Born Metrics on Compact Complex Surfaces

by

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Chapter 1

Introduction

Para-hypercomplex manifolds have several applications in geometric models of string theory and integrable systems [8], [15], [16], particularly when associated with some compatible neutral metric. Such neutral metrics fall into two categories: para-hyperHermitian and Born. Initially the more important of these cases was believed to be the para-hyperHermitian metrics, and over the past 20 years the theory of these metrics has been developed in papers such as [2], [3], [4], [9], [10], and [11], particularly in the case of compact 4-manifolds. More recently, in [6] a new model for spacetime in string theory was proposed making use of Born reciprocity; the model employs a geometric structure which the authors dub "Born Geometry", consisting of a para-hypercomplex structure and a neutral metric of Born type. The goal of this project will be to develop some of the basic theory for Born metrics, and construct examples, as at this point in time very little is known about these structures other than that they have potentially useful applications.

In chapter 2 we review integrability of almost complex and almost product structures. We then have a brief discussion of para-hypercomplex manifolds and their characterization in terms of pairs of complex structures.

In chapter 3 we introduce the compatible metrics of a para-hypercomplex manifold, of which there are four types, and we will notice that three of the classes of compatible metrics are essentially equivalent, in a sense which we will make clear. Once we have reduced to the two important classes of compatible metrics we will compare and contrast their properties, including the fact that para-hyperHermitian metrics exist only in manifolds of dimension $4n$ while there are examples of Born manifolds for any even dimension. We prove that every para-hypercomplex manifold admits a Born metric, which is in contrast with the para-hyperHermitian case in which there are several examples of para-hypercomplex manifolds without para-hyperHermitian metrics. We will also give some examples of Born and para-hyperHermitian metrics on vector spaces and on tangent bundles of flat para-Kähler manifolds.

Since the para-hyperHermitian metrics are the more well-studied of the two classes of compatible metrics, chapter 4 will give a summary of the results found in [2], [3], [4], [10], and [11] on para-hyperHermitian metrics in the case of compact 4-manifolds. We will begin the section by explaining the para-hyperHermitian characterization in terms of differential forms which is commonly used in the case of complex surfaces. We will then look at some topological constraints on the complex surfaces admitting para-hyperHermitian structures, as well as those admitting para-hyperKähler structures. We remark that any compact complex surface admitting a para-hyperKähler metric admits one which is flat, making such surfaces ideal canditates for constructing para-hyperHermitian structures on their tangent bundles.

In chapter 5 we look at the known examples of para-hyperHermitian surfaces, and adapt their metrics to give Born metrics with the same para-hypercomplex structures. We notice that there is often more freedom in our choice of Born metric for a given para-hypercomplex structure than in the para-hyperHermitian case, and in particular we find examples of para-hypercomplex surfaces which have previously shown to prohibit para-hyperHermitian strucures, but which do admit Born metrics.

Chapter 2

Para-Hypercomplex Structures

Definition 2.1. Let M be a smooth manifold. An *almost complex structure* on M is a tangent bundle endomorphism J such that $J^2 = -\mathrm{Id}$. An *almost product structure* on M is a tangent bundle endomorphism P such that $P^2 = \mathrm{Id}$ and $P \neq \pm\mathrm{Id}$. We say an almost complex structure or almost product structure A is *integrable* if

$$N^A(X,Y) = -A^2[X,Y] + A[AX,Y] + A[X,AY] - [AX,AY] = 0 \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$. (N^A is the Nijenhuis tensor of A) An integrable almost complex structure is called a *complex structure*, and an integrable almost product structure is called a *product structure*.

Remark 2.2. It can be shown that the vanishing of the Nijenhuis tensor of A (2.1) is equivalent to the eigenbundles of A being closed under the Lie bracket when A is an almost complex structure or an almost product structure.

Definition 2.3. An *almost para-hypercomplex structure* on a manifold M is a collection (J_1, J_2, J_3) where J_1 is an almost complex structure, J_2, J_3 are almost product structures, and

$$J_1 J_2 = -J_2 J_1 = J_3. \tag{2.2}$$

Notice that any almost para-hypercomplex manifold is immediately an almost complex manifold as J_1 is an almost complex structure on M. In particular this forces any para-hypercomplex manifold to be even dimensional as the eigenvectors of any almost complex structure come in complex conjugate pairs.

Definition 2.4. A *para-hypercomplex structure* on M is an almost para-hypercomplex structure such that J_1, J_2, J_3 are simultaneously integrable, i.e. the Nijenhuis tensor $N^{J_i} = 0$ for $i = 1, 2, 3$.

We now show that while para-hypercomplex structures are defined by a complex structure J_1 and two product structures J_2, J_3, they are in fact determined entirely by J_1 and J_2.

Lemma 2.5. *Let (M, J_1, J_2, J_3) be an almost para-hypercomplex manifold. The $+1$ and -1 eigenspaces of J_2 are isomorphic at each point via the complex structure J_1*

Proof. If $v \in TP_x$ is such that v is in the $+1$ eigenspace of J_2, then $J_2(J_1 v) = J_2 J_1 J_2 v = -J_1 v$, so $J_1 v$ is in the -1 eigenspace of J_2. Similarly, if u is in the -1 eigenspace of J_2, $J_2(J_1 u) = -J_1 J_2 u = J_1 u$, so $J_1 u$ is in the $+1$ eigenspace of J_2. □

Proposition 2.6. *Let (M, J_1, J_2, J_3) be an almost para-hypercomplex manifold, and suppose that two of J_1, J_2, J_3 are integrable. Then all three of J_1, J_2, J_3 are integrable, and (M, J_1, J_2, J_3) is para-hypercomplex.*

Proof. We will give the proof in the case that J_1, J_2 are integrable; the other cases are similar. Let $T_2^{\pm}$ and $T_3^{\pm}$ be the $\pm$ eigenbundles of J_2 and J_3, respectively. Any section of T_3^+ can be written as $u + J_1 u$ for some $u \in \Gamma(T_2^+)$, since $\dim T_3^+ = \dim T_2^+$, and $J_1 J_2(u + J_1 u) = J_1 u + J_1 J_2 J_1 u = J_1 u + u$ for $u \in \Gamma(T_2^+)$. Further, the map $u \mapsto u + J_1 u$ must be invertible since $J_1 u$ and u were shown in Lemma 2.5 to be linearly independent when $u \neq 0$. We can also write a section of T_3^+ as $J_1 u - u$ for $u \in \Gamma(T_2^-)$, since if $u \in \Gamma(T_2^-)$, $J_1 u \in \Gamma(T_2^+)$ and $J_1 u - u = J_1 u + J_1(J_1 u)$.

Now suppose that $u, v \in \Gamma(T_2^+)$. Then

$$[u + J_1 u, v + J_1 v] = [u, v] + [J_1 u, v] + [u, J_1 v] + [J_1 u, J_1 v]$$

using the vanishing of the Nijenhuis tensor N^{J_1}, and subsequently the vanishing of $-J_1 N^{J_1}$,

$$[u, v] + [J_1 u, v] + [u, J_1 v] + [J_1 u, J_1 v] = [u, v] + J_1[u, v] - J_1[J_1 u, J_1 v] + [J_1 u, J_1 v].$$

Since J_2 is integrable, $u, v \in \Gamma(T_2^+)$ and $J_1 u, J_1 v \in \Gamma(T_2^-)$, so we have $[u, v] \in \Gamma(T_2^+)$ and $[J_1 u, J_1 v] \in \Gamma(T_2^-)$. From this we can see that $[u, v] + J_1[u, v]$ and $[J_1 u, J_1 v] - J_1[J_1 u, J_1 v]$ are both sections of T_3^+, and therefore J_3 is integrable. □

Example 2.1. A particularly simple example of such a para-hypercomplex manifold is $\mathbb{C}^n$ with standard complex structure J_1, and $J_2 : \frac{\partial}{\partial z_i} \mapsto \frac{\partial}{\partial \bar{z}_i}, i = 1, \ldots, n$, where $z_i, i = 1, \ldots, n$ are the holomorphic coordinates on $\mathbb{C}^n$ induced by J_1. J_1 is clearly integrable, and J_2 is also integrable as the sections of the $+1$ eigenbundle are all of the form

$$\sum_{i=1}^{n} f_i \left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i} \right),$$

with $f_i \in C^\infty(\mathbb{C}^n)$ for $i = 1, \ldots, n$, and

$$\left[\sum_{i=1}^{n} f_i \left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i} \right), \sum_{i=1}^{n} g_i \left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i} \right) \right] = \sum_{i=1}^{n} \sum_{j=1}^{n} \Bigg[f_i \left(\frac{\partial g_j}{\partial z_i} + \frac{\partial g_j}{\partial \bar{z}_i} \right) \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j} \right) - g_j \left(\frac{\partial f_i}{\partial z_j} + \frac{\partial f_i}{\partial \bar{z}_j} \right) \left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i} \right) \Bigg],$$

which is also a section of the $+1$ eigenbundle of J_2. This induces a para-hypercomplex structure when we apply the construction from Proposition 2.6.

It was shown in [3, Lemma 1] that a necessary and sufficient condition for the existence of a para-hypercomplex structure on a manifold X is the existence of two complex structures J_+, J_- such that $J_- J_+ + J_+ J_- = 2p\mathrm{Id}$ for some $p \in \mathbb{R}, |p| > 1$, where J_+ and J_-

induce the same orientation on X. In this case, the para-hypercomplex structure is given by

$$J_1 = J_+, \qquad J_2 = \frac{1}{2\sqrt{p^2-1}}[J_+, J_-], \qquad J_3 = -\frac{1}{\sqrt{p^2-1}}(pJ_+ + J_-).$$

Chapter 3

Metrics on Para-Hypercomplex Manifolds

We now want to consider (not necessarily positive) metrics g on para-hypercomplex manifolds which are compatible with the para-hypercomplex structure, in the sense that

$$g(J_i\cdot, J_i\cdot) = \chi_i g(\cdot, \cdot), i = 1, 2, 3 \tag{3.1}$$

where $\chi_i \in \mathbb{C}$ is a constant for each i. It is clear that any such χ_i will be ± 1, as

$$\chi_i^2 g(\cdot, \cdot) = g(J_i^2\cdot, J_i^2\cdot) = g(\cdot, \cdot) \text{ for } i = 1, 2, 3.$$

We call (χ_1, χ_2, χ_3) the *orthogonality* of g. Notice that by (2.2), $\chi_3 = \chi_1\chi_2$, so there are four possible orthogonalities of g:

$$(1, -1, -1), (-1, 1, -1), (-1, -1, 1), \text{ and } (1, 1, 1).$$

Of these choices, the one most commonly seen in the current literature is the para-hyperHermitian case.

3.1 Para-HyperHermitian metrics

Definition 3.1. An *almost para-hyperHermitian manifold* is an almost para-hypercomplex manifold (M, J_1, J_2, J_3) equipped with a compatible metric g of orthogonality $(1, -1, -1)$. We call g a *para-hyperHermitian metric* on M. If (J_1, J_2, J_3) are simultaneously integrable, we say (M, J_1, J_2, J_3, g) is a *para-hyperHermitian manifold*.

Notice that a para-hyperHermitian metric on a manifold of dimension $2n$ must have neutral signature (n, n), as for any $p \in M$ and $X \in TM_p$ with $g_p(X, X) > 0$, $g_p(J_3X, J_3X) < 0$ and vice versa, meaning that the spaces $P_p(M) := \{X \in TM_p : g_p(X, X) > 0\}$ and $N_p(M) := \{X \in TM_p : g_p(X, X) < 0\}$ are isomorphic. Para-hyperHermitian manifolds have been studied in [2], [3], [4], [9], [10], and [11], among others. The appeal of the para-hyperHermitian case is that each of the J_i induce a fundamental 2-form Ω_i on the manifold given by $\Omega_i(\cdot, \cdot) := g(\cdot, J_i\cdot)$, allowing direct use of de Rham and Dolbeaut cohomologies, as well as techniques from Hermitian geometry. These bilinear forms are skew since for any $X, Y \in \Gamma(TM)$, $\Omega_i(Y, X) = g(Y, J_iX) = \chi_i g(J_iY, J_i^2X) = \chi_i a_i g(X, J_iY) = \chi_i a_i \Omega_i(X, Y)$, where a_i is such that $J_i^2 = a_i\mathrm{Id}$ for $i = 1, 2, 3$. Since in the para-hyperHermitian case $-\chi_i = a_i$ for each $i = 1, 2, 3$, Ω_i is a differential 2-form.

Example 3.1. Consider $\mathbb{C}^{2n}$ with the same para-hypercomplex structure (J_1, J_2, J_3) as given in Example 2.1. Consider the metric on $(\mathbb{C}^{2n}, J_1, J_2, J_3)$ given by

$$g := \sum_{k=1}^{n} i\left(dz_{2k}d\bar{z}_{2k-1} - dz_{2k-1}d\bar{z}_{2k}\right).$$

Clearly

$$g(J_1\cdot, J_1\cdot) = \sum_{k=1}^{n} i\left(d(iz_{2k})d(-i\bar{z}_{2k-1}) - d(iz_{2k-1})d(-i\bar{z}_{2k})\right) = g(\cdot, \cdot)$$

and

$$g(J_2\cdot, J_2\cdot) = \sum_{k=1}^{n} i\left(d\bar{z}_{2k}dz_{2k-1} - d\bar{z}_{2k-1}dz_{2k}\right) = -g(\cdot, \cdot),$$

so g is a para-hyperHermitian metric on $(\mathbb{C}^{2n}, J_1, J_2, J_3)$.

The following result was given by Kamada in [11] for local frame fields, but we will give the simplified version for a vector space, which appears in [2].

Proposition 3.2 (Kamada)**.** *Let V be a vector space with a para-hypercomplex structure, and let $V^{\pm} \subset V$ be the ± 1-eigenspace of J_2. Then there is a bijective correspondence between non-degenerate skew bilinear forms on $V^{\pm}$ and para-hyperHermitian metrics on V.*

Proof. Let h be a non-degenerate skew 2-form on V^+. We can extend h to V by letting $h(Y^-, X) = h(X, Y^-) = 0$ for all $Y^- \in V^-, X \in V$. We define the bilinear form g by $g(X, Y) := h(X, J_1Y) + h(Y, J_1X)$ for any $X, Y \in V$. It is immediate from the definition of g that it is symmetric, and

$$g(J_1X, J_1Y) = h(J_1X, -Y) + h(J_1Y, -X) = h(Y, J_1X) + h(X, J_1Y) = g(X, Y),$$

so J_1 is an isometry with respect to g. For any $X, Y \in V$ we write

$$X = X^+ + X^-, \qquad Y = Y^+ + Y^-, \qquad \text{where } X^+, Y^+ \in V^+, X^-, Y^- \in V^-.$$

Then

$$g(X, Y) = h(X^+, J_1Y^-) + h(Y^+, J_1X^-)$$

since J_1 maps V^+ to V^-, so the non-degeneracy of g follows immediately from the non-degeneracy of h. This also shows us that

$$\begin{aligned} g(J_2X, J_2Y) &= h(J_2X^+, J_1J_2Y^-) + h(J_2Y^+, J_1J_2X^-) \\ &= -h(X^+, J_1Y^-) - h(Y^+, J_1X^-) \\ &= -g(X, Y), \end{aligned}$$

so g is para-hyperHermitian.

Conversely, if g is a para-hyperHermitian metric on V, then we can define h by the restriction of $\overline{h}(X, Y) := \frac{1}{2}g(J_1X, Y)$ to V^+. Then

$$h(Y, X) = \frac{1}{2}g(J_1Y, X) = \frac{1}{2}g(-Y, J_1X) = -\frac{1}{2}g(J_1X, Y) = -h(X, Y),$$

so h is skew, and since V^+ is a g-isotropic subspace of V, for any $X^+ \in V^+$ there must be $Y^- \in V^-$ such that $g(Y^-, X^+) \neq 0$ by the non-degeneracy of g. It follows that $-J_1 Y^- \in V^+$ and $h(-J_1 Y^-, X^+) \neq 0$, so h is non-degenerate.

Since V^+ and V^- are isomorphic via J_1, given any non-degenerate skew 2-form h^+ on V^+, $h^-(\cdot, \cdot) := h^+(J_1 \cdot, J_1 \cdot)$ is a non-degenerate skew 2-form on V^-, and $h^-(J_1 \cdot, J_1 \cdot) = h^+(\cdot, \cdot)$, so the above proof also holds for skew 2-forms on V^-. □

Remark 3.3. In particular, this Proposition shows that any para-hypercomplex manifold M which admits a para-hyperHermitian structure has dimension divisible by four, since non-degenerate skew bilinear forms exist only in vector spaces of even dimension [12, Theorem 8.1], and $\dim TM_x = 2 \dim TM_x^{\pm}$ for each $x \in M$.

For our next example of a para-hyperHermitian manifold we will first need to discuss another related structure.

Definition 3.4. An *almost para-Hermitian manifold* is a manifold with an almost product structure P and a metric g such that $g(\cdot, \cdot) = -g(P\cdot, P\cdot)$. M is *para-Hermitian* if in addition P is integrable.

Remark 3.5. If (M, J_1, J_2, J_3, g) is an almost para-hyperHermitian manifold, then (J_2, g) and (J_3, g) are immediately para-Hermitian structures on M.

Similarly to the Hermitian case, there is a 2-form ω associated to the para-Hermitian structure, which is given by $\omega(\cdot, \cdot) = g(\cdot, P\cdot)$.

Definition 3.6. A *para-Kähler manifold* is a para-Hermitian manifold such that the associated 2-form ω is closed.

This now allows us to state the following Proposition from [9]:

Proposition 3.7 (Ianuş, Vîlcu). *If (M, P, g) is an almost para-Hermitian manifold, then TM is an almost para-hyperHermitian manifold, which is integrable if and only if M is para-Kähler and flat with respect to the Levi-Civita connection of g.*

In this case the almost para-hypercomplex structure is determined by

$$J_1 X^h = X^v, \qquad J_2 X^h = (PX)^v, \qquad -J_1^2 = J_2^2 = \mathrm{Id},$$

where X is a vector field of M, X^v is the vertical lift of X to TM, and X^h is the horizontal lift of X to TM induced by the Levi-Civita connection corresponding to the almost para-Hermitian metric. The almost para-hyperHermitian metric on TM is the *Sasaki metric* of g, which is defined as the unique metric G on TM such that

$$G(X^h, Y^h) = G(X^v, Y^v) = g(X, Y) \text{ and } G(X^h, Y^v) = 0$$

for any $X, Y \in \Gamma(TM)$, and is given by

$$G(Z, W) = (g(\pi_* Z, \pi_* W) + g(KZ, KW)) \circ \pi, \qquad Z, W \in \Gamma(TTM) \tag{3.2}$$

where $K : \Gamma(TTM) \mapsto \Gamma(TM)$ is the map defined by $K(X^h) = 0$ and $(K(X^v))^v = X^v$ for $X \in \Gamma(TM)$.

3.2 Born metrics

The other case of interest is the Born metrics, which are the main topic of this paper.

Definition 3.8. An *almost Born manifold* is a para-hypercomplex manifold (M, J_1, J_2, J_3) equipped with a compatible metric g of orthogonality $(-1, 1, -1)$. We call g a *Born metric* on M. If (J_1, J_2, J_3) are simultaneously integrable, we call (M, J_1, J_2, J_3, g) a *Born manifold*.

Notice that Born metrics are also always of neutral signature (n, n), which can be seen using the same argument as in the para-hyperHermitian case.

In the Born case, while J_3 still induces a 2-form $\Omega_3(\cdot,\cdot) := g(\cdot, J_3\cdot)$, J_1 and J_2 each induce a metric $h_i(\cdot,\cdot) := g(\cdot, J_i\cdot), i = 1, 2$ instead of the 2-forms obtained in the para-hyperHermitian case, as $a_1 = \chi_1, a_2 = \chi_2$. Notice that

$$h_1(J_1\cdot, J_1\cdot) = g(J_1\cdot, J_1^2\cdot) = -g(J_1\cdot,\cdot) = -g(\cdot, J_1\cdot) = -h_1(\cdot,\cdot)$$
$$h_1(J_2\cdot, J_2\cdot) = g(J_2\cdot, J_1J_2\cdot) = -g(J_2\cdot, J_2J_1\cdot) = -g(\cdot, J_1\cdot) = -h_1(\cdot,\cdot)$$

and

$$h_2(J_1\cdot, J_1\cdot) = g(J_1\cdot, J_2J_1\cdot) = -g(J_1\cdot, J_1J_2\cdot) = g(\cdot, J_2\cdot) = h_2(\cdot,\cdot)$$
$$h_2(J_2\cdot, J_2\cdot) = g(J_2\cdot, J_2^2\cdot) = g(J_2\cdot,\cdot) = g(\cdot, J_2\cdot) = h_2(\cdot,\cdot)$$

so h_1, h_2 are in fact also compatible with the para-hypercomplex structure, with orthogonality $(-1,-1,1)$ and $(1,1,1)$ respectively. Since the map taking g to h_i is clearly invertible, we see that there is a natural correspondence between metrics with orthogonality $(1,1,1)$, metrics with orthogonality $(-1,-1,1)$, and Born metrics. While this makes metrics of orthogonality $(1,1,1)$ and $(-1,-1,1)$ less interesting on their own, this correspondence becomes very useful when constructing Born metrics, as will be seen in Theorem 3.9.

Example 3.2. Again, consider $\mathbb{C}^n$ with the para-hypercomplex structure (J_1, J_2, J_3) given in Example 2.1. Then consider the metric on $\mathbb{C}^n$ given by $g := \sum_{k=1}^{n} (dz_k^2 + d\bar{z}_k^2)$. For this metric, we have

$$g(J_1\cdot, J_1\cdot) = \sum_{k=1}^{n} \left(i^2 dz_k^2 + (-i)^2 d\bar{z}_k^2\right) = -g(\cdot,\cdot)$$

and

$$g(J_2\cdot, J_2\cdot) = \sum_{k=1}^{n} \left(d\bar{z}_k^2 + dz_k^2\right) = g(\cdot,\cdot),$$

so g is a Born metric on $(\mathbb{C}^n, J_1, J_2, J_3)$.

Theorem 3.9. *Every para-hypercomplex manifold admits a Born structure.*

Proof. Let (M, J_1, J_2, J_3) be a para-hypercomplex manifold, and let h be a Hermitian metric with respect to J_1. It is a standard result in complex geometry that such a metric always exists (see for Example [14, section 11.1]). We can then define the symmetric bilinear form h_2 by $h_2(\cdot,\cdot) = h(\cdot,\cdot) + h(J_2\cdot, J_2\cdot)$. h_2 is clearly non-degenerate as h is positive definite and any sum of positive definite bilinear forms is positive definite. J_1 is an isometry with respect to h_2 as it is an isometry with respect to h, and J_2 is an isometry with respect to h_2 as $h_2(J_2\cdot, J_2\cdot) = h(J_2\cdot, J_2\cdot) + h(\cdot,\cdot) = h_2(\cdot,\cdot)$. This shows that there is a Born structure g such that h_2 is the associated metric with orthogonality $(1, 1, 1)$. □

Remark 3.10. Notice that the technique of averaging to get a metric with the correct orthogonality may result in a degenerate form in the case where the desired metric is of mixed signature, so the connection between a Born metric and its associated metric of orthogonality $(1, 1, 1)$ is essential to the proof of Theorem 3.9, as $(1, 1, 1)$ is the only orthogonality for which a positive metric is possible. In particular, the proof fails in the para-hyperHermitian case, and there are multiple cases of para-hypercomplex manifolds which do not admit a para-hyperHermitian metric, either due to dimensional constraints (see Proposition 3.2), or due to topological constraints on certain associated vector bundles [2].

If (M, P, g) is a flat para-Kähler manifold, recall that Proposition 3.7 gives a para-hypercomplex structure on TM. It follows from Theorem 3.9 that this para-hypercomplex manifold must admit a Born metric. The following example illustrates this in a simple case.

Example 3.3 (Tangent bundle of a complex torus)**.** Consider the complex n-torus X with product structure

$$P : \frac{\partial}{\partial z_i} \mapsto \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_i} \mapsto \frac{\partial}{\partial z_i}$$

and para-Kähler metric

$$g = \sum_{i=1}^{n} \left(dz_i^2 - d\bar{z}_i^2 \right).$$

One can easily see that the curvature of the Levi-Civita connection for g is zero, as its associated matrix-valued one-form is everywhere 0. This also tells us that if we take local coordinates $\{z_i, \bar{z}_i, w_i, \bar{w}_i : i = 1, \dots n\}$ about p for some $p \in TX$ where $w_i = \frac{\partial}{\partial z_i}$, then TTX_p decomposes into the horizontal space

$$H_p^{TX} = \text{Span}\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} : i = 1, \dots n\}$$

and the vertical space

$$V_p^{TX} = \text{Span}\{\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_i} : i = 1, \dots n\}.$$

We notice that the metric $h := \sum_{i=1}^{n} (dz_i d\bar{z}_i)$ is a positive metric on X such that P is an isometry, and also that h has the same vertical and horizontal subspaces as g, as its Levi-Civita connection also has everywhere 0 matrix-valued one-form. If we now look at the Sasaki metric $\tilde{h}$ of h as defined in (3.2), we obtain a metric of orthogonality $(1, 1, 1)$ with respect to the para-hypercomplex structure as given in Proposition 3.7, and therefore $\mu(\cdot, \cdot) := \tilde{h}(\cdot, J_2 \cdot)$ is a Born metric on TX.

From this point onward, we will restrict our attention to para-hypercomplex structures on compact complex surfaces.

Chapter 4

A Summary of Results on Para-HyperHermitian Surfaces

When dealing with the case of complex surfaces, authors commonly use a reformulated characterization of the para-hyperHermitian structure in terms of differential forms rather than the one given in Definition 3.1 with a metric and a para-hypercomplex structure.

Proposition 4.1 (Kamada, [11])**.** *A para-hyperHermitian structure on a compact 4-manifold X is characterized by $\Omega_1, \Omega_2, \Omega_3 \in \Gamma(\Lambda^2 T^* X)$ and $\theta \in \Gamma(T^* X)$, where $\Omega_1, \Omega_2, \Omega_3$ satisfy the identities*

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \qquad \Omega_\alpha \wedge \Omega_\beta = 0, \alpha \neq \beta, \qquad d\Omega_\alpha = \theta \wedge \Omega_\alpha, \alpha = 1, 2, 3.$$

The Ω_i are the *fundamental forms* of the para-hyperHermitian structure, and θ is the *Lee form*.

Remark 4.2. One can switch between this characterization and the definition via the equations

$$\Omega_1(\cdot,\cdot) = g(\cdot, J_1\cdot), \qquad \Omega_2(\cdot,\cdot) = g(\cdot, J_2\cdot), \qquad \Omega_3(\cdot,\cdot) = g(\cdot, J_3\cdot),$$
$$\Omega_3(J_1\cdot,\cdot) = \Omega_2(\cdot,\cdot), \qquad \Omega_1(J_2\cdot,\cdot) = \Omega_3(\cdot,\cdot), \qquad \Omega_2(J_3\cdot,\cdot) = -\Omega_1(\cdot,\cdot).$$

We know from the theory of Hermitian metrics that Ω_1 is a real $(1,1)$-form in the holomorphic decomposition with respect to J_1, and we can see from the commutation relations of J_1, J_2, J_3 that Ω_2 and Ω_3 are the real and imaginary parts, respectively, of a $(2,0)$-form $\Omega_2+i\Omega_3$, which is nowhere-vanishing by the non-degeneracy of Ω_2, Ω_3. Requiring the existence of a nowhere-vanishing $(2,0)$-form imposes strong topological restrictions on the complex surfaces which admit a para-hyperHermitian structure, as given in the following Theorem [4, Theorem 8].

Theorem 4.3. *If a compact complex surface admits a para-hyperHermitian structure, it has a nowhere-vanishing $(2,0)$-form, and is therefore one of the following: a complex torus, a K3 surface, a primary Kodaira surface, a Hopf surface, a Inoue surface of type S^0, S^+, or S^-, or a properly elliptic surface with odd first Betti number.*

Proposition 3.2 gives us a further restriction on the existence of a para-hyperHermitian metric on a para-hypercomplex surface (X, J_1, J_2, J_3), as there is a non-degenerate skew 2-form on V^+ if and only if $\Lambda^2 V^+$ is trivial (i.e. V^+ is orientable), where V^+ is the $+1$-eigenbundle of J_2. This follows from the fact that $\Lambda^n E$ has rank one for any rank-n vector bundle E, and any nowhere-vanishing section of a rank-one vector bundle induces a global trivialization. Since for any line bundle L over a manifold M there is a double cover of M so that L pulls back to an orientable bundle [13, Orientation Covers, p.393], we can deduce the following result given in [4].

Proposition 4.4. *any para-hypercomplex structure on a compact complex surface either admits a para-hyperHermitian metric, or its lift to a double cover admits a para-hyperHermitian metric. In particular, any compact surface admitting a para-hypercomplex structure will belong to the list given in Theorem 4.3, or has a double cover belonging to this list.*

Remark 4.5. Notice that the existence of a nowhere-vanishing 2-form on V^+ is equivalent to the existence of a para-hyperHermitian metric, while the existence of a nowhere-vanishing $(2,0)$-form on X in Theorem 4.3 is merely a necessary condition.

A subclass of para-hyperHermitian structures which is of particular interest is the class for which the Lee form $\theta = 0$. These are known as para-hyperKähler manifolds, as the requirement that $\theta = 0$ forces each of the fundamental forms to be closed. This means that the $(2,0)$-form $\Omega_2 + i\Omega_3$ belongs to the canonical bundle since every closed $(2,0)$-form is holomorphic, and so any compact complex surface which admits a para-hyperKähler metric has a holomorphically trivial canonical bundle, further restricting the possible surfaces to complex tori, primary Kodaira surfaces, and K3 surfaces. It was shown in [18] that in fact K3 surfaces do not admit para-hyperKähler metrics, and in [11] that all complex 2-tori and primary Kodaira surfaces do.

Chapter 5

Examples of Born Surfaces

We will now give explicit constructions of Born structures on specific complex surfaces. We notice that of the possible surfaces given in Theorem 4.3 and Proposition 4.4 all but the K3 surfaces and Enriques surfaces (defined as surfaces which have a K3 surface as a double cover) have universal covers which are open subsets of $\mathbb{C}^2$, and so can be expressed as quotients of these open subsets by free and properly discontinuous group actions. To help with finding Born metrics on these surfaces we will show a condition on an almost para-hypercomplex structure on such a quotient to be para-hypercomplex.

Proposition 5.1. *Let M be a smooth manifold which is a quotient of an open subset $U \subset \mathbb{C}^n$ by a free and properly discontinuous group action. Then any para-hypercomplex structure (J_1, J_2, J_3) on U which descends as an almost para-hypercomplex structure on M is integrable on M.*

Proof. Since (J_1, J_2, J_3) is integrable on U, the eigenbundles of J_1, J_2, J_3 are closed under the Lie bracket. Therefore the only way that any of J_1, J_2, J_3 can fail to be integrable on M is if the transition functions of M do not preserve the eigenbundles of $J_i, i = 1, 2, 3$. But since an atlas for M can be chosen so that transition functions are given by applying the group action, and since $J_i, i = 1, 2, 3$, are well-defined under such transitions as they descend to M, their eigenbundles are preserved under the group action and therefore (J_1, J_2, J_3) is a para-hypercomplex structure on M. □

The above condition allows us to only check integrability on the universal cover for any manifold of the above type.

5.1 Complex 2-Tori

A complex 2-torus is a quotient of $\mathbb{C}^2$ by the free abelian subgroup generated by $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, where $\text{span}_{\mathbb{R}}(\Lambda) = \mathbb{C}^2$. The standard complex structure on $\mathbb{C}^2$ descends to the 2-torus, so it is naturally a complex surface. The following examples give two families of Born structures on the 2-torus $X = \mathbb{C}^2/\Lambda$.

Example 5.1. Consider the linear map $J_2 : T\mathbb{C}^2 \to T\mathbb{C}^2$ given by

$$J_2\left(\frac{\partial}{\partial z_i}\right) := \frac{\partial}{\partial \bar{z}_i}, i = 1, 2 \qquad\qquad J_2^2 = \text{Id},$$

where z_1, z_2 are the standard holomorphic coordinates. Then clearly J_2 descends to a tangent bundle endomorphism on any 2-torus X, so J_1, J_2 induce a para-hypercomplex structure on X where J_1 is the standard complex structure of X. Now consider the symmetric bilinear form

$$g := p(z_1, z_2)(dz_1^2 + d\bar{z}_1^2) + q(z_1, z_2)(dz_2^2 + d\bar{z}_2^2) + r(z_1, z_2)(dz_1 dz_2 + d\bar{z}_1 d\bar{z}_2),$$

where p, q, r are smooth functions on X such that $4pq - r^2$ is nowhere vanishing. (This requirement guarantees that the metric is non-degenerate.) Since

$$dz_i dz_j(J_1\cdot, J_1\cdot) = -dz_i dz_j(\cdot,\cdot), d\bar{z}_i d\bar{z}_j(J_1\cdot, J_1\cdot) = -d\bar{z}_i d\bar{z}_j(\cdot,\cdot),$$
$$\text{and } dz_i dz_j(J_2\cdot, J_2\cdot) = d\bar{z}_i d\bar{z}_j(\cdot,\cdot),$$

g has orthogonality $(-1, 1, -1)$, and therefore is a Born structure on $(X, J_1, J_2, J_1 J_2)$.

Example 5.2. Let $J_2' : T\mathbb{C}^2 \to T\mathbb{C}^2$ be the linear map given by

$$J_2'\left(\frac{\partial}{\partial z_1}\right) = \frac{\partial}{\partial \bar{z}_2}, \qquad J_2'\left(\frac{\partial}{\partial z_2}\right) = \frac{\partial}{\partial \bar{z}_1}, \qquad J_2'^2 = \mathrm{Id}.$$

J_2' also descends to a tangent bundle endomorphism on any 2-torus X, so J_1, J_2' induce a para-hypercomplex structure on X. Consider

$$g' := p(z_1, z_2)(dz_1^2 + d\bar{z}_2^2) + q(z_1, z_2)(dz_2^2 + d\bar{z}_1^2) + r(z_1, z_2)(dz_1 dz_2 + d\bar{z}_1 d\bar{z}_2),$$

where p, q, r are smooth functions on X such that $4pq - r^2$ is nowhere vanishing. We can again see that g' is a Born metric on $(X, J_1, J_2', J_1 J_2')$.

Remark 5.2. These examples demonstrate another notable difference between Born metrics and para-hyperHermitian metrics on complex surfaces. In [2, Proposition 13], the authors showed that any two para-hyperHermitian metrics g, h on a given para-hypercomplex surface are conformally equivalent, i.e. $g = fh$ for some smooth non-vanishing function f. (This follows immediately from Proposition 3.2 and the fact that the space of skew bilinear forms on a 2-dimensional vector space has dimension 1.) However, this is clearly not the case with Born metrics on the 2-torus, as for both para-hypercomplex structures given we have found multiple non-conformally equivalent Born metrics.

5.2 Hopf Surfaces

A Hopf surface is a compact complex surface with universal covering $A = \mathbb{C}^2 \setminus \{(0,0)\}$. A primary Hopf surface is a Hopf surface of the form A/Γ, where Γ is the group generated by $(z_1, z_2) \mapsto (az_1 + \lambda z_2^m, bz_2)$, where a, b are complex numbers with $0 < |a| \leq |b| < 1$, $m \in \mathbb{Z}^+$, and $\lambda(a - b^m) = 0$ [1]. All primary Hopf surfaces are diffeomorphic, and it is not difficult to see that the Hopf surface with $\lambda = 0, a = \frac{1}{2}, b = \frac{1}{2}$ is diffeomorphic to $S^1 \times S^3$, since A is diffeomorphic to $\mathbb{R}^+ \times S^3$, and quotienting $\mathbb{R}^+$ by the action of a $\frac{1}{2}$-scaling gives S^1. Here we give examples of para-hypercomplex structures on specific classes of Hopf surfaces, as well as a Born metric in the case that $b = \bar{a}$.

Example 5.3. Let X be the Hopf surface A/Γ where Γ is the free group generated by $\varphi : (z_1, z_2) \mapsto (az_1, \bar{a}z_2)$ with $0 < |a| < 1$. The linear map $J_2 : T\mathbb{C}^2 \to T\mathbb{C}^2$ given by

$$J_2\left(\frac{\partial}{\partial z_1}\right) := \frac{\partial}{\partial \bar{z}_2}, \qquad J_2\left(\frac{\partial}{\partial z_2}\right) := \frac{\partial}{\partial \bar{z}_1} \qquad J_2^2 = \mathrm{Id},$$

descends to X, as the differential of the group action is given by

$$\varphi_*\left(\frac{\partial}{\partial z_1}\right) = a\frac{\partial}{\partial z_2}, \quad \varphi_*\left(\frac{\partial}{\partial z_1}\right) = \bar{a}\frac{\partial}{\partial z_2}, \quad \varphi_*\left(\frac{\partial}{\partial \bar{z}_1}\right) = \bar{a}\frac{\partial}{\partial \bar{z}_1}, \quad \varphi_*\left(\frac{\partial}{\partial \bar{z}_2}\right) = a\frac{\partial}{\partial \bar{z}_2},$$

which clearly commutes with J_2. Let g be the symmetric bilinear form

$$g := \frac{dz_1 dz_2 + d\bar{z}_1 d\bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2}.$$

If $(z_1', z_2') = (az_1, \bar{a}z_2)$,

$$\begin{aligned} g_{(z_1', z_2')} &= \frac{dz_1' dz_2' + d\bar{z}_1' d\bar{z}_2'}{z_1'\bar{z}_1' + z_2'\bar{z}_2'} \\ &= \frac{|a|^2 dz_1 dz_2 + |a|^2 d\bar{z}_1 d\bar{z}_2}{|a|^2 z_1\bar{z}_1 + |a|^2 z_2\bar{z}_2} \\ &= g_{(z_1, z_2)} \end{aligned}$$

so g is well-defined as a symmetric bilinear form on X. Additionally,

$$g(J_1\cdot, J_1\cdot) = \frac{-dz_1 dz_2 - d\bar{z}_1 d\bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2} = -g(\cdot,\cdot)$$

and

$$g(J_2\cdot, J_2\cdot) = \frac{d\bar{z}_2 d\bar{z}_1 + dz_2 dz_1}{z_1\bar{z}_1 + z_2\bar{z}_2} = g(\cdot,\cdot),$$

so (X, J_1, J_2, J_1J_2, g) is a Born manifold, where J_1 is the complex structure on X induced by the standard complex structure of $\mathbb{C}^2$.

Example 5.4. Let X be the Hopf surface given by A/D where D is the free group generated by $(z_1, z_2) \mapsto (az_1, bz_2)$, with $a, b \in \mathbb{R}$ and $0 < |a| \le |b| < 1$. In this case we have a para-hypercomplex structure on X determined by the complex structure on X and the product structure $J_2 : \frac{\partial}{\partial z_i} \mapsto \frac{\partial}{\partial \bar{z}_i}$, which can easily be seen to be well defined, as J_2 and φ_* have the same eigenvectors for any $\varphi \in D$. Together with the complex structure J_1 induced from the standard complex structure of $\mathbb{C}^2$, J_2 determines a para-hypercomplex structure on X.

Remark 5.3. Born metrics also exist in this second case using a Hermitian metric (as can be found in [17]) and the construction of Theorem 3.9. However, explicit Hermitian metrics are very complicated for these Hopf surfaces, so explicit examples of Born metrics have not been included for this class of Hopf surfaces.

5.3 Primary Kodaira Surfaces

A primary Kodaira surface is the quotient of $\mathbb{C}^2$ by the group action generated by $\varphi_i : (z_1, z_2) \mapsto (z_1+a_i, \bar{a}_i z_1+z_2+b_i)$, where $i = 1, 2, 3, 4$, and $a_i, b_i \in \mathbb{C}$ are such that $a_1 = a_2 = 0$ and $a_3\bar{a}_4 - a_4\bar{a}_3 = mb_2$ for some $m \in \mathbb{Z}$ [7, p. 46]. A primary Kodaira surface can easily be seen as a principal bundle over an elliptic curve whose fibre is also an elliptic curve. Primary Kodaira surfaces all have holomorphically trivial canonical bundle, and first Betti number equal to 3, so they do not admit Kähler metrics.

The following example gives an explicit Born structure on a primary Kodaira surface.

Example 5.5. The endomorphism $J_2 : T\mathbb{C}^2 \to T\mathbb{C}^2$ given by

$$J_2(\frac{\partial}{\partial z_1}) = e^{i\theta}\frac{\partial}{\partial \bar{z}_1} + 2\text{Re}(z_1)e^{i\theta}\frac{\partial}{\partial \bar{z}_2}, \qquad J_2(\frac{\partial}{\partial z_2}) = -e^{i\theta}\frac{\partial}{\partial \bar{z}_2}, \qquad J_2^2 = \text{Id},$$

where $\theta \in \mathbb{R}$ is a constant, descends to any primary Kodaira surface, as for any choice of φ a generator of the Kodaira group action, and any vector field $X = a(z_1, z_2)\frac{\partial}{\partial z_1} +$

$b(z_1,z_2)\frac{\partial}{\partial z_2}+c(z_1,z_2)\frac{\partial}{\partial \bar z_1}+d(z_1,z_2)\frac{\partial}{\partial \bar z_2}\in\Gamma(T\mathbb{C})$,

$$\begin{aligned}
J_2(\varphi_*X)_{(z_1,z_2)} &= J_2\Bigg(a\circ\varphi^{-1}(z_1,z_2)(\frac{\partial}{\partial z_1}+\bar a_i\frac{\partial}{\partial z_2})+b\circ\varphi^{-1}(z_1,z_2)\frac{\partial}{\partial z_2}\\
&\qquad + c\circ\varphi^{-1}(z_1,z_2)(\frac{\partial}{\partial \bar z_1}+a_i\frac{\partial}{\partial \bar z_2})+d\circ\varphi^{-1}(z_1,z_2)\frac{\partial}{\partial \bar z_2}\Bigg)_{(z_1,z_2)}\\
&= \Bigg[a\circ\varphi^{-1}(z_1,z_2)e^{i\theta}\left(\frac{\partial}{\partial \bar z_1}+(2\mathrm{Re}(z_1)-\bar a_i)\frac{\partial}{\partial \bar z_2}\right)-b\circ\varphi^{-1}(z_1,z_2)e^{i\theta}\frac{\partial}{\partial \bar z_2}\\
&\quad +c\circ\varphi^{-1}(z_1,z_2)e^{-i\theta}\left(\frac{\partial}{\partial z_1}+(2\mathrm{Re}(z_1)-a_i)\frac{\partial}{\partial \bar z_2}\right)-d\circ\varphi^{-1}(z_1,z_2)e^{-i\theta}\frac{\partial}{\partial z_2}\Bigg]_{(z_1,z_2)}\\
&= \Bigg[a\circ\varphi^{-1}(z_1,z_2)e^{i\theta}\left(\frac{\partial}{\partial \bar z_1}+(2\mathrm{Re}(z_1-a_i)+a_i)\frac{\partial}{\partial \bar z_2}\right)-b\circ\varphi^{-1}(z_1,z_2)e^{i\theta}\frac{\partial}{\partial \bar z_2}\\
&\quad +c\circ\varphi^{-1}(z_1,z_2)e^{-i\theta}\left(\frac{\partial}{\partial z_1}+(2\mathrm{Re}(z_1-a_i)+\bar a_i)\frac{\partial}{\partial z_2}\right)-d\circ\varphi^{-1}(z_1,z_2)e^{-i\theta}\frac{\partial}{\partial z_2}\Bigg]_{(z_1,z_2)}\\
&= \varphi_*(J_2X)_{(z_1,z_2)}
\end{aligned}$$

Now consider the symmetric bilinear form

$$h:=|z_1|^2dz_1d\bar z_1-\bar z_1dz_1d\bar z_2-z_1dz_2d\bar z_1+dz_2d\bar z_2$$

over $\mathbb{C}^2$. It is not difficult to see that h is Hermitian, and

$$\begin{aligned}
h_{\varphi(z_1,z_2)} =&|z_1+a_i|^2dz_1d\bar z_1-(\bar z_1+\bar a_i)dz_1(d\bar z_2+a_id\bar z_1)\\
&-(z_1+a_i)d\bar z_1(dz_2+\bar a_idz_1)+(dz_2+\bar a_idz_1)(d\bar z_2+a_id\bar z_1)\\
=&(|z_1|^2+a_i\bar z_1+z_1\bar a_i+|a_i|^2-a_i(\bar z_1+\bar a_i)-\bar a_i(z_1+a_i)+|a_i|^2)dz_1d\bar z_1\\
&+(\bar a_i-\bar z_1-\bar a_i)dz_1d\bar z_2+(a_i-z_1-a_i)dz_2d\bar z_1+dz_2d\bar z_2\\
=&h_{(z_1,z_2)},
\end{aligned}$$

so h descends to a Hermitian metric on our Kodaira surface. Now via Theorem 3.9, we have that the metric g given by

$$g(\cdot,\cdot):=h(J_2\cdot,\cdot)+h(\cdot,J_2\cdot)=\mathrm{Re}\left(-\bar z_1^2e^{i\theta}dz_1^2+2\bar z_1e^{i\theta}dz_1dz_2-e^{i\theta}dz_2^2\right)$$

is a Born structure on our Kodaira surface with respect to the para-hypercomplex structure given above.

5.4 Inoue Surfaces

Here we give explicit examples of para-hypercomplex structures on Inoue surfaces of types S^+ and S^-.

Inoue surfaces are quotients of $\mathbb{H} \times \mathbb{C}$, where $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. An Inoue surface S of type S^+ corresponding to the matrix $N \in \mathrm{SL}_2(\mathbb{Z})$ with real eigenvalue $\alpha > 1$ is given by the quotient of $\mathbb{H} \times \mathbb{C}$ by the group action generated by

$$\begin{aligned}
\varphi_0 :&(z_1, z_2) \mapsto (\alpha z_1, z_2 + t) \\
\varphi_i :&(z_1, z_2) \mapsto (z_1 + a_i, z_2 + b_i z_1 + c_i), && i = 1, 2 \\
\varphi_3 :&(z_1, z_2) \mapsto \left(z_1, z_2 + \frac{b_1 a_2 - a_1 b_2}{r}\right)
\end{aligned}$$

where z_1 and z_2 are the standard holomorphic coordinates of $\mathbb{H}$ and $\mathbb{C}$ respectively, (a_1, a_2) and (b_1, b_2) are eigenvectors of N, $t \in \mathbb{C}$, and $c_i, r \in \mathbb{R}$.

Example 5.6. Consider the tangent bundle endomorphism $J_2 : T(\mathbb{H} \times \mathbb{C}) \to T(\mathbb{H} \times \mathbb{C})$ given by

$$J_2(\frac{\partial}{\partial z_i}) = \frac{\partial}{\partial \bar{z}_i}, i = 1, 2, \qquad J_2^2 = \mathrm{Id}$$

Since J_2 and $(\varphi_i)_*, i = 0, 1, 2, 3$, are defined in the same way for each point in $\mathbb{H} \times \mathbb{C}$, in order for J_2 to descend to the Inoue surface S it is enough to show that J_2 commutes with $(\varphi_i)_*$ as $\mathbb{C}$-linear maps on $\mathrm{Span}\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}\right\}$ for each $i = 0, 1, 2, 3$. As matrices over this basis, the maps are

$$J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad (\varphi_0)_* = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\varphi_i)_* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_i & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b_i & 1 \end{bmatrix}, i = 1, 2, \qquad (\varphi_3)_* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus we can easily see that J_2 descends to a tangent bundle endomorphism over S so, together with the complex structure J_1 inherited from $\mathbb{H} \times \mathbb{C}$, J_2 determines a para-hypercomplex structure on S.

Example 5.7. One can also show that J_2 is a well-defined product structure on an Inoue surface of type S^-, which is given by the quotient of $\mathbb{H} \times \mathbb{C}$ by $\varphi_1, \varphi_2, \varphi_3$, and $\psi_0 := (z_1, z_2) \mapsto (\alpha z_1, -z_2)$, where φ_i are the same as in the S^+ case for $i = 1, 2, 3$.

Remark 5.4. Notice that given any Inoue surface S of type S^-, there exists an Inoue surface $\overline{S}$ of type S^+ with $t = 0$ which is a double cover of S. (In fact, $\overline{S}$ is defined using the same $\varphi_1, \varphi_2, \varphi_3$ as S.) It was shown in [2] that Inoue surfaces of type S^+ admit para-hyperHermitian structures associated to the para-hypercomplex structure in Example 5.6, but while an Inoue surface of type S^- inherits the para-hypercomplex structure from its covering S^+ Inoue surface, the para-hyperHermitian metric does not descend, and since all para-hyperHermitian structures associated to the para-hypercomplex structure of Example 5.6 are conformally equivalent, the para-hypercomplex structure from Example 5.7 does not admit a para-hyperHermitian structure.

5.5 Hyperelliptic Surfaces

We finish by giving an example of a Born structure on a family of hyperelliptic surfaces.

Example 5.8. Let T_1 and T_2 be the tori $\mathbb{C}/(1, 2\alpha)$, $\mathbb{C}/(1, \beta)$ with $\alpha, \beta \in \mathbb{H}$, and consider the hyperelliptic surface $X = (T_1 \times T_2)/\Gamma$, where Γ is the group generated by $\varphi : (z_1, z_2) \mapsto (z_1 + \alpha, -z_2)$. Consider the Born structure on $T_1 \times T_2$ given in Example 5.1 with $p = q = 1, r = 0$. (Recall that a product of any two complex 1-tori is a complex 2-torus.) φ^* acts on $\Gamma(T\mathbb{C}^2)$ by

$$\varphi_* \left(\frac{\partial}{\partial z_1} \right) = \frac{\partial}{\partial z_1}, \qquad \varphi_* \left(\frac{\partial}{\partial \bar{z}_1} \right) = \frac{\partial}{\partial \bar{z}_1}$$
$$\varphi_* \left(\frac{\partial}{\partial z_2} \right) = -\frac{\partial}{\partial z_2} \qquad \varphi_* \left(\frac{\partial}{\partial \bar{z}_2} \right) = -\frac{\partial}{\partial \bar{z}_2}$$

so the product structure J_2 descends to X as φ^* is diagonal with respect to an eigenbasis of J_2. Also, g descends to X as

$$\begin{aligned} \varphi^* g &= d\varphi(z_1)^2 + d\varphi(z_2)^2 + d\varphi(\bar{z}_1)^2 + d\varphi(\bar{z}_2)^2 \\ &= dz_1^2 + (-1)^2 dz_2^2 + d\bar{z}_1^2 + (-1)^2 d\bar{z}_2^2 \\ &= g, \end{aligned}$$

so g is also a Born metric on (X, J_1, J_2, J_3).

Remark 5.5. Note that while there are hyperelliptic surfaces which admit Born structures, a hyperelliptic surface cannot have a nowhere-vanishing $(2, 0)$-form, and so by Theorem 4.3 no hyperelliptic surface admits a para-hyperHermitian metric.

Chapter 6

Further questions

Recall that given a Born manifold (M, J_1, J_2, J_3, g), there is an associated metric $h_2(\cdot,\cdot) := g(\cdot, J_2)$ with orthogonality $(1,1,1)$. Note that h_2 may be positive or indefinite. However, in the proof of Theorem 3.9, Born metrics constructed always have associated $(1,1,1)$-metric which is positive. In the case where the associated metric is positive, the associated metric is in particular Hermitian with respect to J_1. Given this fact, a natural question to ask is when two distinct Hermitian metrics will yield the same Born metric upon applying the algorithm of Theorem 3.9. Understanding this would allow us to make use of the already well-developed classification results for Hermitian metrics in studying Born metrics.

In the case where the associated metric is indefinite, it is still unknown whether a Born metric with indefinite associated $(1,1,1)$-metric exists given a para-hypercomplex manifold. As there are often topological restrictions on manifolds which admit indefinite metrics, the answer to this question is less clear than in the positive case. It could be that, as with the para-hyperHermitian metrics, Born metrics with indefinite associated $(1,1,1)$-metrics occur only on a subset of para-hypercomplex manifolds.

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