Fixed Points of Free Self-Maps of the Ball

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Classical Fixed Point Theorem

Let $d \in \mathbb{N}$ and write $\mathbb{B}_d$ for the unit ball of $\mathbb{C}^d$. We will write $\mathbb{D}$ for the unit disc as an exception. We will denote by ρ the Poincare metric on $\mathbb{D}$.

Theorem (Rudin '78, Hervé '63)

If $f : \mathbb{B}_d \to \mathbb{B}_d$ is an analytic function, such that $f(0) = 0$, then the fixed points of f and of the linear map $f'(0)$ coincide. In particular, the fixed point set of f is the intersection of $\mathbb{B}_d$ with a linear subspace of $\mathbb{C}^d$.

Key ingredient: Every point in $\partial\mathbb{B}_d$ is a complex extreme point. Let $U \subset \mathbb{C}^d$ be an open set. A point $x \in \partial U$ is called complex extreme if the only vector $y \in \mathbb{C}^d$ satisfying $x + \mathbb{D}y \subset \overline{U}$ is $y = 0$.

Application

Let $V, W \subset \mathbb{B}_d$ be analytic subvarieties cut out by multipliers on the Drury-Arveson space. For simplicity, let us restrict ourselves to the case when $0 \in V, W$ and V and W span $\mathbb{C}^d$. Let $\mathcal{M}_V$ and $\mathcal{M}_W$ be the associated operator algebras. A natural question is to what extent does the geometric data determine the operator-algebraic one and vice versa.

Theorem (Davidson, Ramsey and Shalit '14)

Assume that V spans $\mathbb{C}^d$ and so does W. Then the algebras $\mathcal{M}_V$ and $\mathcal{M}_W$ are completely isometrically isomorphic if and only if there exists an automorphism of $\mathbb{B}_d$ that maps V onto W.

Idea of the proof: One shows that the isomorphism induces two analytic maps $f, g\colon \mathbb{B}_d \to \mathbb{B}_d$, such that $f \circ g|_V = \mathrm{Id}_V$ and $g \circ f|_W = \mathrm{Id}_W$. Let $h = f \circ g$, then V is in the fixed points of h and thus by the fixed point theorem h is the identity.

Some Hyperbolic Geometry

The result of Rudin and Hervé tells us about the hyperbolic geometry of the unit ball of $\mathbb{C}^d$. The application we have in mind requires a brief tour of hyperbolic geometry of convex bounded domains in $\mathbb{C}^d$.
Let $U \subset \mathbb{C}^d$ be a bounded open domain, then one can define hyperbolic metrics on U. The two most prominent are the Cartheodory and the Kobayashi metric. We will assume that U is the unit ball of a norm on $\mathbb{C}^d$ and thus is convex. In the case of convex sets the two metrics coincide (Lempert '82). The Caratheodory metric is defined by

$$c_U(z,w) = \sup \{\rho(f(z), f(w)) \mid f \in \mathrm{Hol}(U, \mathbb{D})\}.$$

In particular, if U is the unit ball of $\|\cdot\|$, then $c_U(0,w) = \rho(0, \|w\|)$.

Some Hyperbolic Geometry

Let U be a ball of a norm in $\mathbb{C}^d$ and $z, w \in U$. A complex geodesic connecting z to W is an analytic map $\gamma\colon \mathbb{D} \to U$, such that for every $a, b \in \mathbb{D}$, $c_U(f(a), f(b)) = \rho(a, b)$ and there exist two points $a_0, b_0 \in \mathbb{D}$, such that $f(a_0) = z$ and $f(b_0) = w$.

Example: Let $0 \neq z \in U$, then the function $\gamma(t) = tz/\|z\|$ is a complex geodesic connecting 0 to z.

The question of the uniqueness of geodesics is a difficult one, however, if $z \in U$ is such that $z/\|z\|$ is complex extreme, then the geodesic described above is the unique one connecting 0 to z.

Some Hyperbolic Geometry

The results of Vigué generalize the classical result of Rudin and Hervé and shed light on the connection with hyperbolic geometry.

Theorem (Vigué '84, '85)

Let U be as above and $f : U \to U$ an analytic map. If $z, w \in U$ are fixed by f, then there exists a complex geodesic connecting z to w, that is fixed by f. Alternatively, if z is a fixed point of f and $v \in T_z U$ is a tangent vector fixed by df, then there is a complex geodesic through z that is tangent at z to v and consisting of fixed points of f.

Free Functions

Let $\mathbb{M}_d = \sqcup_{n=1}^{\infty} M_n(\mathbb{C})^{\oplus d}$ and consider the free unit ball $\mathfrak{B}_d = \{X \in \mathbb{M}_d \mid XX^* < I\}$. An nc-map $f \colon \mathfrak{B}_d \to \mathbb{M}_k$ is a graded function that repects direct sums and similarities. The latter condition will be used extensively. We will write $\mathfrak{B}_d(n) = \mathfrak{B}_d \cap M_n(\mathbb{C})^{\oplus d}$.

What can be said about the fixed points of an nc map $f \colon \mathfrak{B}_d \to \mathfrak{B}_d$?

On each level we have an analytic self-map $f \colon \mathfrak{B}_d(n) \to \mathfrak{B}_d(n)$. Each $\mathfrak{B}_d(n)$ is a unit ball of a norm, however, most points on the boundary are not complex extreme.

Main Result

Theorem (S. '18)

Let $f : \mathfrak{B}_d \to \mathfrak{B}_d$ *be an nc map, such that* $f(0) = 0$ *and assume that the fixed points set of* f *on* $\mathfrak{B}_d(1) = \mathbb{B}_d$ *is the intersection of* $\mathbb{B}_d$ *with a subspace* $V \subset \mathbb{C}^d$*. Then the fixed point set of* f *on level* n *is* $\mathfrak{B}_d(n) \cap V \otimes M_n(\mathbb{C})$*.*

Ingredients of the proof:

- The matrix span of a subset $S \subset \mathbb{M}_d$.
- Perron-Frobenius theory of completely positive maps.
- Hyperbolic geometry.

Matrix Span

For a set of points $\mathcal{S} \subset \mathbb{M}_d$, we define the matrix span

$$\text{mat-span}(\mathcal{S})(n) = \text{Span}\left\{ I_d \otimes T(X) \mid X \in \mathcal{S},\ T \in \mathcal{L}(M_n(\mathbb{C})) \right\}.$$

A particular case is the case of a singleton $\mathcal{S} = \{X\}$. In this case we have a different description of the matrix span. For every point $X \in \mathbb{M}_d(n)$ we define the set $X^\perp \subset (\mathbb{C}^d)^*$ as the set of functionals φ, such that $\varphi \otimes I_{M_n}(X) = 0$. Then
$\text{mat-span}(X)(k) = \bigcap_{\varphi \in X^\perp} \ker\left(\varphi \otimes I_{M_k}\right)$.

Example: The point $P = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$ satisfies
$\text{mat-span}(P) = \mathbb{M}_d$.

Perron-Frobenius Theory of Completely Positive Maps

To every point $X \in \mathfrak{B}_d(n)$ we can associate a completely positive map $\Psi_X \colon M_n \to M_n$ via $\Psi_X(T) = XTX^* = \sum_{j=1}^{d} X_j T X_j^*$.
A point $X \in \mathbb{M}_d(n)$ is called irreducible if the map $\mathbb{C}\langle z_1, \ldots, z_d \rangle \to M_n$ induced by $z_j \mapsto X_j$ is surjective. In particular, if X is irreducible, then the completely positive map Ψ_X is irreducible, i.e, it does not map the positive cone in M_n into a face (Farenick '96).
The Perron-Frobenius theorem for positive maps on finite-dimensional C^*-algebras was proved by Evans and Høegh-Krohn in '78. It states that if $\Psi \colon A \to A$ is irreducible, then there exists $0 < x \in A$, such that $\Psi(x) = rx$, where $r > 0$ is the spectral radius of Ψ. Furthermore, r is a simple eigenvalue of Ψ.

Irreducible Points and Similarities

This theorem leads us to the following interesting observation:

Lemma

Let $X \in \mathfrak{B}_d(n)$ be irreducible, then there exists $S \in \mathrm{GL}_n(\mathbb{C})$, such that if $Y = S^{-1}XS$, then $Y \in \mathfrak{B}_d(n)$ and $Y/\|Y\|$is a coisometry ($YY^* = \|Y\|^2 I$).

How does this help?
The coisometries are complex extreme points of $\mathfrak{B}_d(n)$.

What Happens if f fixes an Irreducible Point?

Now we can describe what happens if f fixes an irreducible point X, under the assumption that $f(0) = 0$. Since X is irreducible we have that $\dim X^{\perp} \leq d - 2$.

1. Observe that if $X \in \mathfrak{B}_d(n)$ is such that $X/\|X\|$ is a coisometry, then $f(tX) = tX$, for every $t \in \mathbb{D}$.
2. One can prove that if $X/\|X\|$ is a coisometry and $\Delta f(0,0)(X) = X$, then $f(tX) = tX$, for every $t \in \mathbb{D}$.
3. Since f is nc, if it fixes X, then it fixes a similarity orbit of X, so we may assume that $X/\|X\|$ is a coisometry.
4. If $f(X) = X$, then $\Delta f(0,0)$ fixes mat-span$(X) \cap \mathfrak{B}_d(n)$.
5. By (2) we have that for every coisometry $Y \in$ mat-span(X), we have that $f(tY) = tY$, for every $t \in \mathbb{D}$.
6. Since every irreducible point $Z \in \mathfrak{B}_d \cap$ mat-span(X) is similar to a scalar multiple of some coisometry, we have that Z is fixed by f and thus all of mat-span(X) is fixed by f.

Idea of the Proof

We just saw that fixing just a single irreducible point implies that a whole subspace of points is fixed.

Example: Let $r > 0$, if $f : \mathfrak{B}_d \to \mathfrak{B}_d$ is an nc map, such that $f(0) = 0$ and $f(rP) = rP$, then f is the identity map. In fact, if f fixes any irreducible point with lineary independent coordinates, then f is the identity.

What happens to reducible points? One can think of a point of the form $\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}$ as the point $X \oplus Y$ with the extra information of a tangent vector Z at it.

This consideration allows us to show that the irreducible points determine the fixed points

Application of the Main Result

We will consider subvarieties of $\mathfrak{B}_d$. Analogously to the commutative case, these are the zero loci of weak-* closed ideals of the algebra of bounded nc function on $\mathfrak{B}_d$ that we view as multipliers of the Fock space. We will say that a subvariety $\mathfrak{V} \subset \mathfrak{B}_d$ is non-degenerate, if mat-span$(\mathfrak{V}) = \mathbb{M}_d$.

Theorem (S. '18)

Let $\mathfrak{V}, \mathfrak{W} \subset \mathfrak{B}_d$ be non-degenerate subvarieties. Let $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ be the associated operator algebras of nc functions. Furthermore, assume that $H^\infty(\mathfrak{V}) \cong H^\infty(\mathfrak{W})$ completely isometrically. If $\mathfrak{V}$ (and thus $\mathfrak{W}$) has a scalar point, then there exists an automorphism of $\mathfrak{B}_d$ that maps $\mathfrak{V}$ onto $\mathfrak{W}$.

Examples of "Bad" Varieties

For completeness, we include an example of a subvariety that has no scalar points.

Example: Consider the subvariety $\mathfrak{V}$ cut out by the polynomials x^2, y^2 and $xy + yx = 1/4$. This subvariety clearly has no scalar points. However, $\frac{1}{2}P \in \mathfrak{V}(2)$.

The lowest non-empty level of a subvariety that does not contain any scalar points consists entirely of irreducible points. We do not know yet whether the claim of the last theorem is true for this more general case.

Thank You!