and we get that $\mathrm{Hom}_A(\mathfrak{m}, A/\mathfrak{m}) \cong \mathrm{Ext}^1(A/\mathfrak{m}, A/\mathfrak{m})$. However, every homomorphism of A-modules from $\mathfrak{m}$ to $A/\mathfrak{m}$ factors through $\mathfrak{m}/\mathfrak{m}^2$ and thus it tells you that $\mathrm{Ext}^1(A/\mathfrak{m}.A/\mathfrak{m})$ is the tangent space at x. So the upper triangular matrices with the same entry on the diagonal know the derivations.

Lemma 2.11. *Let $\Omega \subset \mathbb{M}(V)$ be an nc domain, and $f, g\colon \Omega \to \mathbb{M}(W)$ be two nc functions*

(1) For every $X \in \Omega(n)$, $Y \in \Omega(m)$ the map $\Delta f(X,Y)\colon M_{n,m}(V) \to M_{n,m}(W)$ is a linear map.

(2) For every $\alpha, \beta \in \mathbb{C}$ we have $\Delta(\alpha f + \beta g)(X,Y)(Z) = \alpha\Delta f(X,Y)(Z) + \beta\Delta g(X,Y)(Z)$.

(3) If $\varphi\colon V \to W$ is a linear map, then the collection $\{\varphi \otimes I_{M_n}\}_{n\in\mathbb{N}}$ is an nc function and we have that $\Delta\varphi(X,Y)(Z) = (\varphi \otimes I_{M_{n,m}})(Z)$.
(4) If W is an algebra we can take pointwise products of nc functions and in this case

$$\Delta(fg)(X,Y)(Z) = f(X)\Delta g(X,Y)(Z) + \Delta f(X,Y)(Z)g(Y).$$

Proof. To prove the first item, note first that for every Z, there exists $r > 0$, such that $\left(\begin{smallmatrix} X & rZ \\ 0 & Y \end{smallmatrix}\right) \in \Omega_{n+m}$ since Ω_{n+m} is open. Now set $\Delta f(X,Y)(Z) = \frac{1}{r}\Delta f(X,Y)(rZ)$ Now one checks that this is well-defined and homogeneous see [18, Proposition 2.4]. Additivity is proved in [18, Proposition 2.6]. The rest is trivial. $\square$

Proposition 2.12. *Let $\Omega_i \subset \mathbb{M}(V_i)$, for $i = 1,2$ be nc domains. Let $f\colon \Omega_1 \to \Omega_2$ and $g\colon \Omega_2 \to \mathbb{M}(W)$ be nc functions, then for every choice of our usual suspects X ,Y and Z*

$$\Delta(g\circ f)(X,Y)(Z) = \Delta g(f(X),f(Y))\left(\Delta f(X,Y)(Z)\right).$$

Proof.

$$\begin{pmatrix} g(f(X)) & \Delta(g\circ f)(X,Y)(Z) \\ 0 & g(f(Y)) \end{pmatrix} = g\left(f\left(\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}\right)\right) =$$

$$g\left(\begin{pmatrix} f(X) & \Delta f(X,Y)(Z) \\ 0 & f(Y) \end{pmatrix}\right) = \begin{pmatrix} g(f(X)) & \Delta g(f(X),f(Y))\left(\Delta f(X,Y)(Z)\right) \\ 0 & g(f(Y)) \end{pmatrix}.$$

$\square$

Corollary 2.13. *Let $f\colon \Omega \to \mathbb{M}(W)$ be an nc function and let W be a Banach algebra. Write $\mathbf{GL}_n(W)$ for the units of $M_n(W)$. Then*

- *The set $\mathbf{GL}(W) = \sqcup_{n=1}^{\infty}\mathbf{GL}_n(W)$ is an nc domain and $\iota(X) = X^{-1}$ is an nc map on $\mathbf{GL}(W)$.*
- *Let $\Omega^{\times} = \{X \in \Omega \mid f(X) \in \mathbf{GL}(W)\}$. This is an nc set and we have that*

$$\Delta f^{-1}(X,Y)(Z) = -f(X)^{-1}\Delta f(X,Y)(Z)f(Y)^{-1}.$$

Proof. Observe that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}.$$

Thus $\Delta\iota(A,B)(C) = -A^{-1}CB^{-1}$ and by the chain rule we have that

$$\Delta f^{-1}(X,Y)(Z) = -f(X)^{-1}\Delta f(X,Y)(Z)f(Y)^{-1}.$$

$\square$

Now for the difference-differential formula that gives this operator its name

Proposition 2.14. *Let $\Omega \subset \mathbb{M}(V)$ be an nc domain, $f\colon \Omega \to \mathbb{M}(W)$ an nc function, $X \in \Omega(n)$, $Y \in \Omega(m)$ and $S \in M_{n,m}(\mathbb{C})$. That is*

$$Sf(X) - f(Y)S = \Delta f(X,Y)(XS - SY).$$

Proof. Let $T = \left(\begin{smallmatrix} I_n & S \\ 0 & I_m \end{smallmatrix}\right)$, then

$$T^{-1}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}T = \begin{pmatrix} X & XS - SY \\ 0 & Y \end{pmatrix}$$

Hence

$$\begin{pmatrix} f(X) & \Delta f(X,Y)(XS-SY) \\ 0 & f(Y) \end{pmatrix} =$$

$$f(T^{-1}(X \oplus Y)T) = T^{-1}(f(X) \oplus f(Y))T = \begin{pmatrix} f(X) & f(X)S - Sf(Y) \\ 0 & f(Y) \end{pmatrix}.$$

□

The following corollary appears as part of the proof of [1, Theorem 4.10], which as the authors themselves point out is a modification of [14, Proposition 2.5]. Agler and McCarthy prove in [1, Theorem 4.10] that local boundedness implies continuity in the case when $V = \mathbb{C}^d$ and $W = B(\mathcal{H}, \mathcal{K})$ and the proof is neat. Since we are going to discuss Tay-lo-Taylor series, we will do it in a bit more technical way in the next section.

Corollary 2.15. *Let V and W be Banach spaces. Let $\Omega \subset \mathbb{M}(V)$ be an nc domain and $f \colon \Omega \to \mathbb{M}(W)$ be a du-continuous nc function, then f is Gateaux differentiable. The Gateaux derivative of f at X is $\Delta f(X,X)$. In particular, if $V = \mathbb{C}^d$ and $W = \mathbb{C}^e$, then f is analytic as a function in the coordinates on every level.*

Proof. Let $X \in \Omega(n)$ and $Z \in M_n(V)$. We want to show that $\lim_{t\to 0} \frac{1}{t}\left(f(X+tZ) - f(X)\right)$ exists. Since $\Omega(n)$ is open, for every Z of small enough norm $\left(\begin{smallmatrix} X+tZ & Z \\ 0 & X \end{smallmatrix}\right) \in \Omega_{2n}$, for $t \in \mathbb{D}$. Now note that by the difference-differential formula with $S = I$ we have that for every $s \in \mathbb{D}$

$$f(X+tZ) - f(X) = \Delta f(X+tZ, X)(tZ) = t\Delta f(X+tZ,X)(Z).$$

Therefore, $\lim_{t\to 0} \frac{1}{t}\left(f(X+tZ) - f(X)\right) = \lim_{t\to 0} \Delta f(X+tZ,X)(Z)$, but the the right hand limit exists by the continuity of f. Indeed,

$$\begin{pmatrix} f(X) & \Delta f(X,X)(Z) \\ 0 & f(X) \end{pmatrix} = \lim_{t\to 0} f\left(\begin{pmatrix} X+tZ & Z \\ 0 & X \end{pmatrix}\right) =$$

$$\lim_{t\to 0} \begin{pmatrix} f(X+tZ) & \Delta f(X+tZ,X)(Z) \\ 0 & f(X) \end{pmatrix}.$$

We see from this that $\lim_{t\to 0} \frac{1}{t}\left(f(X+tZ) - f(X)\right) = \Delta f(X,X)(Z)$. The last claim follows from Hartogs' theorem. □

It only remains to understand what kind of map is $(X,Y) \mapsto \Delta f(X,Y)$. Note that

Proposition 2.16 ([18, Proposition 2.15, 2.17])**.** *Let $\Omega \subset \mathbb{M}(V)$ be an nc domain and $f \colon \Omega \to \mathbb{M}(W)$ an nc function.*

(1) Let $X \in \Omega(n)$, $Y \in \Omega(m)$, $Z \in M_{n,m}(V)$, $S \in \mathbf{GL}_n(\mathbb{C})$ and $T \in \mathbf{GL}_m(\mathbb{C})$. Then

$$\Delta f(S^{-1}XS, T^{-1}XT)(S^{-1}ZT) = S^{-1}\Delta f(X,Y)(Z)T.$$

(2) Let $X_i \in \Omega(n_i)$, $Y_i \in \Omega(m_i)$ and $Z_{ij} \in M_{n_i,m_j}(V)$, fro $i,j = 1,2$. Then

(a)

$$\Delta f(X_1 \oplus X_2, Y_1)\left(\begin{pmatrix} Z_{11} \\ Z_{21} \end{pmatrix}\right) = \begin{pmatrix} \Delta f(X_1,Y_1)(Z_{11}) \\ \Delta f(X_2,Y_1)(Z_{21}) \end{pmatrix}.$$

(b)

$$\Delta f(X_1, Y_1 \oplus Y_2)\left(\begin{pmatrix} Z_{11} & Z_{12} \end{pmatrix}\right) = \begin{pmatrix} \Delta f(X_1,Y_1)(Z_{11}) & \Delta f(X_1,Y_2)(Z_{12}) \end{pmatrix}.$$

(c)
$$\Delta f(X_1 \oplus X_2, Y_1 \oplus Y_2)\left(\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\right) = \begin{pmatrix} \Delta f(X_1, Y_1)(Z_{11}) & \Delta f(X_1, Y_2)(Z_{12}) \\ \Delta f(X_2, Y_1)(Z_{21}) & \Delta f(X_2, Y_2)(Z_{22}) \end{pmatrix}.$$

Proof. To prove the first item note that
$$\begin{pmatrix} S^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} S^{-1}XS & S^{-1}ZT \\ 0 & T^{-1}YT \end{pmatrix}.$$
Set $Q = S \oplus T$, then we have that
$$\begin{pmatrix} S^{-1}f(X)S & S^{-1}\Delta f(X,Y)(Z)T \\ 0 & T^{-1}f(Y)T \end{pmatrix} = Q^{-1} f\left(\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}\right) Q =$$
$$f\left(Q^{-1}\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} Q\right) = \begin{pmatrix} S^{-1}f(X)S & \Delta f(S^{-1}XS, T^{-1}YT)(S^{-1}ZT) \\ 0 & T^{-1}f(Y)T \end{pmatrix}.$$
(c) of the second item follows from (a) and (b). To prove (a) one notes that
$$\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} X_1 & 0 & Z_{11} \\ 0 & X_2 & Z_{12} \\ 0 & 0 & Y_1 \end{pmatrix} = \begin{pmatrix} X_1 & Z_{11} & 0 & 0 \\ 0 & Y_1 & 0 & 0 \\ 0 & 0 & X_2 & Z_{12} \\ 0 & 0 & 0 & Y_1 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$
Now one uses the fact that f respects intertwinners. $\square$

This proposition describes Δf as an nc function of order one. In the next section, we will meet higher order nc functions and discuss the Taylor-Taylor expansion.

3. Higher-order nc-functions, analyticity and Taylor-Taylor series

The treatment in this section is going to be a bit more sketchy. This is due to the fact that this is quite technical and there is a very good source, namely [18, Chapter 3]. We have sen in the previous section that $\Delta f(X, Y)$ can be viewed as an nc function of two variables. The value of this function at $X \in \Omega(n)$ and $Y \in \Omega(m)$ is a linear map from $M_{n,m}(V) \to M_{n,m}(W)$. Let us consider a lightly bigger upper triangular matrix. Let $X_i \in \Omega(n_i)$, for $i = 1, 2, 3$, $Z_{12} \in M_{n_1,n_2}(V)$ and $Z_{23} \in M_{n_2,n_3}(V)$. Consider the matrix
$$P = \begin{pmatrix} X_1 & Z_{12} & 0 \\ 0 & X_2 & Z_{23} \\ 0 & 0 & X_3 \end{pmatrix}.$$
If $f \colon \Omega \to \mathbb{M}(W)$ is an nc function, then we have that
$$f(P) = \begin{pmatrix} f\left(\begin{pmatrix} X_1 & Z_{12} \\ 0 & X_2 \end{pmatrix}\right) & \Delta f\left(\begin{pmatrix} X_1 & Z_{12} \\ 0 & X_2 \end{pmatrix}, X_3\right)\left(\begin{pmatrix} 0 \\ Z_{23} \end{pmatrix}\right) \\ 0 & f(X_3) \end{pmatrix}.$$
We know what the upper-left corner is. We can also decompose the matrix differently to get the lower left corner. The bottom line is that
$$f(P) = \begin{pmatrix} f(X_1) & \Delta f(X_1, X_2)(Z_{12}) & ? \\ 0 & f(X_2) & \Delta f(X_2, X_3)(Z_{23}) \\ 0 & 0 & f(X_3) \end{pmatrix}.$$

What is the question mark? The second derivative, namely $\Delta^2 f(X_1, X_2, X_3)(Z_{12}, Z_{23})$. Also note that we have got

$$\Delta f\left(\begin{pmatrix} X_1 & Z_{12} \\ 0 & X_2 \end{pmatrix}, X_3\right)\left(\begin{pmatrix} 0 \\ Z_{23} \end{pmatrix}\right) = \begin{pmatrix} \Delta^2 f(X_1, X_2, X_3)(Z_{12}, Z_{23}) \\ \Delta f(X_2, X_3)(Z_{23}) \end{pmatrix}$$

$$\Delta f\left(X_1, \begin{pmatrix} X_2 & Z_{23} \\ 0 & X_3 \end{pmatrix}\right)\left(\begin{pmatrix} Z_{12} & 0 \end{pmatrix}\right) = \begin{pmatrix} \Delta f(X_1, X_2)(Z_{12}) & \Delta^2 f(X_1, X_2, X_3)(Z_{12}, Z_{23}) \end{pmatrix}.$$

This formula is analogous to the commutation of derivatives in different directions.

One can show that in fact $\Delta^2 f(X_1, X_2, X_3)$ is a multilinear map of two arguments. In other words it is a map from $M_{n_1,n_2}(V) \otimes M_{n_2,n_3} \to M_{n_1,n_3}(W)$. I ignore the question "what tensor?" on purpose ,since if $V = \mathbb{C}^d$ they are all the same. The content of the following definition is [18, Proposition 3.1] and the original definition of an nc function of order k is in the beginning of [18, Section 3.1].

Definition 3.1. Let $\Omega_0, \ldots, \Omega_k \subset \mathbb{M}(V)$ be nc domains. A function f on $\Omega_0 \times \cdots \times \Omega_k$ is called a graded function of order k with values in W, if for every $k+1$ tuple of $X_i \in \Omega_i(n_i)$, for $j = 0, \ldots, k$, $f(X_0, \ldots, X_k)\colon M_{n_0,n_1}(V) \otimes \cdots \otimes M_{n_{k-1},n_k}(V) \to M_{n_0,n_k}(W)$ is a linear map. The function F is called an nc function of order k, if it respects intertwiners. That is, if $Z_j \in M_{n_{j-1},n_j}(V)$, for $j = 1, \ldots, k$ and $T \in M_{m,n_0}$ is such that $TX_0 = X_0'T$, for some $X_0' \in \Omega_0(m)$, then

$$Tf(X_0, \ldots, X_k)(Z_1, \ldots, Z_k) = f(X_0', X_1, \ldots, X_k)(TZ_1, \ldots, Z_k).$$

If $T \in M_{m,n_j}(\mathbb{C})$ is such that $TX_j = X_j'T$, then for every $Z_j' \in M_{n_{j-1},m}(V)$

$$\begin{aligned} f(X_0, \ldots, X_k)(Z_1, \ldots, Z_j'T, Z_{j+1}, \ldots, Z_k) = \\ f(X_0, \ldots, X_{j-1}, X_j', X_{j+1}, \ldots, X_k)(Z_1, \ldots, Z_j', TZ_{j+1}, \ldots, Z_k). \end{aligned}$$

The difference-differential operator can be extended to nc function of higher order. It then sends functions of order k to functions of order $k+1$. The idea of the proof is a computation similar to the one we have performed to get Δ^2. In particular, for every nc function we get $\Delta^k f$ — an nc function of order k. We get

Theorem 3.2 ([18, Theorem 3.11]). *Let $f\colon \Omega \to \mathbb{M}(W)$ be an nc function and let $k \in \mathbb{N}$, $X_i \in \Omega(n_i)$ for $i = 0, \ldots, k$ and $Z_j \in M_{n_{j-1},n_j}$ for $j = 1, \ldots, k$, such that*

$$P = \begin{pmatrix} X_0 & Z_1 & 0 & \cdots & 0 \\ 0 & X_1 & Z_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & X_{k-1} & Z_k \\ 0 & \cdots & \cdots & 0 & X_k \end{pmatrix} \in \Omega.$$

Then

$$f(P) = \begin{pmatrix} f(X_0) & \Delta f(X_0, X_1)(Z_1) & \Delta^2 f(X_0, X_1, X_2)(Z_1, Z_2) & \cdots & \Delta^k f(X_0, \ldots, X_k)(Z_1, \ldots, Z_k) \\ 0 & f(X_1) & \Delta f(X_1, X_2)(Z_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & f(X_{k-1}) & \Delta f(X_{k-1}, X_k)(Z_k) \\ 0 & \cdots & \cdots & 0 & f(X_k) \end{pmatrix}$$

Then we have the higher-order difference differential formula.

Theorem 3.3 ([18, Theorem 3.19]). *If f is an nc function of order k on $\Omega_0 \times \cdots \Omega_k$. Then we have that*

$$f(X_0, \ldots, X_{k-1}, X)(Z_1, \ldots, Z_k) - f(X_0, \ldots, X_{k-1}, Y)(Z_1, \ldots, Z_k) = \Delta f(X_0, \ldots, X_{k-1}, Y, X)(Z_1, \ldots, Z_k, X - Y).$$

In particular, if f is the k-th derivative of our favorite nc function, then we get

Corollary 3.4. *Let $f\colon \Omega \to \mathbb{M}(W)$ be an nc function, let $X, Y \in \Omega(n)$ and $Z_1, \ldots, Z_k \in M_n(V)$, then*

$$\Delta^k f(Y, \ldots, Y, X)(Z_1, \ldots, Z_k) - \Delta^k f(Y, \ldots, Y, Y)(Z_1, \ldots, Z_k) = \Delta^{k+1} f(Y, \ldots, Y, X)(Z_1, \ldots, Z_k, X - Y).$$

Now we can use that to get the noncommutative version of Taylor polynomial of a function. Let $X, Y \in \Omega(n)$ and recall that from the first-order difference-differential formula we get

$$f(X) = f(Y) + \Delta f(Y, X)(X - Y).$$

Now we can apply Corollary 3.4 repeatedly to obtain

$$f(X) = f(Y) + \Delta f(Y, Y)(X - Y) + \Delta^2 f(Y, Y, X)(X - Y, X - Y) =$$
$$f(Y)+\Delta f(Y,Y)(X-Y)+\Delta^2 f(Y,Y,Y)(X-Y,X-Y)+\Delta^3 f(Y,Y,Y,X)(X-Y,X-Y,X-Y) = \cdots$$

We conclude that

Theorem 3.5 ([18, Theorem 4.1]). *Let $f\colon \Omega \to \mathbb{M}(W)$ be an nc function and let $X, Y \in \Omega(n)$, then for every $k \in \mathbb{N}$ we have*

$$f(X) = f(Y)+\sum_{r=1}^{n} \Delta^r f(Y, \ldots, Y)(X-Y, \ldots, X-Y)+\Delta^{k+1} f(Y, \ldots, Y, X)(X-Y, \ldots, X-Y).$$

Now we can get the automatic analyticity result. Let us simplify the notations a bit first. For $k \in \mathbb{N}$, let J_k be the Jordan block of order k, namely

$$J_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in M_k.$$

FOr $Z \in M_n(V)$ let us write $J_{k,Z} = J_k \otimes Z$.

Theorem 3.6 ([18, Theorem 7.2]). *Let $\Omega \subset \mathbb{M}(V)$ be an nc domain and let $f\colon \Omega \to \mathbb{M}(W)$ be du locally bounded, then f is Gateaux differentiable and the Gateaux derivative at $X \in \Omega(n)$ is $\Delta f(X, X)$. Furthermore, for every $Z \in M_n(V)$ and every $k \in \mathbb{N}$ we have*

$$\frac{1}{n!} \frac{d^k}{dt^k} f(X + tZ) = \Delta^k f(X, \ldots, X)(Z, \ldots, Z).$$

Proof. Fix $k \in \mathbb{N}$ and let $X \in \Omega(n)$ and $Z \in M_n(V)$, since Ω is an nc domain we can find $r > 0$, such that $P = X^{\oplus k} + rJ_{k,Z} \in \Omega$. Now we can find $\epsilon > 0$, such that for every $|t| < \epsilon$ the matrix $P_t = X^{\oplus k} + tE_{kk} \otimes Z + rJ_{k,Z} \in \Omega$. Shrinking ϵ further we may assume that f is bounded on a neighbourhood of $X^{\oplus k} + rJ_{k,Z}$ that contains the P_t for all $|t| < \epsilon$. Note that

$$\Delta^k f(X, \ldots, X, X + tZ)(Z, \ldots, Z) = \frac{1}{r^k} \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix} f(P_t) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I \end{pmatrix}.$$

Since we have an operator space on our hands $\Delta^k f(X, \ldots, X, X+tZ)(Z, \ldots, Z)$ is bounded for $|t| < \epsilon$. Now apply Theorem 3.5 to get

$$f(X + tZ) = f(X) + \sum_{j=1}^{k-1} t^j \Delta^j f(X, \ldots, X)(Z, \ldots, Z) + \\ + t^k \Delta^k f(X, \ldots, X, X + tZ)(Z, \ldots, Z).$$

Since the remainder is bounded by $C|t|^k$ we get that f is $k - 1$ times differntiable in the direction of Z and the derivatives have the formula we desired.

$\square$

It is a fact that in our case the function F is also Frechet differentiable, i.e,

$$\lim_{\|Z\| \to 0} \frac{\|f(X + Z) - f(X) - \Delta f(X, X)(Z)\|}{\|Z\|} = 0.$$

Furtheremore, one can pass from the TT "polynomial" with remainder to TT series, namely if $Y \in \Omega(n)$ and let $r > 0$ be such that f is bounded on the ball around Y with radius r, then for every $\|X - Y\| < r$ we have

$$f(X) = \sum_{k=0}^{\infty} \Delta^k f(Y, \ldots, Y)(X - Y, \ldots, X - Y).$$

For the proof see [18, Theorem 7.4 and Corollary 7.5]. If $X \in \Omega(mn)$ close enough to $Y^{\oplus m}$, then again

$$f(X) = \sum_{k=0}^{\infty} \Delta^k f(Y^{\oplus m}, \ldots, Y^{\oplus m})(X - Y^{\oplus m}, \ldots, X - Y^{\oplus m}).$$

In particular, if f is locally bounded in the uniform topology one can find $r > 0$ and the series will converge uniformly and absolutely on compacta in $\mathfrak{B}_r(Y)$. Recall that $\Delta f(Y^{\oplus m}, Y^{\oplus m})$ is a linear map $M_{nm}(V) \to M_{nm}(W)$ and by the properties of an nc map of order 1, we have that $\Delta f(Y^{\oplus m}, Y^{\oplus m}) = I_{M_m} \otimes \Delta f(Y, Y)$. Similarly one can simlify higher order nc functions to get that the TT series have the form

$$f(X) = \sum_{k=0}^{\infty} \Delta^k f(Y, \ldots, Y) \left((X - Y^{\oplus m})^{\odot_n k} \right)$$

Here $\odot_n k$ stands for the product of $M_{nm}(V)$ matrices as $m \times m$-matrices over the tensor algebra of $M_n(V)$ and we apply $\Delta^k f(Y, \ldots, Y)$ to each coordinate.

Remark 3.7. The ultimate reference on Taylor-Taylor expansions and higher-order nc functions is the book [18]. The original ideas, however, are in [36].

4. Polynomials and rational functions

Classically, of all analytic function, the polynomials and rational functions are the simplest in many ways. In this section, we will discuss the noncommutative analogs of these two basic classes. Let us start with polynomials, namely elements of $\mathbb{C}\langle z_1, \ldots, z_d \rangle$. It is rather straightforward that for a polynomial p and every $n \in \mathbb{N}$ the restriction of p to $\mathbb{M}_d(n)$ is a matrix polynomial in the coordinates on $\mathbb{M}_d(n)$. It is also clear that since this function is $\mathbf{GL}_n(\mathbb{C})$-equivariant, not every matrix polynomial in the coordinates is a restriction of an nc polynomial. In fact, equivariance is not enough. Equivariant matrix polynomials were studied by Procesi and many others. The algebra of all matrix polynomials on $\mathbb{M}_d(n)$ that are $\mathbf{GL}_n(\mathbb{C})$-equivariant is called the algebra of matrix concomitants of size n. We shall denote this algebra by $\mathfrak{C}_{d,n}$.

To understand this algebra we need some definitions. Let us fix $n \in \mathbb{N}$ and set $z_{ij}^{(k)}$, for $1 \leq i, j \leq n$ and $1 \leq k \leq d$, the coordinates on $\mathbb{M}_d(n)$. For every $k = 1, \ldots, n$, let $X_k = \left(z_{ij}^{(k)}\right)_{i,j=1}^{n}$. We call the X_k the generic matrices. For every word $\alpha \in \mathbb{W}_d$, the free monoid on d letters, we write X^α for the monomial in the generic matrices. Finally, let $\mathfrak{T}_{d,n} \subset \mathbb{C}[z_{ij}^{(k)}]$ be the subalgebra generated by $\operatorname{tr}(X^\alpha)$, where α runs over $\mathbb{W}_d$.

Theorem 4.1 ([26]). *We have that $\mathfrak{T}_{d,n} = \mathbb{C}[z_{ij}^{(k)}]^{\mathbf{GL}_n(\mathbb{C})}$, i.e, $\mathfrak{T}_n$ is precisely the algebra of similarity invariant polynomials on $\mathbb{M}_d(n)$. Furthermore, the center of $\mathcal{C}_{d,n}$ is $\mathfrak{T}_{d,n}$ and $\mathcal{C}_{d,n}$ is generated over $\mathfrak{T}_{d,n}$ by finitely many monomials X^α.*

Remark 4.2. Since $\mathbf{GL}_n(\mathbb{C})$ is reductive, by Hilbert's theorem the algebra $\mathfrak{T}_{d,n}$ is Noetherian. In fact, it corresponds to the categorical quotient of $\mathbb{M}_d(n)$ by similarities.

Definitely our nc polynomials are inside $\mathcal{C}_{d,n}$. How can we tell them apart? In fact, the image of the free algebra under the restriction is the subalgebra generated by X^α without the traces. This is the so-called algebra of generic matrices. It was studied in relation with polynomial identities. The following theorem describes nc polynomials in another way.

Theorem 4.3 ([18, Theorem 6.1] and [19]). *Let $f \colon \mathbb{M}_d \to \mathbb{M}_1$ be an nc function and assume that on every level f is given by a matrix concomitant (in particular, for every $n \in \mathbb{N}$, $f|_{\mathbb{M}_d(n)}$ is a matrix polynomial in the coordinates). Then f is an nc polynomial if and only the degree of the matrix polynomials $f|_{\mathbb{M}_d(n)}$ is uniformly bounded.*

Proof. If f is an nc polynomial, there is nothing to prove. So the question is how to prove the converse direction. This result was first proved by Kaliuzhnyi-Verbovetskyi and Vinnikov in [18] using Taylor-Taylor series. Recently, it was given a new proof by Klep and Špenko using matrix concomitants. I will sketch both of them.

First proof: Let us denote the bound on the degree by m. Since we obtain the difference-differential operators by applying f to upper-triangular matrices, we get that $\Delta^k f(Y_0, \ldots, Y_k)(Z_1, \ldots, Z_k)$ is a polynomial in the coordinates of $Y_0, \ldots, Y_k, Z_1, \ldots, Z_k$. In particular, if we apply the

Taylor-Taylor formula, we get

$$f(Y+tZ)=\sum_{r=0}^{k-1} t^r \Delta^r f(Y,\ldots,Y)(Z,\ldots,Z)+t^k\Delta^k f(Y,\ldots,Y,Y+tZ)(Z,\ldots,Z).$$

For fixed Y and Z this is a polynomial in t of degree at most m, hence the remainder vanishes for $k>m$. Now if we write the Taylor-Taylor expansion around some Y we have

$$f(X)=\sum_{r=0}^{m}\Delta^r f(Y,\ldots,Y)(X-Y,\ldots,X-Y).$$

This is, however, independent of the size of Y and we can take Y to be zero and obtain immediately that f is an nc polynomial.

Second Proof: SInce f is nc and its restriction to each level is a matrix polynomial in the coordinates, we have that $f|_{\mathbb{M}_d(n)}$ is a matrix concommitant. Therefore, using the fact that trace is linear, we can write

$$f|_{\mathbb{M}_d(n)}=\sum_{|\alpha|\leq m} h_{n,\alpha}(X)X^{\alpha}.$$

Here the $h_{n,\alpha}$ are some polynomials in traces of monomials, but we remember the n, since these polynomials might change from level to level. We also require that $\deg \operatorname{tr}(h_{n,\alpha})+|\alpha|\leq m$. Let us take $n\geq m+1$. Since f is nc we have that for every $X,Y\in\mathbb{M}_d(n)$

$$\sum_{|\alpha|\leq m}\operatorname{tr}(h_{2n,\alpha}(X\oplus Y))X^{\alpha}\oplus Y^{\alpha}=f(X\oplus Y)=f(X)\oplus f(Y)=$$

$$=\left(\sum_{|\alpha|\leq m}\operatorname{tr}(h_{n,\alpha}(X))X^{\alpha}\right)\oplus\left(\sum_{|\alpha|\leq m}\operatorname{tr}(h_{n,\alpha}(Y))Y^{\alpha}\right).$$

Comparing coefficients we see that

$$\operatorname{tr}(h_{2n,\alpha}(X\oplus Y))=\operatorname{tr}(h_{n,\alpha}(X))=\operatorname{tr}(h_{n,\alpha}(Y)).$$

However, Procesi in [26] proved that $\mathbb{M}_n$ does not satisfy any notrivial trace identity of degree less than n. Hence these are constants and we are done. □

Remark 4.4. The advantage of the second proof is that it allows us to consider other groups instead of $\mathbf{G}\mathrm{L}_n(\mathbb{C})$. For example in the real setting with transposes of the variables allowed, we can consider O_n and the result still holds.

The next step after considering polynomials is considering rational functions. Classically, we can always localize a commutative domain (a ring with no zero divisors) at every multiplicative subset and in particular, the complement of the zero ideal. The result is the field of fractions of our domain. In the noncommutative setting things are far from nice. If a noncommutative domain satisfies the Ore conditions, then one can construct a localization and it will have similar properties to the classical one. However, the free algebra does not. What can be done? Where can we get rational nc functions? P. M. Cohn (see for example [9] and [10]) realized that localizing at elements alone is not enough and one must localize at matrices. An $n\times n$ matrix T is called full, if one cannot write as $T=PQ$, where P and Q are $n\times k$ and $k\times n$ matrices, respectively, where $k<n$. Then

one localizes (in some sense) at all of the full matrices to obtain the free skew field. One should also consult [29] for a more modern treatment.

However, this is not a point of view I'd like to take. I would like a more function-theoretic way of looking at things. In algebraic geometry, localizing at a single element is encoded by considering the algebraic functions on the Zariski open set — the complement of the zeroes of f. So one can consider the fraction field as the algebraic functions with some domain of definition. The approach of Amitsur [4] is to consider all possible syntactically correct words one can write with the variables of our free algebra, scalars, $+$, $\cdot$ and $^{-1}$. For example $((z_1 - z_2)^{-1} + z_2 z_3 z_2)^{-1} - 2$ or $(z_1 z_2 - z_2 z_1)^{-1}$ or 0^{-1}. Anything goes! Now for each expression, we try to evaluate it on elements of some rings. If one can find a d-tuple of elements of some ring R, such that our expression can be evaluated on, we add this d-tuple to its domain. Now clearly 0^{-1} is a silly expression and its domain is empty. We now throw away all expressions with empty domains. On the remaining we define an equivalence relation, by $r_1 \sim r_2$ if $\operatorname{dom}(r_1) \cap \operatorname{dom}(r_2) \neq 0$ and they are equal as functions on any d-tuple in the intersection. A noncommutative rational function is an equivalence class. Now one has to work hard to prove that the resulting structure is a skew field (a.k.a. division ring). This skew-field is the free skew field.

There are several questions that arise instantly. First and foremost what rings do we take? It is proved in [17] that one can consider only matrix rings of all sizes. From now one when discussing the domain of a rational function we will restrict ourselves to the domain in $\mathbb{M}_d$ and denote this particular nc set by $\operatorname{dom}(r)$. The second question is what is "the" about the skew field that we have constructed? There is no uniqueness for the ring of fractions and one can construct embeddings of the free algebra into various skew-fields that are not embeddable one into the other. The correct notion is the notion of the *universal skew-field of fraction*. We say that a skew-field K is the universal skew field of fractions of a ring R, if for every skew-field L and a homomorphism $\varphi \colon R \to L$, there exists a subring $R \subset K_0 \subset K$, and a homomorphism $\widetilde{\varphi} \colon K_0 \to L$ extending φ, such that for every $0 \neq x \in K_0$, we have that $x^{-1} \in K_0$ if and only if $\widetilde{\varphi}(x) \neq 0$. The homomorphism $\widetilde{\varphi}$ is unique in an appropriate sense. The universal skew-field of fractions is unique up to a unique isomorphism. The notation for the free skew-field is $\mathbb{C}\langle\!\!\langle z_1, \ldots, z_d \rangle\!\!\rangle$.

Remark 4.5. The notion of noncommutative rational functions came first to the world of automata in [30] and [31]. For example, a language is accepted by a weighted automaton if and only if the characteristic series of the language represents a rational function (see [8].

The nicest part about noncommutative rational functions is that you can simplify the complicated expressions by passing to matrices. Let me make this more precise in the case when the domain of our rational function contains the origin.

Theorem 4.6 ([8, Theorem 2.4], [39, Theorem 3.10], [16, Theorem 3.1])**.** *Let r be a rational expression with the origin in its domain. Then this function admits a realization, namely there exists two vectors $\xi, \eta \in \mathbb{C}^n$ and matrices $A_1, \ldots, A_d \in M_n$, such that*

$$r(z_1, \ldots, z_d) = r(0) + \xi^T \left(I - \sum_{j=1}^{n} z_j A_j \right)^{-1} \eta.$$

Furthermore, there exists a unique minimal (in terms of n) realization of r and if $L = I - \sum_{j=1}^{n} z_j A_j$ is the pencil of the minimal realization, then

$$\mathrm{dom}(r) = \{X \in \mathbb{M}_d \mid \det L(X) \neq 0\}.$$

Here for $X \in \mathbb{M}_d(k)$ we have that $L(X) = I_n \otimes I_k - \sum_{j=1}^{n} A_j \otimes X_j$.

This is basically, writing the Schur complement of the matrix $\begin{pmatrix} r(0) & \xi^T \\ \eta & -L \end{pmatrix}$. Realizations and in particular, unique minimal realizations exist for every rational function and can be constructed using [15, Algorithm 4.3].

Recall that an algebra A is called stably finite if for every matrix $T \in M_n(A)$, one-sided invertibility implies invertibility. In particular, every C^*-algebra with a faithful trace is stably finite (see [15, Lemma 5.1]). We have the following complementary answer to our first question. Let us write, for an algebra A and a rational expression r, $\mathrm{dom}_A(r)$ for the domain of r in A.

Theorem 4.7 ([15]). *The following are equivalent for an algebra A.*

(1) A is stably finite.

(2) for every two equivalent rational expressions r_1 and r_2 and every $X \in \mathrm{dom}_A(r_1) \cap \mathrm{dom}_A(r_2)$, $r_1(X) = r_2(X)$.

In fact, one can also work over the real numbers and consider self-adjoint rational expressions and then one can find a realization that is self-adjoint, namely $r = r(0) + \xi^* L^{-1}\xi$ and coefficients of L are self-adjoint. This is the content of [15, Theorem 4.9].

Now let us be more specific. Let $(A.\tau)$ be a C^*-probability space, namely A is a C^* and τ is afaithful tracial sate. One views τ as an expectations and the elements of A as free random variables. The noncommutative joint distribution of $a_1, \ldots, a_d$ is the linear functional on $\mathbb{C}\langle z_1, \ldots, z_d\rangle$ defiend by $p \mapsto \tau(p(a_1, \ldots, a_d)$. In particualr, if $a = a^*$ is self-adjoint one can find a probability measure μ_a on the real line defined by $\tau(a^k) = \int_{-\infty}^{\infty} t^k d\mu_a$. This measure knows the moments of a and thus knows the distribution. One define the Cauchy transform of a probability measure on the real line

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t} d\mu(t).$$

This is an analytic function from the upper-half plane ($\mathbb{C}_+$) to the lower half-plane ($\mathbb{C}_-$). In particular, we have that $G_{\mu_a}(z) = \tau((z-a)^{-1})$. The measure can be recovered from the function using the Stieltjes inversion formula. The same game can be played with $B \subset A$ a subalgebra and a conditional expectation $E\colon A \to B$.

Given $a_1, \ldots, a_d \in A_{sa}$ and a polynomial $p \in \mathbb{C}\langle z_1, \ldots, z_d\rangle$, such that $p(a_1, \ldots, a_d)$ is self-adjoint, can we calculate the distribution of the polynomial in the a's using the distributions of the a's? The answer is yes and it involves free convolution. In their paper Helton, Mai and Speicher have given an answer for rational functions as well.

Theorem 4.8 ([15, Theorem 6.10 and Theorem 6.11]). *Let r be a self-adjoint rational expression and $a_1, \ldots, a_d \in A_{sa}$, such that $(a_1, \ldots, a_d) \in \mathrm{dom}_A(r)$. LEt $r(0) + \xi^* L^{-1}\xi$ be the minimal realization for r and set $\Lambda = \begin{pmatrix} r(0) & \xi^* \\ \xi & -L \end{pmatrix}$. Then $r(a_1, \ldots, a_d) = r(0) + \xi^* L(a_1, \ldots, a_d)^{-1}\xi$*

and we have that for every $z \in \mathbb{C}_+$

$$(z1_A - r(a_1, \ldots, a_d) = \begin{pmatrix} 1_A & 0 \end{pmatrix} \left(\begin{pmatrix} z1_A & 0 \\ 0 & 0 \end{pmatrix} - \Lambda(a_1, \ldots, a_d) \right)^{-1} \begin{pmatrix} 1_A \\ 0 \end{pmatrix}.$$

Furtermore,

$$G_{r(a_1,\ldots,a_d)}(z) = \lim \epsilon \to 0+ \begin{pmatrix} 1 & 0 \end{pmatrix} G_{\Lambda(a_1,\ldots,a_d)} \left(\begin{pmatrix} z & 0 \\ 0 & i\epsilon I \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

5. The Fock space and the free semi-group algebra with nc functions

Recall that classically, $H^\infty(\mathbb{D})$ – the algebra bounded analytic functions on the unit disc can be given the structure of a weak-* closed operator algebra, by letting them act as multiplication operators on $H^2(\mathbb{D})$. The latter is the space of all analytic functions on the disc with square summable Taylor coefficients at the origin. That is $H^2(\mathbb{D})$ is the Hilbert space obtained by defining on $\mathbb{C}[z]$ the inner product with orthonormal basis given by the monomials.

Furthermore, $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space (RKHS). That is for every $z \in \mathbb{D}$, the evaluation at z functional is a bounded functional on $H^2(\mathbb{D})$. This is, however, a Hilbert space and thus there exists a vector $k_z \in H^2(D)$, such that evaluation of $f \in H^2(\mathbb{D})$ at z is given by $f(z) = \langle f, k_z \rangle$. The function $k(z,w) = k_z(w) = \langle k_z, k_w \rangle$ is called the reproducing kernel of the RKHS. In the case of $H^2(\mathbb{D})$ we have that $k(z,w) = \frac{1}{1-z\overline{w}}$, the so-called Szego kernel.

One other algebra of note is $A(\mathbb{D})$ – the norm closure of the polynomials in $H^\infty(\mathbb{D})$. Equivalently, this is the algebra of all analytic functions on $|D$ that extend to a continuous function on $\overline{D}$. The celebrated von Neumann inequality tells us that for every $T \in B(\mathcal{H})$, such that $\|T\| \leq 1$ and every polynomial $p \in \mathbb{C}[z]$

$$\|p(T)\| \leq \sup_{z \in \mathbb{D}} |p(z)| = \|p\|_\infty.$$

In particular, this means that this inequality is true if replace p with $f \in A(\mathbb{D})$. This ties into Sz.-Nagy-Foias dilation theory. Every contraction T admits a unitary power dilation, namely if $T \in B(\mathcal{H})$ is a contraction, then there exists a Hilbert space $\mathcal{K}$, a unitary $U \in B(\mathcal{K})$ and an isometry $V \colon \mathcal{H} \to \mathcal{K}$, such that for every $n \in \mathbb{N}$, $T^n = V^* U^n V$. In more modern language we have a commutative diagram

$$\begin{tikzcd}
C(\mathbb{T}) \arrow[r, "z \mapsto U"] & B(\mathcal{K}) \arrow[d, "V^{-}V"] \\
A(\mathbb{D}) \arrow[u] \arrow[r, "z \mapsto T"'] & B(\mathcal{H})
\end{tikzcd}.$$

This picture is a the motivations behind the Stinespring dilation and Arveson's extension theorems.

If we have a d-tuple of commuting contractions this is already difficult, since for $d > 2$ there is no simultaneous commuting unitary dilation of our d-tuple. Examples for this were given by Parrott and Varopolous.

One way to generalize this was to consider rows of operators, namely think of $(T_1, \dots, T_d)$ as an operator from $\mathcal{H}^{\oplus d}$ to $\mathcal{H}$. If this operator is contractive we say that it is a row contraction. Frazho, Bunce, and Popescu constructed isometric dilations of row contractions (not necessarily commuting) and Popescu also has constructed a Cuntz dilation. The algebra that replaces $A(\mathbb{D})$ is the non-commutative disc algebra $\mathcal{A}_d$. Let us define the objects we need.

Definition 5.1. Let $\mathcal{F}_d^2 = \oplus_{k=0}^{\infty} \left(\mathbb{C}^d\right)^{\otimes k}$ be the Fock space and let L_j be the j-th creation operator, namely the operator that send $v_1 \otimes \cdots \otimes v_k$ to $e_j \otimes v_1 \otimes \cdots \otimes v_k$, where e_j is the j-th element of the standard basis. Let us write $\mathcal{A}_d$ for the norm closed operator algebra generated by the L_j and $\mathcal{L}_d$ for the weak-* closed operator algebra generated by them.

Note that $\mathcal{F}_2^d$ is the ℓ^2 space on the free monoid on d letter. Also note that $\begin{pmatrix} L_1 & \cdots & L_d \end{pmatrix}$ is a row isometry and $\sum_{j=1}^{d} L_j L_j^* = I - P_0$, where P_0 is the projection onto the vector associated to the empty word — the vacuum vector. The algebras $\mathcal{A}_d$ and $\mathcal{L}_d$ were studied extensively by Arias and Popescu [5], Popescu [23, 24], and Davidson and Pitts [11–13]. Davidson and Pitts have shown that the algebras $\mathcal{L}_d$ are a noncommutative generalization of $H^\infty(\mathbb{D}) = \mathcal{L}_1$. For example, one has inner-outer factorization in $\mathcal{L}_d$m namely if $f \in \mathcal{L}_d$, there exists an isometry $v \in \mathcal{L}_d$ and an operator with dense range $g \in \mathcal{L}_d$, such that $f = vg$. $\mathcal{L}_d$ is inverse closed, does not contain any non-scalar normal operators and $\mathcal{L}_d' = \mathcal{R}_d$, the weak-* closed algebra generated by the right creation operators.

We already saw that the Fock space $\mathcal{F}_d^2$ is a Hilbert space of nc functions on the free ball $\mathfrak{B}_d$ with square summable Taylor-Taylor coefficients at the origin. Since we have the vacuum vector $1 \in \mathcal{F}_d^2$, we can identify every $f \in \mathcal{L}_d$ with an nc functions $f1 \in \mathcal{F}_2^d$. However, there is another way but for this, we need cp nc kernels.

Definition 5.2 ([7]). Let $\Omega \subset \mathbb{M}(V)$ be an nc set and let $K \colon \Omega(n) \times \Omega(m) \to \operatorname{Hom}(M_{n,m}, M_{n_m})$ be a function. We say that K is a cp nc kernel on Ω if

(1) The function $ZZ, W) \mapsto K(Z, W^*)$ is a first-order nc function.
(2) For every $Z \in \Omega$ the map $K(Z, Z)$ is positive.

Example 5.3. Let $X \in \mathbb{M}_d(n)$ and $\Omega = \{X^{\oplus k}\}_{k \in \mathbb{N}}$. Let K be a cp nc kernel on Ω, then $K(X^{\oplus k}, X^{\oplus k}) = K(X, X) \otimes I_{M_k}$, hence $K(X, X)$ is cp.

Example 5.4. Let k be a classical rperoducing kernel on some set $X \subset \mathbb{C}^d$ and let $\Omega \subset \mathbb{M}_d$ be the minimal nc set containing X, namely Ω consits of all possible direct sums of points of X, then k extends naturally to a cp nc kernel on Ω by setting

$$k\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}, \begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_m \end{pmatrix}\right)(P) = (k(z_i, w_j)) \circ P.$$

Here $\circ$ stands for the Schur product.

Example 5.5. The noncommutative Szego kernel is defined on $\mathfrak{B}_d$ by

$$K(Z, W)(T) = \sum_{\alpha \in \mathbb{W}_d} Z^\alpha T W^{\alpha *} = \sum_{n=0}^{\infty} \Psi_{Z,W}^n(T).$$

Here $\Psi_{Z,W}(T) = \sum_{j=1}^{d} Z_j T W_j^*$. In particular, for every $Z \in \mathfrak{B}_d$, the map $\Psi_Z = \Psi_{Z,Z}$ is a cp map and since Z is a strict row contraction, Ψ is strictly contractive. So the series converges and $K(Z,Z) = (\mathrm{Id} - \Psi_Z)^{-1}$.

Just like in the classical case we have

Theorem 5.6 ([7]). *Let K be a cp nc kernel on Ω, then there exists a Hilbert space of nc functions on Ω with reproducing kernel K. Conversely, if we have a Hilbert space of nc functions on Ω, such that for every $X \in \Omega(n)$ and every $v, y \in \mathbb{C}^n$, the functionals $f \mapsto \langle f(X)v, y\rangle$ are bounded, then there exists a cp nc kernel K on Ω, such that this space is the nc RKHS of K.*

Lastly, if K is locally bounded, then so are the functions in the nc RKHS associated to K.

Given a kernel K on Ω and $X \in \Omega(n)$ and $v, y \in \mathbb{C}^n$ we define the follwoing nc function that corresponds to the classic k_z

$$K_{X,v,y}(Z)u = K(Z,X)(uv^*)y.$$

It is easy to check that this is an nc function.

Example 5.7. Let K be the nc Szego kernel, then for every $X \in \mathfrak{B}_d(n)$ and every $v, y \in \mathbb{C}^n$, we have that

$$K_{X,v,y}(Z)u = \sum_{\alpha \in \mathbb{W}^d} Z^\alpha u v^* X^{\alpha *} y = \sum_{\alpha} \langle y, X^\alpha v\rangle Z^\alpha u.$$

Note that since X is a strict row contraction this functtion is an element of the Fock space.

Proposition 5.8 ([27]). *The Fock space and the nc RKHS associated to the Szego kernel are unitarily equivalent.*

Classically, every RKHS of functions on X has ana algebra of multipliers. This is an algebra of functions f on X, such that for every g in our RKHS, fg is also in the RKHS. For every f in the multiplier algebra, we immediately have $M_f^* K_{X,v,y} = K_{X,v,f(X)^*y}$. Furthermore, if we have an nc function f and one can define a bounded operator $TK_{X,v,y} = K_{X,v,f(X)^*y}$, then f is a multiplier and $T = M_f^*$. Lastly, every multiplier is bounded in the sense that $\sup_{X\in\Omega} \|f(X)\| < \infty$. The proof is just like in the classical case.

Lemma 5.9 ([7]). *LEt f be an nc function on Ω, then f is a multiplier with $\|f\| \leq 1$ if and only if the following is a cp nc kernel*

$$K_f(Z,W)(T) = K(Z,W)(T) - f(Z)K(Z,W)(T)f(W)^*.$$

Proof. It is easy to check that $K_f(Z,W^*)$ is a first-order nc function. If f is a multiplier of norm less that 1, then for every $K_{X,v,y}$, we have that $\|K_{X,v,f(X)^*y}\| \leq \|K_{X,v,y}\|$. Now we calculate

$$\langle K_{X,v,y}, K_{X,v,y}\rangle = \langle K_{X,v,y}(X)v, y\rangle = \langle K(X,X)(vv^*)y, y\rangle.$$

Hence for every $v, y \in \mathbb{C}^n$ we have that

$$\langle K(X,X)(vv^*)f(X)^*y, f(X)^*y\rangle \leq \langle K(X,X)(vv^*)y, y\rangle$$

Or in other words, $K(X,X)(vv^*) \geq f(X)K(X,X)(vv^*)f(X)^*$. Since this is true for every rank 1 projection it is true for every positive matrix. □

Corollary 5.10. *Let us assume that for every $n \in \mathbb{N}$ and every $X \in \Omega(n)$, there exists $A \geq 0$, such that $K(X,X)(A) = I$. Then for every multiplier f on the nc RKHS associated to K we have that $\sup_{X\in\Omega} \|f(X)\| \leq \|M_f\|$.*

Proof. Let f be a a multiplier, such that $\|M_f\| = 1$. Then K_f is a positive cp nc kernel. Hence for every $n \in \mathbb{N}$ and every $X \in \Omega(n)$ we choose $A \geq 0$, such that $K(X,X)(A) = I$ and get that $0 \leq K_f(X,X)(A) = I - f(X)f(X)^*$. Conclude that $\|f(X)\| \leq 1$. □

Example 5.11. Let K be the nc Szego kernel. The above is true for K, since for every X, $K(X,X) = (\mathrm{Id} - \Psi_X)^{-1}$. Thus $K(X,X)\,(I - \Psi_X(I)) = I$ for every $X \in \mathfrak{B}_d$. Since every X is a strict row contraction $I - \Psi_X(I) > 0$. Hence every multiplier on the Fock space is a bounded nc function on $\mathfrak{B}_d$.

In [20] Mittal and Paulsen prove that every multiplier algebra is weak-* closed. An analogous result is true for nc RKHS

Proposition 5.12 ([27]). *Let K be a cp nc kernel on Ω and $\mathcal{H}_K$ be the associated nc RKHS. Assume that for every $X \in \Omega(n)$ we have that*

$$\mathrm{Span}\{h(X)v \mid v \in \mathbb{C}^n,\ h \in \mathcal{H}_K\} = \mathbb{C}^n.$$

Then bounded WOT onvergence of multipliers is pointwise convergence on Ω. In particular, the multiplier algebra is weak- closed.*

Proof. Let f_α be a bounded net of multipliers that converges to f in WOT. Then for every $X \in \Omega(n)$, $v, y \in \mathbb{C}^n$ we have

$$\langle f_\alpha(X)h(X)v, y\rangle = \langle f_\alpha h, K_{X,v,y}\rangle \to \langle fh, K_{X,v,y}\rangle = \langle f(X)h(X)v, y\rangle.$$

Conversely, if for every point $f_\alpha(X) \to f(X)$, then we have that $\langle f_\alpha h, K_{X,v,y}\rangle \to \langle fh, K_{X,v,y}\rangle$ for every h and every choice of X, v and y as above. It is easy to check that linear combinations of kernel functions are again kernel functions, hence the kernel functions are dense in $\mathcal{H}_K$. Now since the net f_α is bounded we have that $f_\alpha \to f$ in WOT.

The last claim is a consequence of Krein-Smulian combined with the fact that a multiplier is determined by its values. □

In fact, more is true in the case of the Fock space.

Theorem 5.13. *The multiplier algebra of the nc RKHS associated to the nc Szego kernel is $H^\infty(\mathfrak{B}_d)$ — the algebra of all bounded nc function on $\mathfrak{B}_d$. Furthermore, $H^\infty(\mathfrak{B}_d)$ is unitarily equivalent to $\mathcal{L}_d$.*

Some ingredients that we need are the theory of finite-dimensional representations of $\mathcal{L}_d$ developed by Davidson and Pitts.

Theorem 5.14 ([11]). *Let $\mathrm{Hom}_{cc}(\mathcal{L}_d, M_n)$ be the space of completely contractive (and of course unital) homomorphisms of $\mathcal{L}_d$ to M_n. Then the map that sends π to $\begin{pmatrix} \pi(L_1) & \cdots & \pi(L_d) \end{pmatrix}$ is a surjection onto $\overline{\mathfrak{B}_d(n)}$. Moreover, if $X \in \mathfrak{B}_d(n)$, then the fiber over X is the singleton $\{\Phi_X\}$, where Φ_X is the completely contractive and weak-* continuous representation of evaluation at X.*

In fact, more is true for the boundary. Recall that a row contraction is called pure if $\Psi_X^n(I) \to 0$ in WOT. It is a necessary and sufficient condition to guarantee a weak-* continuous Ł$_d$-functional calculus.

Theorem 5.15 ([25, 28]). *Let $X \in \partial\mathfrak{B}_d(n)$ is pure if and only if $X = S^{-1}YS$ for some $S \in \mathbf{GL}_n(\mathbb{C})$ and $Y \in \mathfrak{B}_d(n)$. Hence over every pure point the fiber is a singleton and contains only the evaluation at this point.*

here is one important property of the classical Szego kernel that we haven't discussed yet and that is the (complete) Nevanlinna-Pick property (NP). Recall that a kernel k on a set X has NP, if for every $x_1, \ldots, x_n \in X$ and $w_1, \ldots, w_n \in \mathbb{C}$. There exists a multiplier of norm at most 1, such that $f(x_i) = w_i$ for $i = 1, \ldots, n$ if and only if the matrix $((1 - w_i\overline{w_j})k(x_i, x_j))$ is positive. The complete property is the same only the values are allowed to be matrices. For $\mathcal{L}_d$ we have the following theorem of Davidson and Pitts

Theorem 5.16 ([12, Theorem 2.1]). *Let $\mathcal{I} \subset \mathcal{L}_d$ be a weak-* closed right ideal, then $\mathcal{L}_d/\mathcal{I}$ is completely isometrically isomorphic to $P_{\mathcal{M}^\perp}\mathcal{L}_d$, where $\mathcal{M} = \overline{\mathcal{I}1} \subset \mathcal{F}_d^2$. In other words for every $n \in \mathbb{N}$ and every $F \in M_n(\mathcal{L}_d)$ we have that*

$$\operatorname{dist}(F, M_n(\mathcal{I})) = \|(P_{\mathcal{M}^\perp} \otimes I)F\|.$$

How does this relate to the nc kernels business? Ball, Marx and Vinnikov define the (complete) Nevanlinna-Pick property in [6]. To define it we need the notion of a full nc envelope of a set of points in $\mathbb{M}(V)$. Let $\Omega \subset \mathbb{M}(V)$ be an nc set, we say that Ω is full if $\Omega = \widetilde{\Omega}$ and for every point $\left(\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right) \in \Omega$ we have that $X \in \Omega$. Alternatively, one can say that Ω is closed under injective intertwinners. The full envelope of a subset of $S \subset \mathbb{M}(V)$ is the smallest full nc set that contains S. We will denote the full envelope by S_{full}. Let Ω be an nc set and let $Z \in \Omega$, the relative full

Definition 5.17. Let K be a cp nc kernel on Ω, then K has the NP property if for every point $Z \in \Omega$ and every function f_0 that extends to $\{Z\}_{full} \cap \Omega$, there exists a multiplier of norm at most 1 on the nc RKHS associated to K, such that $f(Z) = f_0(Z)$ if and only if the following map is completely positive

$$K_{f_0}(Z, Z)(T) = K(Z, Z)(T) - f_0(Z)K(Z, Z)(T)f_0(Z)^*.$$

For the complete property, we allow tensoring with matrices or more generally operator in some $B(\mathcal{E})$.

Note that we can consider a single point since we are allowed to take direct sums. As should be in good generalizations we have that

Theorem 5.18 ([6, Corollary 5.6], [27, Theorem 4.7]). *The nc Szego kernel has the complete NP property.*

Proof. One shows [27, Lemma 4.4] that f_0 extends to $\{Z\}_{full} \cap \mathfrak{B}_d$ if and only if $f_0(Z)$ is in the algebra generated by Z. Since this is the case one can find a polynomial p that agrees with f on Z. Now let $\mathcal{I}$ be the kernel of the evaluation at Z. By the theorem of Davidson and Pitts the distance of p to $\mathcal{I}$ is at most 1 (since we have the norm of the compression encoded in K_{f_0}). Therefore, there exists $g \in \mathcal{I}$, such that $\|p - g\| \leq 1$. Now set $f = p - g$. ☐

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