Vector Bundles on Non-Kähler Elliptic Surfaces

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Eric Boulter was the sole author for the material in Chapters 1, 2, 3, 4, 5, and 7 which were written under the supervision of Dr. Ruxandra Moraru. Of these, Chapters 4 and 5 are based on a manuscript [Bou21] written for publication.

The material in Chapter 6 is based on a manuscript [BM22] written for publication by Eric Boulter and Dr. Ruxandra Moraru. It consists of the content of [BM22] for which Eric Boulter made the primary contribution.

Abstract

This thesis studies two problems relating to moduli spaces of vector bundles on non-Kähler elliptic surfaces. The first project involves the holomorphic symplectic structure on smooth and compact moduli spaces of sheaves on Kodaira surfaces. We show that these moduli spaces are neither Kähler nor simply connected. Comparing to other known examples of compact holomorphic symplectic manifolds, this shows that if the moduli spaces are deformation equivalent to a known example, then they are Douady spaces of points on a Kodaira surface.

The second problem deals with the interplay between singularities of moduli spaces of rank-2 vector bundles and existence of stable Vafa–Witten bundles on non-Kähler elliptic surfaces. By constructing a Vafa–Witten bundle in each filtrable Chern class of the elliptic surface when the base has genus $g \geq 2$, we show that such a moduli space is smooth as a ringed space if and only if every bundle in the moduli space is irreducible.

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Chapter 1

Introduction

Moduli spaces of holomorphic vector bundles form a foundational part of algebraic and complex geometry, as a classification tool as well as for the geometry of the spaces themselves. For instance, moduli spaces of vector bundles often inherit interesting geometric structures from their base manifold (such as a holomorphic symplectic or holomorphic Poisson structure [Muk84, Bot95]).

In the simplest cases of smooth projective curves of genus 0 and 1, complete classifications of indecomposable vector bundles were described by Grothendieck [Gro57] and Atiyah [Ati57], respectively. (Here indecomposable refers to bundles which cannot be written as a direct sum of bundles of smaller rank.) Over curves of higher genus, attempting to study indecomposable bundles leads to a moduli space which is non-separated and wildly singular; to resolve this issue, the family of bundles is restricted to the smaller family of *(semi)-stable bundles*, which are bundles E satisfying the property that for any subsheaf $\mathcal{F} \subseteq E$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(E)$,

$$\frac{\deg(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} < (\leq) \frac{\deg(E)}{\mathrm{rank}(E)}.$$

Moduli spaces of stable bundles were constructed for curves of genus at least 2 by Mumford [Mum63] and the topological invariants of these moduli spaces were described by

Narasimhan and Seshadri via a correspondence between stable vector bundles and solutions to the Hermitian Yang–Mills equations [NS65].

The problem of classifying vector bundles becomes more complicated when the base manifold is no longer a curve. While in the curve case there is a canonical choice of degree map corresponding to the natural ordering of the Néron–Severi group, surfaces generally do not have such a canonical ordering. This problem can be resolved by defining an ordering in terms of a choice of *polarisation* (either an ample line bundle H in the algebraic case, or a Gauduchon metric g in the holomorphic case) [Tak72, Gie77, Kob82]. On a complex n-manifold X, a choice of polarisation corresponds to a choice of positive $(1,1)$-form ω on X (either the Kähler form associated to the ample line bundle H, or the 2-form induced from g by the complex structure), and the degree map is defined by

$$\deg_\omega \mathcal{E} := \frac{i}{2\pi}\int_X F_h \wedge \omega^{n-1},$$

where F_h is the curvature of the Chern connection for some Hermitian metric h on $\det(\mathcal{E})$. (While the exact value of F_h may vary by a $\partial\bar{\partial}$-exact form depending on the choice of h, the overall value is independent of h due to the fact that $\partial\bar{\partial}\omega^{n-1} = 0$.)

Moduli spaces of vector bundles over surfaces that admit fibrations are of particular interest as the fibration structure can be leveraged to reduce stability problems to the stability of the restrictions to fibres. Moduli spaces of vector bundles were studied over ruled surfaces in [AB96, AB97], and on Kähler elliptic surfaces in [FM88a, FM88b].

The methods for studying moduli spaces of vector bundles on elliptic surfaces were extended to non-Kähler elliptic surfaces by Braam–Hurtubise [BH89] and Moraru [Mor03] in the Hopf case, and Teleman [Tel98] for bundles with trivial determinant on more general elliptic fibre bundles.

One important application of moduli spaces of vector bundles is to the classification of holomorphic symplectic manifolds. A holomorphic symplectic manifold is a complex $2m$-dimensional manifold equipped with a closed holomorphic 2-form η such that η^m is a nowhere-vanishing holomorphic m-form. Holomorphic symplectic manifolds of dimension 2 have been fully classified into complex tori, K3 surfaces and Kodaira surfaces, but examples in higher dimensions are sparse. One of the main tools for constructing such examples is the fact that any smooth moduli space of vector bundles over a holomorphic symplectic surface is itself holomorphic symplectic [Muk84]. In fact, every known deformation class of Kähler holomorphic symplectic manifolds contains an example that can be obtained as a moduli space of vector bundles over a holomorphic symplectic surface (i.e. a K3 surface or a complex 2-torus) [O'G97, Yos01], or as a desingularisation of such a moduli space [O'G99, O'G03].

In the non-Kähler case, there are three known ways of constructing compact holomorphic symplectic manifolds from primary Kodaira surfaces. Beauville showed in [Bea83] that the Douady space of points on a Kodaira surface is again holomorphic symplectic. Bogomolov and Guan constructed simply connected holomorphic symplectic manifolds of non-Kähler type (now known as Bogomolov-Guan manifolds) by exploiting a natural foliation structure on the Douady space of points of a Kodaira surface [Gua95, Bog96]. The final method is due to Toma, who showed that a moduli space $\mathcal{M}_{r,\delta,c}(X)$ of g-stable sheaves of rank r, determinant δ, and second Chern class c on a non-algebraic compact holomorphic symplectic surface is a compact holomorphic symplectic manifold if and only if every g-semi-stable sheaf $\mathcal{F}$ with $\operatorname{rank}(\mathcal{F}) = r, \det(\mathcal{F}) = \delta, c_2(\mathcal{F}) = c$ is g-stable [Tom01]. In the case that X is a Kodaira surface, this only occurs if the moduli space is *stably irreducible*. We say that a torsion-free coherent sheaf $\mathcal{E}$ is *irreducible* if every non-zero coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ has $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{E}$. A moduli space of sheaves is stably irreducible if it is non-empty and every sheaf represented by the moduli space is irreducible. In fact, for such moduli spaces, every sheaf is automatically stable independent of a choice of metric.

Closely related to vector bundles are the concepts of Higgs bundles, Vafa–Witten pairs, and more generally stable V-pairs. Let X be a compact n-dimensional complex manifold endowed with a Gauduchon metric g whose fundamental form is ω, and let V be a fixed holomorphic vector bundle on X. Consider a pair (E, ϕ) consisting of a holomorphic vector bundle E on X and a holomorphic section $\phi \in H^0(X, \mathcal{E}nd(E) \otimes V)$ such that $\phi \wedge \phi = 0$. The section ϕ is called a *Higgs field*. The V-pair (E, ϕ) is then said to be *g-stable* if for any proper subsheaf $\mathcal{S} \subset E$ such that $\phi(\mathcal{S}) \subset \mathcal{S} \otimes V$ (that is, $\mathcal{S}$ is *ϕ-invariant*), we have

$$\frac{\deg_g(\mathcal{S})}{\operatorname{rank} \mathcal{S}} < \frac{\deg_g(E)}{\operatorname{rank} E},$$

in which case ϕ is a *g-stable Higgs field*.

Some important special cases of stable V-pairs are *Higgs bundles*, where $V = \mathcal{T}_X^*$, and Vafa–Witten pairs, where $V = K_X$. Higgs bundles were first introduced by Hitchin [Hit87] on algebraic curves and generalised to higher dimensional varieties by Simpson [Sim92]. They also correspond to a special class of solutions of the Kapustin-Witten equations on compact Kähler surfaces [Tan17]. Although Higgs bundles over curves have been extensively studied over the past thirty-five years, less is known in the higher dimensional case. Some results have nonetheless been obtained in higher dimensions by Biswas and Bottacin [Bis94, Bis11, Bot00], among others.

The Vafa–Witten equations are a close relative of the Hitchin–Simpson equations for Higgs bundles and have been studied in [VW94, Tan17, TT20]. Via a Kobayashi–Hitchin type correspondence, solutions to these equations are given by stable K_X-pairs (E, ϕ) consisting of a vector bundle E and a global section $\phi \in H^0(X, \mathcal{E}nd_0 E \otimes K_X)$ [LT06].

In this thesis, we focus on two problems related to vector bundles on elliptic surfaces. The first problem considers smooth and compact moduli spaces of stable sheaves on a primary Kodaira surface. As primary Kodaira surfaces have a natural holomorphic symplectic structure, these moduli spaces are also holomorphic symplectic [Muk84, Tom01]. In the case that these moduli spaces have dimension two, it was shown that they are again primary Kodaira surfaces by Aprodu, Moraru, and Toma [AMT12]. However, it is still unknown what the possible deformation types are for higher dimensional moduli spaces of stable sheaves on Kodaira surfaces. With respect to this problem, the goal of the thesis is to rule out all but one of the known deformation families for holomorphic symplectic manifolds.

The second problem addressed in this thesis is the smoothness of moduli spaces of stable vector bundles over general non-Kähler elliptic surfaces. While stable vector bundles are guaranteed to correspond to smooth points in a moduli space for primary Hopf surfaces or primary Kodaira surfaces [BM05b], singularities at stable points may occur for other non-Kähler elliptic surfaces whose canonical bundles admit non-trivial sections. By standard deformation theory arguments, obstructions to deformations of a vector bundle E on a manifold X are valued in

$$H^2(X, \mathcal{E}nd_0 E) = H^0(X, \mathcal{E}nd_0 E \otimes K_X)^\vee.$$

The latter group is parameterised by solutions to the Vafa–Witten equations over E, so finding singularities of moduli spaces of stable vector bundles is equivalent to finding solutions to the Vafa–Witten equations.

The second goal of this thesis is thus to find smoothness conditions for moduli spaces of rank-2 vector bundles on a non-Kähler elliptic surface $\pi : X \to B$ by studying existence criteria for solutions to the Vafa–Witten equations.

Chapter 2 begins with a construction of non-Kähler elliptic surfaces and a review of some useful geometric invariants. This chapter also includes an overview of the known deformation families of compact holomorphic symplectic manifolds for comparison with those constructed in Chapter 5.

Chapter 3 presents the concept of slope stability for Gauduchon manifolds, which is the main stability criterion used in this thesis. In the case of non-Kähler elliptic surfaces, we describe the destabilising bundle structure for reducible bundles and show the existence of stably irreducible moduli spaces of vector bundles with arbitrary dimensions using a numerical existence criterion due to Brînzănescu [Brî96, Lemma 4.30]. The main result of this chapter is the following:

Proposition. *Let $\pi : X \to B$ be a non-Kähler principal elliptic surface, and set*

$$\nu(X) := \begin{cases} 0, & \text{if } NS(X) \text{ is finite,} \\ \min\limits_{\alpha \in NS(X), \alpha^2 \neq 0} (-\alpha^2/2), & \text{otherwise.} \end{cases}$$

If $r \geq 2$, $\delta \in \operatorname{Pic}(X)$ satisfies $c_1^2(\delta) = -2\nu(X)$, and

$$\left(\frac{1-r}{r}\right)\nu(X) \leq c < \left(\frac{2-r}{r-1}\right)\nu(X),$$

the moduli space $\mathcal{M}_{r,\delta,c}(X)$ of stable sheaves with rank r, determinant δ, and second Chern class c is stably irreducible whenever it is non-empty. In this case, $\mathcal{M}_{r,\delta,c}(X)$ has dimension

$$2r^2\Delta(r,\delta,c) = 2rc + 2(r-1)\nu(X).$$

Since there are primary Kodaira surfaces X with arbitrary whole number values of $\nu(X)$, this means that there are stably irreducible moduli spaces of vector bundles on Kodaira surfaces in every even dimension. We end the chapter by extending the definition of slope stability to V-pairs where V is an arbitrary vector bundle, including some simple examples.

Chapter 4 discusses the spectral construction for non-Kähler elliptic surfaces. This construction associates, to every vector bundle on an elliptic surface $\pi : X \to B$, a divisor in its relative Jacobian $J(X)$. This invariant gives a natural way of describing a moduli space of sheaves $\mathcal{M}_{2,\delta,c}$ as a flat family lying over the set $\mathbb{P}_{\delta,c}$ of spectral curves known as the *graph map*. In particular, if X is a primary Hopf surface or a primary Kodaira surface, the graph map corresponds to an algebraically completely integrable system with respect to the natural Poisson structure on the moduli space [BM05b]. In the case of rank-2 sheaves which are *stably irreducible*, meaning that any sheaf with the same Chern classes is irreducible, we show that the corresponding spectral curve has smooth irreducible components. This result is applied in the main results of both Chapters 5 and 6.

In Chapter 5, we restrict ourselves to the case where moduli spaces of stables sheaves are compact holomorphic symplectic manifolds, particularly stably irreducible moduli spaces of stable sheaves on primary Kodaira surfaces. The main result of this chapter is the following:

Theorem. *Let $\pi : X \to B$ be a primary Kodaira surface, and let $\mathcal{M}_{2,\delta,c}(X)$ be a moduli space of stable sheaves of rank 2, determinant δ, and second Chern class c such that $\mathcal{M}_{2,\delta,c}(X)$ is stably irreducible. If $\dim \mathcal{M}_{2,\delta,c}(X)$ is positive, then no connected component of $\dim \mathcal{M}_{2,\delta,c}(X)$ is simply connected or Kähler.*

This result demonstrates that if $\mathcal{M}_{2,\delta,c}(X)$ is deformation equivalent to one of the other known examples of holomorphic symplectic manifolds, then it must be deformation equivalent to a Douady space of points over a primary Kodaira surface. This result is obtained by studying the Lagrangian fibration structure on the moduli space of sheaves induced by its graph map. The remainder of the chapter considers methods by which we could exploit this Lagrangian fibration structure to determine further topological invariants of the moduli spaces and compare them to Douady spaces of points over Kodaira surfaces.

Chapter 6 studies solutions to the Hitchin–Simpson and Vafa–Witten equations over non-Kähler elliptic surfaces. We begin with some general computations regarding stable V-pairs for V an arbitrary vector bundle before specialising to the case of $V = K_X$ or $V = \mathcal{T}_X^*$. In the Vafa–Witten case, we discuss the correspondence between the existence of Vafa–Witten bundles and the deformations of the underlying vector bundles. The main result of this chapter is the following:

Theorem. *Let $\pi : X \to B$ be a non-Kähler elliptic surface with B of genus at least 2, and let $\mathcal{M}_{2,\delta,c}(X)$ be the moduli space of stable vector bundles of rank 2, determinant δ, and second Chern class c on X with $\delta \in \mathrm{Pic}(X)$ and $c \in \mathbb{Z}$ such that $\Delta(2, \delta, c) > 0$. Then, there is a non-trivial solution (E, ϕ) to the Vafa–Witten equations with $E \in \mathcal{M}_{2,\delta,c}$ if and only if $\mathcal{M}_{2,\delta,c}(X)$ contains a reducible vector bundle. In particular, if every stable bundle in $\mathcal{M}_{2,\delta,c}(X)$ is good in the sense of Friedman [Fri98, Definition 16], then $\mathcal{M}_{2,\delta,c}(X)$ is stably irreducible.*

By a good vector bundle we mean a vector bundle E such that $H^2(X, \mathcal{E}nd_0(E)) = 0$, or equivalently that E corresponds to a smooth point (as a ringed space) of dimension $h^1(X, \mathcal{E}nd_0(E)$ in the corresponding moduli space. This result is an improvement on [BM05b, Proposition 4.2], which previously obtained a weaker sufficient condition for unobstructed deformations involving a bound on the number of allowed jumps in the graph of a vector bundle. In the process of obtaining the above result, we completely classify the solutions (E, ϕ) to the Vafa–Witten equations on non-Kähler elliptic surfaces $\pi : X \to B$ when E has rank 2 and is such that $\operatorname{rank} \pi_*(L \otimes E) \leq 1$ for all $L \in \mathrm{Pic}(X)$. In this case, the set of Higgs fields on E is equal to the set of global sections of a line bundle on B determined by the maximal destabilising bundles of E.

The Appendix of this thesis gives a brief overview of some results in homological algebra and deformation theory which will be used at various points in the document; this may be useful for those less familiar with common notation and terminology used in homological algebra.

Chapter 2

Geometry of Relevant Examples

As the focus of this thesis is the subject of non-Kähler elliptic surfaces, we begin with a survey of basic results on these surfaces in Section 2.1, both to familiarise the reader with these surfaces and to fix notation. Sections 2.3 and 2.2 survey known results on compact holomorphic symplectic manifolds in preparation for Chapter 5, in which we study compact moduli spaces of sheaves on Kodaira surfaces, which admit a holomorphic symplectic structure.

2.1 Non-Kähler elliptic surfaces

Through the course of this thesis, the main class of manifolds we study consists of elliptic surfaces that are not Kähler.

Definition 2.1. An *elliptic surface* is a compact complex surface X together with a surjective holomorphic map $\pi : X \to B$, such that B is a complex curve and the general fibre of π is a genus-1 curve. We say that an elliptic surface is an elliptic *quasi-bundle* if all smooth fibres are isomorphic and the only singular fibres are multiples of the smooth fibre. A *principal* elliptic surface is an elliptic quasi-bundle with no multiple fibres.

Principal elliptic surfaces can be constructed as follows:

Proposition 2.2 (Teleman [Tel98])**.** *Let $\pi : X \to B$ be a compact principal elliptic surface. Then there is a positive integer d, a line bundle $\Theta \in \mathrm{Pic}^d(B)$ of degree d, and a complex number τ with $|\tau| > 1$ such that $X \cong \Theta^*/(\tau)$, where Θ^* is the principal $\mathbb{C}^*$-bundle corresponding to Θ, and (τ) is the subgroup of $\mathbb{C}^*$ generated by τ, acting via the standard action. Up to biholomorphism, the surface is determined uniquely by B, Θ, and τ. The homeomorphism type of the surface is determined by d and the genus of B.*

A similar result holds for Kähler principal elliptic surfaces with $d = 0$ instead of $d > 0$.

If $\pi : X \to B$ is a non-Kähler principal elliptic surface as described in Proposition 2.2, where B is a genus-g curve, then X has the following invariants:

$$\begin{gathered} H^1(X,\mathbb{Z}) = \mathbb{Z}^{2g+1}, \qquad H^2(X,\mathbb{Z}) = \mathbb{Z}^{4g} \oplus \mathbb{Z}/d\mathbb{Z}, \qquad H^3(X,\mathbb{Z}) = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/d\mathbb{Z} \\ H^{0,1}(X) = \mathbb{C}^{g+1}, \qquad H^{1,1}(X) = \mathbb{C}^{2g}, \end{gathered}$$

where $H^{p,q}(X) = H^q(X, \Omega^p_X)$ [BHPV03]. (See Example A.12 for an explicit computation of the singular cohomology.) The other Hodge numbers can be computed using the relations $H^{p,q}(X) = (H^{2-p,2-q}(X))^\vee$ and

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X),$$

since for all compact complex surfaces, the Hodge-de Rham spectral sequence degenerates at page 1. This gives the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g+1 & \\ g & & 2g & & g \\ & g+1 & & g & \\ & & 1 & & \end{array}.$$

The Hodge numbers can also be directly computed using the following description of the holomorphic cotangent bundle together with the Hirzebruch-Riemann-Roch theorem:

Proposition 2.3. *Let $\pi : X \to B$ be a principal elliptic surface. Then the holomorphic cotangent bundle $\mathcal{T}_X^*$ is isomorphic to $\mathcal{O}_X \oplus \pi^* K_B$ if X admits a Kähler metric, and $\pi^*(\mathcal{A}_K)$ otherwise, where $\mathcal{A}_K$ is the unique non-split extension*

$$0 \to K_B \to \mathcal{A}_K \to \mathcal{O}_B \to 0.$$

Proof. Let $\{U_\alpha, \varphi_\alpha\}$ be an atlas of B such that $\{U_\alpha, \psi_\alpha\}$ is a local trivialisation for Θ^* and set $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ to be the corresponding transition functions. We also set $V_0 = \{[w] \in \mathbb{C}^*/(\tau) : 1 < |w| < |\tau|\}$ and $V_1 = \{[w] \in \mathbb{C}^*/(\tau) : |\tau|^{-1/2} < |w| < |\tau|^{1/2}\}$. Let $f_{\alpha\beta}$ be transition maps for Θ. Then $\{\psi_\alpha^{-1}(\varphi_\alpha(U_\alpha) \times V_i)\}$ gives an atlas for X with transition functions $\eta_{ij,\alpha\beta}(z, w) = (g_{\alpha\beta}(z), f_{\alpha\beta}(z)\tau^k w)$, where z is a local coordinate in B, w is a coordinate of $\mathbb{C}^*$, and

$$k = \begin{cases} 1 & i = 0, j = 1, |w| < 1 \\ -1 & i = 1, j = 0, |w| > |\tau|^{1/2} \\ 0 & \text{otherwise} \end{cases}.$$

If we now compute the pullbacks $\eta_{ij,\alpha\beta}^*(\mathrm{d}\tilde{z})$ and $\eta_{ij,\alpha\beta}^*(\frac{1}{\tilde{w}}\mathrm{d}\tilde{w})$, we get

$$\begin{aligned} \eta_{ij,\alpha\beta}^*(\mathrm{d}\tilde{z}) &= \mathrm{d}(g_{\alpha\beta}(z)) \\ &= g'_{\alpha\beta}\mathrm{d}z, \\ \eta_{ij,\alpha\beta}^*(\frac{1}{\tilde{w}}\mathrm{d}\tilde{w}) &= \frac{1}{f_{\alpha\beta}\tau^k w}\mathrm{d}(f_{\alpha\beta}\tau^k w) \\ &= \frac{1}{w}\mathrm{d}w + \frac{f'_{\alpha\beta}}{f_{\alpha\beta}}\mathrm{d}z, \end{aligned}$$

so $\mathcal{T}_X^*$ has transition maps $\begin{bmatrix} g'_{\alpha\beta} & \frac{f'_{\alpha\beta}}{f_{\alpha\beta}} \\ 0 & 1 \end{bmatrix}$. If the degree of Θ is zero (in which case X admits a Kähler metric), the $f_{\alpha\beta}$ can be chosen to be constant, so $\mathcal{T}_X^*$ splits as a direct sum $\mathcal{O}_X \oplus \pi^*\mathcal{T}_B^*$. Since the transition maps of $\mathcal{T}_X^*$ depend only on z and since $g'_{\alpha\beta}$ are the transition maps of $\pi^*(\mathcal{T}_B^*)$, we find that $\mathcal{T}_X^*$ is the pullback of an extension in $\mathrm{Ext}^1(\mathcal{O}_B, K_B)$. If $\deg \Theta > 0$, then the extension does not split, and since Serre duality gives

$$\mathrm{Ext}^1(\mathcal{O}_B, K_B) = (H^0(B, \mathcal{O}_B))^* \cong \mathbb{C},$$

$\mathcal{T}_X^*$ is isomorphic to the unique non-split extension. □

Remark 2.4. When B is $\mathbb{P}^1$, $\mathcal{A}_K \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and when B has genus 1, $\mathcal{A}_K$ is the Atiyah bundle.

For non-Kähler elliptic surfaces which are not principal, we have the following result.

Proposition 2.5 (Brînzănescu [Brî96])**.** *Every non-Kähler elliptic surface is a quasi-bundle.*

To any non-Kähler elliptic surface $\pi : X \to B$ with multiple fibres $\{F_i = \pi^{-1}(p_i)\}_{1 \le i \le r}$ of orders m_i respectively, we can associate to X a cyclic cover $\rho : C \to B$ of order $m := \mathrm{lcm}_{1 \le i \le r}(m_i)$ which has ramification of order m_i at p_i. There is then a principal elliptic surface $\psi : Y \to C$ such that

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\rho}} & X \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\rho} & B \end{array} \tag{2.1}$$

is a commutative diagram [Brî96, Proposition 3.18].

The Hodge diamond of a non-Kähler elliptic quasi-bundle is the same as that of a non-Kähler principal elliptic surface over the same base, but the torsion components of its singular cohomology may differ.

Remark 2.6. In general, if $\pi : X \to B$ is a non-Kähler elliptic surface possibly with multiple fibres, we can use the natural exact sequence

$$\pi^*\mathcal{T}_B^* \longrightarrow \mathcal{T}_X^* \longrightarrow \Omega^1_{X/B} \longrightarrow 0$$

together with the fact that π is a submersion to show that $\mathcal{T}_X$ is a non-split extension of the sheaf $\Omega^1_{X/B}$ of relative Kähler differentials by $\pi^*\mathcal{T}_B^*$. In particular, since $\pi : X \to B$ is an elliptic surface we have

$$\Omega^1_{X/B} = \omega_{X/B} \simeq \mathcal{O}_X\left(\sum_{i=1}^{r}(m_i - 1)T_i\right),$$

where T_i are the multiple fibres of multiplicity m_i and $\omega_{X/B}$ is the relative dualising sheaf of π which appears in the formulation of relative Serre duality (See Proposition A.12). However, the fact that $\mathcal{T}_X^*$ can be described in this way only uniquely determines the bundle $\mathcal{T}_X^*$ if $\pi : X \to B$ is principal.

Corollary 2.6.1. *The canonical bundle K_X of X is isomorphic to $\pi^* K_B \otimes \omega_{X/B}$.*

Proof. The canonical bundle is defined as

$$K_X := \det(\mathcal{T}_X^*) \simeq \det(\pi^*(\mathcal{T}_B^*)) \otimes \det(\Omega^1_{X/B}) = \pi^*(K_B) \otimes \omega_{X/B}.$$

□

2.1.1 The Picard group

Let $\pi : X \to B$ be a principal elliptic surface. In order to understand the Picard group of X, we break up the problem into studying $\operatorname{Pic}^0(X)$ and the Néron–Severi group $NS(X) := \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$.

Any line bundle in $\operatorname{Pic}^0(X)$ decomposes uniquely as the product of the pullback of a degree zero line bundle on the base and a bundle with *constant factor of automorphy* [Tel98, Proposition 1.6]. The line bundle L_α with constant factor of automorphy α is the quotient of the trivial line bundle $\Theta^* \times \mathbb{C}$ on Θ^* by the $\mathbb{Z}$-action

$$\begin{aligned} \Theta^* \times \mathbb{C} \times \mathbb{Z} &\to \Theta^* \times \mathbb{C}, \\ (z, t, n) &\mapsto (\tau^n z, \alpha^n t). \end{aligned}$$

Using these facts, one can also show that $\pi^*\Theta \simeq L_{\tau^{-1}}$ [Tel98, Proposition 1.7], demonstrating that the first Chern class of a pullback bundle is torsion, and that

$$\mathcal{O}_X(m_i T_i) \simeq \pi^* \mathcal{O}_B(p_i)$$

for any multiple fibre T_i of multiplicity m_i lying over a point $p_i \in B$. Let P_2 be the subgroup of $\operatorname{Pic}(X)$ generated by line bundles associated to divisors on X. We can see from the above computations that $\mathrm{Pic}^{\mathrm{Tors}}(X) \simeq P_2 \times \mathbb{C}^*/(\pi^*\Theta, L_\tau)$, where $\mathrm{Pic}^{\mathrm{Tors}}(X)$ is the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles with torsion first Chern class.

The family of line bundles with constant factor of automorphy is equipped with a universal sheaf $\mathcal{U}$ on $X \times \mathbb{C}^*$ given as the quotient of the trivial line bundle $\Theta^* \times \mathbb{C}^* \times \mathbb{C}$ on $\Theta^* \times \mathbb{C}^*$ by the group action

$$n \cdot (z, \alpha, t) := (\tau^n z, \alpha, \alpha^n t). \tag{2.2}$$

This bundle will play a role in the classification of vector bundles over X via the spectral construction in Chapter 4.

Since $\pi : X \to B$ is an elliptic fibration, we can understand $\mathrm{Pic}(X)/\pi^* \mathrm{Pic}(B)$ as a group corresponding to families of line bundles on the smooth fibre T parameterised by B. Since the line bundles with constant factor of automorphy contain a cyclic subgroup of $\pi^* \mathrm{Pic}(B)$, such families can be described by maps from B to $T^* := \mathrm{Pic}^0(T)$. Up to fixing base points, we obtain the group $\mathrm{Pic}(X)/\pi^* \mathrm{Pic}(B) \simeq T^* \times \mathrm{Hom}(J(B), T^*)$, where $J(B)$ is the Jacobian variety of B. (For a more detailed proof of this result, see [Brî96, Section 3.2].)

Note that there is a one-to-one correspondence between $T^* \times \mathrm{Hom}(J(B), T^*)$ and maps from B to T^*, as the Albanese map of B is $b \mapsto \mathcal{O}_B(b - b_0)$ for a choice of base point $b_0 \in B$. This fact allows us to parameterise $\mathrm{Pic}(X)/\pi^* \mathrm{Pic}(B)$ by the set of sections of the *relative Jacobian* $\pi_J : J(X) \to B$ when $\pi : X \to B$ is principal. (Note that the relative Jacobian of a non-Kähler elliptic surface $\pi : X \to B$ with general fibre T is $J(X) = B \times T^*$, with projection onto the first factor being the associated map to B.) Given a line bundle $\delta \in \mathrm{Pic}(X)$, we associate the section

$$S_\delta = \{(b, \lambda) \in J(X) : \delta|_{\pi^{-1}(b)} \simeq \lambda\},$$

called the *spectral curve* of δ.

Remark 2.7. If π has multiple fibres, for any $\delta \in \mathrm{Pic}(X)$ the line bundle $\tilde{\rho}^*\delta$ has a spectral curve $S_{\tilde{\rho}^*\delta}$ on $J(Y)$, where $\psi : Y \to C$ is the principal elliptic surface defined as in (2.1). We can define the spectral curve of δ as a section of the relative Jacobian $J(X)$ by

$$S_\delta := (\rho \times \mathrm{Id}_{T^*})(S_{\tilde{\rho}^*\delta}).$$

Associated to any line bundle $\delta \in \mathrm{Pic}(X)$, there is also a ruled surface $\mathbb{F}_\delta$ given by the quotient of $J(X)$ by the $\mathbb{Z}/2\mathbb{Z}$-action sending (b, λ) to $(b, \delta|_{\pi^{-1}(b)} \otimes \lambda^{-1})$. The e-invariant

$$e_\delta := \max\{-\sigma^2 : \sigma \text{ is a section of } \mathbb{F}_\delta\} \tag{2.3}$$

of $\mathbb{F}_\delta$, where σ^2 is the self-intersection of the curve σ, appears in the existence criteria for rank-2 vector bundles on X with determinant δ, as shown in [BM05a, Theorem 4.5].

2.2 The Douady space of points on a complex surface

This exposition is based mainly on [dCM00] and [Nak99].

If X is a complex surface, the Douady space of points $X^{[n]}$ is a moduli space parameterizing the coherent $\mathcal{O}_X$-modules with finite support of length n. The Douady space can be defined locally using an analytic GIT-type construction for $\Delta^{[n]}$, where Δ is the holomorphic bi-disk $\{(z_1, z_2) \in \mathbb{C}^2 : |z_\alpha| < 1\}$, and gluing the $\Delta^{[n]}$ via the induced transition maps of a decomposition of X into a union of bi-disks.

In order to define $\Delta^{[n]}$, we consider the set

$$U := \{(A_1, A_2, t) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^n : [A_1, A_2] = 0 \text{ and } |\lambda| < 1 \text{ for eigenvalues } \lambda \text{ of } A_\alpha\}\,.$$

We also set H to be the elements of U satisfying the stability condition that there is no proper subspace of $\mathbb{C}^n$ containing t which is preserved by A_1, A_2. If we quotient by the $\mathrm{GL}_n(\mathbb{C})$-action $M \cdot (A_1, A_2, t) = (MA_1M^{-1}, MA_2M^{-1}, Mt)$, we get $H/\mathrm{GL}_n(\mathbb{C}) \cong \Delta^{[n]}$.

Example 2.1. To construct $\Delta^{[2]}$, we can split into cases depending on whether or not A_i has a unique eigenvalue for $i = 1, 2$. If at least one of A_1, A_2 has two distinct eigenvalues, then A_1, A_2 can simultaneously diagonalised, and since we will be taking the quotient by a $\mathrm{GL}_n(\mathbb{C})$-action, we can assume the corresponding elements of U are of the form

$$\left(A_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, A_2 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}\right)$$

for some $\lambda_1, \lambda_2, \mu_1, \mu_2 \in B_{\mathbb{C}}(0,1)$ and $t_1, t_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ or $\mu_1 \neq \mu_2$. If we assume this element belongs to H, then up to the $\mathrm{GL}_n(\mathbb{C})$-action, we can assume that $t = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. This piece of $\Delta^{[2]}$ therefore corresponds to the complement of the diagonal in $\mathrm{Sym}^2\Delta$.

If instead both A_1 and A_2 have a unique eigenvalue, then they can be simultaneously upper-triagularised and up to the $\mathrm{GL}_n(\mathbb{C})$-action we can assume elements of U have the form

$$\left(A_1 = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}, A_2 = \begin{bmatrix} \mu & \beta \\ 0 & \mu \end{bmatrix}, t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right)$$

for some $\lambda, \mu \in B_{\mathbb{C}}(0,1), \alpha, \beta, t_1, t_2 \in \mathbb{C}$. If we assume that such an element belongs to H, then up to the $\mathrm{GL}_n(\mathbb{C})$-action we will have $t = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and α, β not both zero. After taking the quotient, the resulting set will be $\Delta \times \mathbb{P}^1$, meaning that $\Delta^{[2]} \simeq \mathrm{Bl}_{\mathrm{diag}}(\mathrm{Sym}^2\Delta)$.

In the case that X has dimension 2, $X^{[n]}$ is itself a complex manifold, and a holomorphic symplectic structure on X naturally lifts to $X^{[n]}$ [Bea83]. In the following, we compile some known information about Lagrangian fibrations and holomorphic invariants of Douady spaces over elliptic surfaces.

2.2.1 Lagrangian fibration structure for the Douady space of an elliptic surface

The following is closely adapted from the discussion in [Leh11] of the Hilbert scheme of points on a K3 surface.

If $\pi : X \to B$ is a principal elliptic surface with fibre T, there is an induced Abelian variety fibration on the Douady space $X^{[n]}$ given by the composition $\pi^{[n]} := \varrho \circ \mathrm{Sym}^n(\pi)$, where $\mathrm{Sym}^n(\pi) : \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(B)$ is the induced map of symmetric products and $\varrho : X^{[n]} \to \mathrm{Sym}^n(X)$ is the Douady-Barelet morphism sending a finite subscheme to its support with multiplicity. (As shown in [CC93], $\mathrm{Sym}^n(B)$ is isomorphic to the projectivisation of an indecomposable bundle on B of rank n and degree -1.) We will focus on this fibration in the case of $n = 2$, as in this case ϱ is simply the blow-up of the diagonal Δ in $\mathrm{Sym}^n(X)$. If $(b_1, b_2) \in \mathrm{Sym}^2(B) \setminus \Delta_B$, the fibre $(\pi^{[2]})^{-1}(b_1, b_2)$ is $T \times T$. For a point $(b, b) \in \Delta_B$, the fibre is given by the union of two irreducible components:

$$\mathbb{P}(N_{\mathrm{Sym}^2(X)/\Delta}|_{(\mathrm{Sym}^2(\pi))^{-1}(b,b)}) \cong \mathbb{P}(\mathcal{T}_X|_{T_b})$$

and $\mathrm{Sym}^2(T)$. The symmetric product $\mathrm{Sym}^2(T)$ can naturally be thought of as the set of effective divisors of degree 2, and it has a ruled surface structure given by sending a divisor to its linear equivalence class in $\mathrm{Pic}^2(T) \cong T$. The intersection of these components is given by the diagonal of $\mathrm{Sym}^2(T)$ and the section $\mathbb{P}(T_{X/B}|_{T_b})$ in $\mathbb{P}(TX|_{T_b})$. Since TX is isomorphic to $\pi^*\mathcal{A}$ with $\mathcal{A}$ the Atiyah bundle, $TX|_{T_b}$ is trivial and $T_{X/B}|_{T_b}$ is the mapping onto the second factor.

2.2.2 Topology

Using a result of de Cataldo and Migliorini [dCM00] (due to Göttsche [Göt90] in the projective case), we have that the Betti numbers of $X^{[n]}$ satisfy

$$\sum_{n=1}^{\infty} p(X^{[n]}, t) q^n = \prod_{k=1}^{\infty} \prod_{j=0}^{4} \left(1 - (-t)^{2k-2+j} q^k\right)^{(-1)^{j+1} b_j(X)}, \tag{2.4}$$

where $p(X^{[n]}, t) = \sum_{j \geq 0} b_j(X^{[n]}) t^j$ is the Poincaré polynomial. (Note that truncating the product on the right at $k = n$ gives the correct coefficient for q^i for each $i \leq n$, so the Betti numbers of $X^{[n]}$ for a particular choice of n can be computed with this formula.) In addition, we have from [Bea83] that $\pi_1(X^{[n]}) \cong H_1(X, \mathbb{Z})$.

If X is also Kähler, the formula (2.4) can be refined to also compute the Hodge numbers by

$$\sum_{n=1}^{\infty} h(X^{[n]}, x, y) t^n = \prod_{k=1}^{\infty} \prod_{p=0}^{2} \prod_{q=0}^{2} \left(1 - (-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k\right)^{(-1)^{p+q+1} h^{p,q}(X)}, \tag{2.5}$$

where $h(X^{[n]}, x, y) = \sum_{p \geq 0, q \geq 0} h^{p,q}(X^{[n]}) x^p y^q$ is the Hodge-Poincaré polynomial.

2.3 Compact holomorphic symplectic manifolds

Definition 2.8. A *holomorphic symplectic manifold* is a pair (X, η) consisting of X a $2m$-dimensional complex manifold and $\eta \in H^0(X, \Omega_X^2)$ a closed holomorphic 2-form such that η^m is nowhere-vanishing. A holomorphic symplectic manifold (X, η) is *irreducible* if X is simply connected, admits a Kähler metric, and

$$H^0(Z, \Omega_Z^k) = \begin{cases} 0 & k \text{ is odd,} \\ \mathbb{C}[\eta^{\wedge k/2}] & k \text{ is even,} \end{cases}$$

for all $0 \leq k \leq 2m$.

Remark 2.9. A holomorphic symplectic manifold of dimension $2m$ is irreducible if and only if it is irreducible hyperkähler, meaning it is simply connected and admits a metric with holonomy $\mathrm{Sp}(m)$.

The study of holomorphic symplectic manifolds began with Bogomolov's classification theorem for compact Kähler manifolds with trivial first Chern class.

Theorem 2.10 (Bogomolov [Bog74]). *Let X be a compact Kähler manifold with $c_1(X) = 0$. Up to a finite étale cover $\tilde{X}$, X decomposes uniquely into a product*

$$\tilde{X} = T \times \prod_{i=1}^{k} Y_i \times \prod_{j=1}^{\ell} Z_j,$$

where the Y_i are irreducible Calabi-Yau manifolds, the Z_j are irreducible holomorphic symplectic manifolds, and T is a complex torus.

In the above statement, an irreducible Calabi-Yau manifold Y is a simply connected Kähler manifold of dimension $n \geq 3$ such that K_Y is trivial and $H^0(Y, \Omega_Y^k) = 0$ for $0 < k < n$.

Holomorphic symplectic manifolds are also interesting from the standpoint of their deformation theory. The deformations of an irreducible holomorphic symplectic manifold

(X, η) are always unobstructed, and are determined by the weight-2 Hodge structure together with the *Beauville-Bogomolov-Fujiki form* $q : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{R}$,

$$q(\alpha, \beta) = \lambda \int_X \alpha \wedge \beta \wedge \eta^{m-1} \wedge \overline{\eta}^{m-1} - \frac{\lambda(m-1)}{2m} \frac{\left(\int_X \alpha \wedge \eta^{m-1} \wedge \overline{\eta}^m\right)\left(\int_X \beta \wedge \eta^m \wedge \overline{\eta}^{m-1}\right)}{\int_X \eta^m \wedge \overline{\eta}^m},$$

where λ is the unique positive real number such that q is integral and primitive.

Example 2.2. The simplest example of a compact holomorphic symplectic manifold is a complex-$2n$ torus. The fact that this is holomorphic symplectic follows immediately since $\mathbb{C}^{2n}$ has a translation-invariant holomorphic symplectic structure given by

$$\eta := \sum_{k=1}^{n} \mathrm{d}z_{2k-1} \wedge \mathrm{d}z_{2k}.$$

The Hodge diamond of a $2n$-dimensional complex torus is given by

$$\mathrm{h}^{p,q} = \binom{2n}{p}\binom{2n}{q}$$

since any Dolbeault cohomology class can be represented by a wedge product of p $\mathrm{d}z_i$s and q $\mathrm{d}\overline{z}_j$s. In dimension 2, this gives the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}.$$

Example 2.3. A *K3 Surface* is a simply connected compact complex surface with trivial canonical bundle. Surfaces of this type can be constructed as smooth degree-4 hypersurfaces of $\mathbb{P}^3$ or as double covers of $\mathbb{P}^2$ branched along a smooth sextic curve. The holomorphic symplectic form is given by choosing any non-zero section of the canonical bundle. The Hodge diamond of a K3 surface is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Example 2.4. A *Primary Kodaira Surface* (henceforth referred to as a Kodaira surface) is a holomorphic symplectic surface that admits no Kähler structure. A Kodaira surface can be constructed either as in Proposition 2.2 with a genus-1 base curve B, or as the quotient of the nilpotent Lie group

$$\left\{ \begin{bmatrix} 1 & \overline{z} & w \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : z, w \in \mathbb{C} \right\} \subseteq \mathrm{GL}(3, \mathbb{C})$$

by a subgroup Γ with generators g_1, g_2, g_3, g_4 such that the centre of Γ is $< g_1, g_2 >$ and $g_3 g_4 = g_1^d g_4 g_3$, where d is the same positive integer that appears in the statement of Proposition 2.2. The fundamental group of this surface is Γ, and its abelianisation is $\mathbb{Z}^3 \oplus \mathbb{Z}/d\mathbb{Z}$. The Hodge diamond of a Kodaira surface is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array}.$$

The above three cases together give all possible examples in dimension two by the Enriques–Kodaira classification of complex surfaces. To construct examples in higher dimensions, we can use the following two results:

Theorem 2.11 (Beauville [Bea83])**.** *If S is a holomorphic symplectic surface, then the Douady space of points $S^{[n]}$ is holomorphic symplectic.*

If the surface S is a K3-surface, the resulting Douady space is irreducible. Families in this deformation class are called of K3$^{[n]}$-type. Manifolds of this type have second Betti number $b_2 = 23$ when $n \neq 1$. If S is a complex torus, the Douady space of n points admits a map

$$\begin{aligned} \Sigma : & S^{[n]} \to S \\ & Z \mapsto \sum_{z \in Z} \ell_z(Z) z, \end{aligned}$$

where the sum is taken with respect to the group structure on the torus. The general fibre is irreducible holomorphic symplectic and is known as the *generalised Kummer variety* of $S^{[n]}$. The corresponding deformation class is called Kum$_{n-1}$-type.

Theorem 2.12 (Mukai, [Muk84])**.** *If S is a holomorphic symplectic surface, the moduli space $\mathcal{M}_v^h(S)$ of h-stable sheaves $\mathcal{E}$ with* $\mathrm{ch}(\mathcal{E}) = v$ *is holomorphic symplectic with symplectic form*

$$\eta_{\mathcal{E}} : \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \otimes \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$$

given by the Yoneda product, where h is a polarisation/Kähler metric/Gauduchon metric and we employ the isomorphisms $(\mathcal{T}\mathcal{M}_v^h(S))_{\mathcal{E}} \simeq \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ *and* $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}$*.*

An analogous result also holds for moduli spaces $\mathcal{M}_{r,\delta,c}^h(S)$ of h-stable sheaves $\mathcal{E}$ with $\operatorname{rank} \mathcal{E} = r, \det(\mathcal{E}) = \delta, c_2(\mathcal{E}) = c$.

Remark 2.13. Note that Theorem 2.11 is a special case of this result, as $S^{[n]} \cong \mathcal{M}_{1,\mathcal{O}_S,n}(S)$.

If S is a K3 surface and $\mathcal{M}_{r,\delta,c}(S)$ is compact of dimension $2n$, it is of K3$^{[n]}$-type [O'G97]. If S is a complex torus and $\mathcal{M}_{r,\delta,c}(S)$ is compact of dimension $2n$, there is an Albanese map from $\mathcal{M}_{r,\delta,c}(S)$ to S whose fibre is of Kum$_{n-1}$-type [Yos01].

Outside of the irreducible holomorphic symplectic manifolds obtained via the previous two results, there are two exceptional deformation families found by O'Grady. Examples of type OG10 occur in dimension 10 as smooth compactifications of the moduli space of stable sheaves on a K3 surface with Chern character divisible by 2 [O'G99]. A similar construction in dimension 6 with an Abelian surface instead of a K3 surface gives examples of type OG6 [O'G03]. (The compactification is a symplectic resolution of the corresponding moduli space of semistable sheaves.)

In the non-Kähler case, in addition to the spaces constructed via Theorems 2.11 and 2.12, we have the Bogomolov-Guan manifolds.

Example 2.5 (Bogomolov-Guan manifolds)**.** Let $\pi : S \to B$ be a Kodaira surface. Then $S^{[n]}$ is a holomorphic symplectic manifold with holomorphic symplectic form η and there is a natural Lagrangian fibration $\pi^{[n]} : S^{[n]} \to B^{[n]}$ as discussed in Section 2.2.1. After fixing an elliptic curve structure on B, there is a natural summation map $\Sigma : B^{[n]} \to B$ given by

$$\{x_1, \ldots, x_n\} \mapsto \sum_{i=1}^{n} x_i,$$

where the sum is taken with respect to the group structure on B. The manifold

$$Y := (\pi^{[n]})^{-1}(\Sigma^{-1}(0))$$

is a smooth hypersurface of $S^{[n]}$ and therefore a co-isotropic submanifold. Quotienting by the characteristic foliation of Y [Saw09] with respect to η results in a holomorphic symplectic space of dimension $2n-2$ which admits a smooth and simply connected symplectic resolution. (See [Bog96, Gua95] for more explicit constructions.)

Chapter 3

Slope Stability for Vector Bundles and V-Pairs

In order to study moduli spaces of vector bundles, it is necessary to restrict to those which satisfy a stability condition to obtain a moduli space that is separated. This stability condition involves a bound on the positivity (as defined with respect to a Gauduchon metric or an ample line bundle) of subsheaves of a vector bundle. In this chapter, we begin by giving an overview of stability conditions and degree computations following [LT95]. In the particular case of vector bundles on non-Kähler elliptic surfaces, we discuss reducibility of the vector bundles, which determines whether the stability condition is trivial. For reducible bundles, we show how one can compute destabilising subbundles, and we give a numerical criterion to determine which moduli spaces admit reducible bundles. Finally, we end the chapter by extending our discussion of stability conditions from vector bundles to V-pairs, which consist of a vector bundle E together with a holomorphic section $\phi \in H^0(X, \mathcal{E}nd_0(E) \otimes V)$ for a fixed vector bundle V on X.

3.1 Slope stability for Gauduchon manifolds

When studying moduli spaces of vector bundles on a complex curve C, in order to construct a well-behaved moduli space it is necessary to restrict to those bundles which are *(semi)-stable*: a vector bundle E is (semi)-stable if for every coherent subsheaf $\mathcal{F}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(E)$ we have

$$\frac{\deg(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} < (\leq) \frac{\deg(E)}{\operatorname{rank}(E)},$$

where

$$\deg(\cdot) := \int_C c_1(\cdot).$$

Note that the value of deg is invariant with respect to the form representing c_1 by Stokes' theorem.

For higher-dimensional complex manifolds, we can generalise this slope stability concept in the presence of a Gauduchon metric g, i.e. a Hermitian metric on an n-dimensional complex manifold whose associated 2-form ω_g satisfies $\partial\bar{\partial}\omega_g^{n-1} = 0$. (Up to scaling by a constant, there is a unique Gauduchon metric in the conformal class of any Hermitian metric on a connected compact complex manifold of dimension $n \geq 2$, so every complex manifold admits a Gauduchon metric [Gau84].) Given a Hermitian vector bundle (E, h), we can extend the degree function to

$$\deg_g(E) := \frac{i}{2\pi} \int \operatorname{Tr}(F_h) \wedge \omega_g^{n-1},$$

where F_h is the curvature of the Chern connection on (E, h). Varying the Hermitian metric h will change $\operatorname{Tr}(F_h)$ by a $\partial\bar{\partial}$-exact 2-form, so $\deg_g(L)$ depends only on the Gauduchon metric and the holomorphic isomorphism class of E. If g is a Kähler metric, the above degree formula depends on E only up to its first Chern class, and is therefore a topological invariant. When X is projective, we often ask that the metric g be taken to be the restriction of the Fubini–Study metric under some closed embedding, which is equivalent to asking that the Kähler form of g represents the first Chern class of some ample line bundle H. In this case, H is called a *polarisation* of X. We now say that a vector bundle E is *g-(semi)-stable* if for every coherent subsheaf $\mathcal{F}$ with $0 < \operatorname{rank}\mathcal{F} < \operatorname{rank} E$,

$$\mu_g(\mathcal{F}) := \frac{\deg_g(\mathcal{F})}{\operatorname{rank}\mathcal{F}} < (\leq) \frac{\deg_g(E)}{\operatorname{rank} E} =: \mu_g(E).$$

Stable vector bundles are intimately related to solutions of the Hermitian-Einstein equations which, given a holomorphic vector bundle E on a Gauduchon manifold (X, g), asks for a Hermitian metric on E such that

$$i\Lambda_{\omega_g} F_h = \gamma \mathrm{Id}_E$$

for some $\gamma \in \mathbb{R}$, where Λ_{ω_g} is the adjoint of the map $\cdot \wedge \omega_g$ [LT95]. A vector bundle E admits an irreducible Hermitian-Einstein metric with constant γ if and only if E is stable with

$$\gamma = \frac{2\pi\mu_g(E)}{(n-1)!\mathrm{Vol}_g(X)}.$$

This correspondence was demonstrated in [Don85, Don87] for projective manifolds, [UY86] for compact Kähler manifolds, and [Buc88, LY87] for general compact complex manifolds.

If X is a non-Kähler complex surface, $\deg_g : \mathrm{Pic}(X) \to \mathbb{R}$ is a smooth Lie group homomorphism such that $\deg_g|_{\mathrm{Pic}^0(X)}$ is surjective [LT95, Proposition 1.3.13]. In the case where X is a non-Kähler elliptic surface, this property uniquely determines the degree map up to scaling for line bundles with torsion first Chern class. Recall from Section 2.1.1 that any line bundle with torsion first Chern class on a principal elliptic surface $\pi : X = \Theta^*/\langle\tau\rangle \to B$ can be written as $\pi^* H \otimes L_\alpha$, where $H \in \mathrm{Pic}(B)$, L_α is the line bundle with constant factor of automorphy α, and $\pi^*\Theta \otimes L_\tau \cong \mathcal{O}_X$. Therefore, for any Gauduchon metric on g, the degree map satisfies

$$\deg_g(\pi^* H \otimes L_\alpha) = c\left(\deg(H) - \frac{d \log|\alpha|}{\log|\tau|}\right) \tag{3.1}$$

for some $c \in \mathbb{R}_{>0}$, where $d = \deg(\Theta)$. In the sequel we will assume that for principal elliptic surfaces $\pi : X \to B$, the metric g is chosen so that

$$\deg_g(\pi^* H) = \deg(H)$$

for all $H \in \mathrm{Pic}(B)$. The above still holds if $\pi : X \to B$ has multiple fibres, with the additional constraint that for any multiple fibre T_i with multiplicity m_i, $\deg_g(\mathcal{O}_X(T_i)) = 1/m_i$ for any g which has been normalised as above.

3.2 Reducibility and destabilising bundles

For this section, we assume that X is a compact non-algebraic complex surface. Vector bundles on these surfaces break up into two categories: those which are reducible, and those which are irreducible.

Definition 3.1. A rank-r sheaf $\mathcal{E}$ is *reducible* if there is a subsheaf $\mathcal{F} \subseteq \mathcal{E}$ such that

$$0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}.$$

A rank-r sheaf is *irreducible* if every non-zero subsheaf has rank r.

Remark 3.2. This distinction is unique to non-algebraic complex manifolds; a coherent sheaf defined over an algebraic variety is irreducible if and only if it has rank 1.

Clearly, if a sheaf is irreducible, then it is automatically stable with respect to any Gauduchon metric. Stability of reducible bundles can be investigated using their *maximal destabilising bundles*.

Definition 3.3. Let X be a compact complex surface with Gauduchon metric g, and let E be a vector bundle on X. A locally-free subsheaf $F \subseteq E$ is a *maximal destabilising bundle* for E with respect to g if $0 < \operatorname{rank} F < \operatorname{rank} E$, E/F is torsion-free, and for any positive-degree line bundle L, $\mathrm{Hom}(F \otimes L, E) = 0$.

For certain classes of surfaces, one can compute the destabilising bundles, which simplifies the process for checking stability.

Proposition 3.4 (Friedman [Fri98]). *Let E be a vector bundle on X and let $\mathcal{F} \subseteq E$ be a subsheaf with* $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} E$*. Then:*

1. *There is an effective divisor D so that $\mathcal{F}^{\vee\vee} \otimes \mathcal{O}_X(D)$ is a maximal destabilising bundle.*

2. *If E has rank 2, for any maximal destabilising bundle L of E, there is a finite analytic subspace $Z \subset X$ such that*

$$0 \longrightarrow L \longrightarrow E \longrightarrow \det(E) \otimes L^{-1} \otimes \mathcal{I}_Z \longrightarrow 0$$

is an exact sequence, where $\mathcal{I}_Z$ is the ideal sheaf of Z.

Proposition 3.5 (Brînzănescu–Moraru [BM05b], Proposition 3.3 and Theorem 3.5)**.** *If $\pi : X \to B$ is a non-Kähler elliptic surface and E is a rank-2 vector bundle on X, then E has at most two maximal destabilising bundles, and a maximal destabilising bundle K for E is unique if and only if $\det(E) \otimes K^{-2} \in P_2$. (Here P_2 is the subgroup of $\mathrm{Pic}(X)$ in correspondence with divisors.) If E is an extension of line bundles with two distinct destabilising bundles K_1, K_2, they satisfy the relation*

$$K_1 \otimes K_2 \otimes \det(E)^{-1} \simeq \mathcal{O}_X\left(\sum_{F \in A(E)} -F\right),$$

where $A(E)$ is the set of fibres F of π such that $E|_F$ is not split. When E is not an extension of line bundles, there is an allowable elementary modification of E with the same maximal destabilising bundles [BM05b, Theorem 3.5].

Elementary modifications will be discussed in detail in Section 4.2.

3.3 Stably irreducible moduli spaces

In order to construct compact moduli spaces of stable sheaves, we want to restrict to choices of invariants which do not permit strictly semi-stable sheaves. As shown in [Tom01], the correct condition to impose to get this result for non-Kähler surfaces is to require that the moduli spaces are *stably irreducible*.

Definition 3.6. A moduli space $\mathcal{M}_{r,\delta,c}(X)$ is *stably irreducible* if it is non-empty and every sheaf represented by $\mathcal{M}_{r,\delta,c}(X)$ is irreducible.

Brînzănescu gives a method for finding stably irreducible moduli spaces with the following result:

Proposition 3.7 ([Brî96], Lemma 4.30)**.** *Let X be a compact complex surface, and let $\mathcal{E}$ be a rank-r coherent sheaf with determinant δ and second Chern class c. If $\mathcal{E}$ is reducible, then*

$$\Delta(r,\delta,c) \geq t(r,\delta) := -\frac{1}{2} \sup_{k\in\mathbb{Z}, 0<k<r} \left(\frac{1}{k(r-k)} \sup_{\alpha\in NS(X)} \left(\frac{kc_1(\delta)}{r} - \alpha \right)^2 \right). \tag{3.2}$$

Here $\Delta(r,\delta,c)$ is given by

$$\Delta(r,\delta,c) := \frac{c}{r} - \frac{r-1}{2r^2} c_1^2(\delta). \tag{3.3}$$

The above proposition implies that in order to find stably irreducible moduli spaces of sheaves, it is sufficient to check that the moduli space is non-empty and $\Delta(r,\delta,c) < t(r,\delta)$.

Proposition 3.8. *Let $\pi : X \to B$ be a non-Kähler principal elliptic surface, and set*

$$\nu(X) := \begin{cases} 0, & \text{if } NS(X) \text{ is finite,} \\ \min_{\alpha\in NS(X), \alpha^2\neq 0} (-\alpha^2/2), & \text{otherwise.} \end{cases}$$

If $r \geq 2$, $\delta \in \mathrm{Pic}(X)$ satisfies $c_1^2(\delta) = -2\nu(X)$, and

$$\left(\frac{1-r}{r}\right)\nu(X) \leq c < \left(\frac{2-r}{r-1}\right)\nu(X),$$

then the moduli space $\mathcal{M}_{r,\delta,c}(X)$ is stably irreducible whenever it is non-empty.

Proof. Note that

$$\frac{1}{k(r-k)} \left(\frac{kc_1(\delta)}{r} - \alpha \right)^2$$

is invariant under $(k,\alpha) \mapsto (r-k, c_1(\delta)-\alpha)$ and

$$\left(\frac{kc_1(\delta)}{r} - \alpha\right)^2 \leq \alpha^2 - \frac{k^2}{r^2}c_1(\delta)^2$$

when $k \leq r/2$ by negative semi-definiteness of the intersection product. By choosing δ so that $c_1^2(\delta) = -2\nu(X)$, we obtain

$$\begin{aligned} t(r,\delta) &= -\frac{1}{2} \sup_{k\in\mathbb{Z}, 0<k<r} \left(\frac{1}{k(r-k)} \sup_{\alpha \in NS(X)} \left(\frac{kc_1(\delta)}{r} - \alpha \right)^2 \right) \\ &= -\frac{1}{2} \sup_{k\in\mathbb{Z}, 0<k\leq r/2} \left(\frac{1}{k(r-k)} \sup_{\alpha \in NS(X)} \left(\frac{kc_1(\delta)}{r} - \alpha \right)^2 \right) \\ &= \min_{k\in\mathbb{Z}, 0<k\leq r/2} \left(\frac{k\nu(X)}{r^2(r-k)} \right) \\ &= \frac{\nu(X)}{r^2(r-1)}. \end{aligned}$$

We then note that $\mathcal{M}_{r,\delta,c}(X)$ is stably irreducible if $0 \leq \Delta(r,\delta,c) < t(r,\delta)$, and the result follows after isolating c. □

Remark 3.9. If X is a Kodaira surface, then $\Delta(r,\delta,c) \geq 0$ is a necessary and sufficient condition for $\mathcal{M}_{r,\delta,c}(X)$ to be non-empty [ABT02]. Furthermore, since $\nu(X)$ corresponds to the minimal degree of an isogeny between B and T, we can take $\nu(X)$ to be any non-negative integer by choosing B and T appropriately. As an example of a choice of B and T with $\nu(X) = k$ for some $k \in \mathbb{Z}^+$, take

$$B = \mathbb{C}/\left(\mathbb{Z} + \left(\sqrt{-1}\right)\mathbb{Z}\right), \qquad T = \mathbb{C}/\left(\mathbb{Z} + k\left(\sqrt{-1}\right)\mathbb{Z}\right).$$

Remark 3.10. In the case where $r = 2$ and X has base of genus $g \geq 2$, a sufficient condition for the moduli space $\mathcal{M}_{2,\delta,c}(X)$ to be non-empty is that $\Delta(2,\delta,c) \geq -e_\delta/4$ [BM05a, Theorem 4.5], where e_δ is the invariant associated to the ruled surface $\mathbb{F}_\delta$ as defined in (2.3). Therefore, if $\nu(X) \geq (2-e_\delta)/2$, then there are choices of invariants which give non-empty stably irreducible moduli spaces of sheaves over X.

3.4 Stability for V-pairs

We also define slope stability for V-pairs, which consist of a pair (E, ϕ) with E a holomorphic vector bundle and $\phi : E \to E \otimes V$ a holomorphic map known as the *Higgs field*. In the special cases where $V = \mathcal{T}_X$, $V = K_X$, or $V = \mathcal{T}_X^*$, V-pairs are called *co-Higgs bundles*, *Vafa–Witten pairs*, or *Higgs bundles*, respectively.

Definition 3.11. Let (E, ϕ) be a V-pair. A subsheaf $\mathcal{F} \subseteq E$ is *ϕ-invariant* if $\phi(\mathcal{F}) \subseteq \mathcal{F} \otimes V$. The V-pair (E, ϕ) is *(semi)-stable* if for any ϕ-invariant subsheaf $\mathcal{F}$ of E with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} E$,

$$\mu_g(\mathcal{F}) < (\leq) \mu_g(E).$$

We equivalently say that ϕ is a stable Higgs field for E.

Example 3.1. Let E and V be vector bundles, and suppose E is stable. Then any Higgs field $\phi : E \to E \otimes V$ is automatically stable.

Note that if (E, ϕ) is a V-pair and $\phi = \mathrm{Id}_E \otimes \sigma$ for some section σ of V, then (E, ϕ) is g-stable if and only if E is. Since V-pairs of this type are not interesting from a classification point of view, we typically restrict to V-pairs which are trace-free. Here we are defining the trace by

$$\operatorname{Tr} \sum_{k=1}^{n} A_k \otimes \sigma_k = \sum_{k=1}^{n} \operatorname{Tr} A_k \otimes \sigma_k,$$

where $A_k \in \operatorname{End}(E)$ and $\sigma_k \in H^0(X, V)$.

Example 3.2. Let V be any positive-degree vector bundle on X. Although $\mathcal{O}_X \oplus V$ is not a stable vector bundle,

$$\left(\mathcal{O}_X \oplus V, \phi := \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} : \mathcal{O}_X \oplus V \to V \oplus (V \otimes V) \right)$$

is a stable trace-free V-pair for any $\alpha \in H^0(X, V \otimes V)$, as $\mu(\mathcal{O}_X \oplus V) > 0$ and every ϕ-invariant subsheaf factors through $\mathcal{O}_X$.

Chapter 4

The Spectral Construction for Non-Kähler Elliptic Surfaces

In the case that X is an elliptic fibration, one can obtain an intermediate classification of vector bundles before the moduli space of sheaves using a method known as the *spectral construction*, which assigns to each vector bundle a divisor (the spectral cover) in the relative Jacobian $J(X)$ of X. The map assigning to each vector bundle its spectral cover provides a natural fibration structure on the moduli space of stable vector bundles, simplifying some problems involving the topology of the moduli space. This chapter reviews the spectral construction in the case of rank-2 vector bundles on a non-Kähler elliptic surface as described in [BM05a, BM05b, BM06]. We then discuss the relationship between the *elementary modifications* of a vector bundle and the *jumps* in its spectral cover in order to prove technical results used in Chapters 5 and 6. (The jumps are irreducible components of the spectral cover which correspond to fibres where the restriction to the fibre is not semi-stable.) The most important of these technical results is that, when $\mathcal{M}_{2,\delta,c}(X)$ is a stably irreducible moduli space of sheaves and $\Sigma \subset J(X)$ is the spectral cover of a sheaf in $\mathcal{M}_{2,\delta,c}(X)$, every irreducible component of Σ is smooth.

4.1 The spectral construction

Suppose that X is a non-Kähler principal elliptic surface. If $\mathcal{E} \in \mathrm{Coh}(X)$ is torsion-free, we can associate to $\mathcal{E}$ its Fourier–Mukai transform

$$\mathcal{L}_{\mathcal{E}} = R^1 p_{2*}(\mathcal{U} \otimes {p_1}^* \mathcal{E})$$

on $B \times \mathbb{C}^* \subseteq B \times \mathrm{Pic}^0(X)$, where $\mathcal{U}$ is the universal line bundle on

$$X \times_B (B \times \mathbb{C}^*) \subseteq X \times_B (B \times \mathrm{Pic}^0(X))$$

as discussed in Section 2.1.1, and the p_i's are the morphisms corresponding to the fibred product. The resulting $\mathcal{L}_{\mathcal{E}}$ is a torsion sheaf supported on an effective divisor $\tilde{S}_{\mathcal{E}}$ in $X \times \mathbb{C}^*$, consisting of points (b, a) such that

$$h^1(T, (\mathcal{E} \otimes L_a)|_{\pi^{-1}(b)}) \neq 0,$$

possibly with higher multiplicity. Since $L_\tau \in \pi^*(\mathrm{Pic}(B))$ and all bundles on B pull back to bundles which are trivial on all fibres of π, the divisor $\tilde{S}_{\mathcal{E}}$ descends to a divisor $S_{\mathcal{E}}$ on the relative Jacobian $J(X) = B \times T^*$ of X, where $T^* := \mathrm{Pic}^0(T)$. We call $S_{\mathcal{E}}$ the *spectral curve* of $\mathcal{E}$.

We can describe the spectral curve more concretely as

$$S_{\mathcal{E}} = \{(b, \lambda) \in B \times T^* : H^1(T, \mathcal{E}|_{\pi^{-1}(b)} \otimes \lambda) \neq 0\},$$

with the multiplicity of (b, λ) given by $h^1(T, \mathcal{E}|_{\pi^{-1}(b)} \otimes \lambda)$.

Remark 4.1. The spectral curve of a sheaf $\mathcal{E}$ can also be constructed using a twisted Fourier–Mukai transform between the category of coherent sheaves on X and the category of twisted sheaves on $J(X)$ as described in [BM06].

Since the restriction morphism

$$\iota^* : H^2(X, \mathbb{Z}) \to H^2(T, \mathbb{Z})$$

is zero [Tel98], $\mathcal{E}|_{\pi^{-1}(b)} \otimes \lambda$ is a degree zero bundle for all $(b, \lambda) \in B \times T^*$. This fact together with the classification of vector bundles on genus 1 curves gives that $S_E \cap \{b\} \times \mathrm{Pic}^0(T)$ contains at most $\mathrm{rank}(\mathcal{E})$ points if and only if $\mathcal{E}|_{\pi^{-1}(b)}$ is semistable. Thus the spectral curve has the form

$$S_{\mathcal{E}} = \overline{C} + \sum_{b \in U_{\mathcal{E}}} \ell_b(\{b\} \times T^*)$$

for some integers ℓ_b, where $\overline{C}$ is an r-section of $J(X) \to B$ and

$$U_{\mathcal{E}} := \{b \in B : \mathcal{E}|_{\pi^{-1}(b)} \text{ is unstable}\}.$$

Remark 4.2. If $\pi : X \to B$ has multiple fibres, the spectral curve of a sheaf $\mathcal{E}$ can be constructed via its cyclic cover

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\rho}} & X \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\rho} & B \end{array}$$

by computing the spectral curve of $\tilde{\rho}^*\mathcal{E}$ in $J(Y)$ and setting $S_{\mathcal{E}}$ to be the image of $S_{\tilde{\rho}^*\mathcal{E}}$ by $\rho \times \mathrm{Id}_{T^*}$.

Definition 4.3. We say that a vector bundle E has a *jump* at b if $E|_{\pi^{-1}(b)}$ is unstable, and the *multiplicity* $\ell(E, b)$ of the jump is the multiplicity ℓ_b of $b \times T^*$ in $S_{\mathcal{E}}$.

In the case of rank-2 sheaves, for each $\delta \in \mathrm{Pic}(X)$, we can define an involution

$$\iota_\delta : (b, \lambda) \mapsto (b, \delta|_{\pi^{-1}(b)} \otimes \lambda^{-1})$$

on $J(X)$. (This involution only depends on the class of δ in $\mathrm{Pic}(X)/\pi^*(\mathrm{Pic}(B))$.) The rank-2 sheaves $\mathcal{E}$ with $\det \mathcal{E} = \delta \otimes \pi^*(\lambda)$ for some $\lambda \in \mathrm{Pic}(B)$ are precisely the sheaves whose spectral curves are invariant under the action of ι_δ. Thus these spectral curves descend to the ruled surface $\mathbb{F}_\delta := J(X)/\iota_\delta$ with induced projection $\rho : \mathbb{F}_\delta \to B$. By [BM05a], this ruled surface can be described as $\mathbb{F}_\delta = \mathbb{P}(V_\delta)$, where

$$V_\delta := q_{1*}\left(\mathcal{O}_{J(X)}\left(S_{\mathcal{O}_X} + S_\delta\right)\right),$$

and $q_1 : J(X) \cong B \times T^* \to B$ is the projection map. The bundle V_δ is a rank-2 semi-stable vector bundle on B of degree $-\frac{c_1(\delta)^2}{2}$ [BM05a, Lemma 3.8].

Remark 4.4. A rank-2 torsion-free sheaf $\mathcal{E}$ on X is irreducible if and only if

$$S_{\mathcal{E}} = \overline{C} + \sum_{i=1}^{k} \{b_i\} \times T^*$$

with $\overline{C}$ reduced and irreducible [BM05a]. If $\overline{C} = \Sigma_1 + \Sigma_2$ for some sections Σ_1 and Σ_2 of $J(X) \to B$, then $\mathcal{E}$ has two destabilising bundles K_1 and K_2 with spectral curves $S_{K_1} = \Sigma_1$ and $S_{K_2} = \Sigma_2$, respectively. If $\overline{C} = 2\Sigma$ is non-reduced, then $\mathcal{E}$ has a unique destabilising bundle K with spectral curve $S_K = \Sigma$ [BM05b, Proposition 3.4].

Definition 4.5. For any rank-2 torsion-free sheaf $\mathcal{E}$ with $\det(\mathcal{E}) = \delta$, the *graph* of $\mathcal{E}$ is the set $S_{\mathcal{E}}/\iota_\delta \subset \mathbb{F}_\delta$, where $\mathbb{F}_\delta$ is the ruled surface associated to δ as defined in Section 2.1.1.

Proposition 4.6 (Brînzănescu–Moraru [BM05a])**.** *Let $\mathcal{E}$ be a rank-2 torsion-free sheaf with determinant δ and second Chern class c, and let G be the graph of $\mathcal{E}$. Then G is an effective divisor linearly equivalent to $A_\delta + \rho^*\mathfrak{b}$, where A_δ is the graph of $\mathcal{O}_X \oplus \delta$, $\rho : \mathbb{F}_\delta \to B$ is the induced projection map, and $\mathfrak{b} \in \mathrm{Pic}^c(B)$.*

Definition 4.7. Given a line bundle $\delta \in \mathrm{Pic}(X)$ and an integer c, the *space of graphs* $\mathbb{P}_{\delta,c}$ is the set of all divisors in $\mathbb{F}_\delta$ linearly equivalent to $A_\delta + \rho^*\mathfrak{b}$ for some $\mathfrak{b} \in \mathrm{Pic}^c(B)$. The *graph map* $\mathcal{G} : \mathcal{M}_{2,\delta,c}(X) \to \mathbb{P}_{\delta,c}$ is the holomorphic map taking each sheaf $\mathcal{E} \in \mathcal{M}_{2,\delta,c}(X)$ to its graph. (Equivalently, $\mathcal{G}$ is the map sending each sheaf to its spectral curve. We use these two definitions interchangeably.)

4.2 Spectral curves and elementary modifications

In order to understand vector bundles with jumps, the main method is to study their *elementary modifications*. Given a rank-2 vector bundle E on a complex manifold X, a smooth effective divisor D, a line bundle λ on D, and a surjective sheaf map $g : E|_D \to \lambda$, the *elementary modification E' of E by* (D, λ) is the unique vector bundle satisfying the exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \iota_*\lambda \longrightarrow 0,$$

where $\iota : D \to X$ is the inclusion map. The invariants of an elementary modification are given by

$$\det(E') = \det(E) \otimes \mathcal{O}_X(-D), \qquad c_2(E') = c_2(E) + c_1(E).[D] + \iota_* c_1(\lambda).$$

In the case that $\pi : X \to B$ is a non-Kähler elliptic surface and D is a prime divisor, the divisor D is of the form $\pi^{-1}(b)$ for some $b \in B$. Since D has torsion first Chern class, the determinant and second Chern class are related by

$$\det(E') = \det(E) \otimes \pi^*(\mathcal{O}_B(-b)), \qquad c_2(E') = c_2(E) + \deg(\lambda)$$

and the elementary modification E' then has $\ell(E', b) = \ell(E, b) + \deg(\lambda)$ in this case.

If a vector bundle E has a jump at b, there is a unique elementary modification of E along $\pi^{-1}(b)$ by a negative degree bundle, called the *allowable elementary modification* of E at b [Mor03, Section 4.1.2]; in particular, since $E|_{\pi^{-1}(b)}$ is unstable, it is of the form $\lambda \oplus (\lambda^* \otimes \det(E)|_{\pi^{-1}(b)})$ for some $\lambda \in \mathrm{Pic}^{-h}(T^*)$ with $h > 0$, with the map $g : E|_{\pi^{-1}(b)} \to \lambda$ given by projection onto the first coordinate.

Proposition 4.8 (Brînzănescu–Moraru [BM05a]). *If $\mathcal{E}$ is a rank-2 torsion-free sheaf on X, then $\mathcal{E}$ has finitely many jumps, and*

$$\sum_{b \in U_{\mathcal{E}}} \ell(\mathcal{E}, b) \leq 2\Delta(\mathcal{E}).$$

Proof. First, suppose that $\mathcal{E}$ is not locally free. In this case, $\mathcal{E}^{\vee\vee}/\mathcal{E}$ is a torsion sheaf supported at m points with multiplicity, and $\mathcal{E}^{\vee\vee}$ is a vector bundle satisfying

$$\Delta(\mathcal{E}^{\vee\vee}) = \Delta(\mathcal{E}) - \frac{m}{r}, \qquad S_{\mathcal{E}^{\vee\vee}} = S_{\mathcal{E}} - \sum_{x \in \mathrm{Supp}(\mathcal{E}^{\vee\vee}/\mathcal{E})} \{\pi^{-1}(\pi(x))\} \times T^*.$$

Since $\Delta(\mathcal{E}^{\vee\vee}) \geq 0$, the support of $\mathcal{E}^{\vee\vee}/\mathcal{E}$ must be finite, so we can reduce to the case of vector bundles. Now suppose that $\mathcal{E}$ is a vector bundle. If $\mathcal{E}|_{\pi^{-1}(b)}$ is unstable, it must split by the classification of rank-2 bundles on elliptic curves with

$$\mathcal{E}|_{\pi^{-1}(b)} = L \oplus (\det(\mathcal{E}|_{\pi^{-1}(b)}) \otimes L^{-1}$$

for some $L \in \text{Pic}^{-k}(T)$ with $k > 0$, and the allowable elementary modification

$$0 \longrightarrow E \longrightarrow \mathcal{E} \longrightarrow \iota_* L \longrightarrow 0$$

is a vector bundle with $c_1^2(E) = c_1^2(\mathcal{E})$ and $c_2(E) = c_2(\mathcal{E}) - k$, where $\iota : \pi^{-1}(b) \to X$ is the inclusion map. Since $\Delta(E) \geq 0$, we must have $k \leq 2\Delta(\mathcal{E})$. We can iterate this process across all unstable fibres to see that there can only be finitely many, as all vector bundles have non-negative discriminant. Therefore, considering both cases, we have that $\mathcal{E}$ admits finitely many jumps and the sum of their multiplicities is at most $2\Delta(\mathcal{E})$. □

By contrast, elementary modifications by positive-degree line bundles are highly non-unique; if a vector bundle E satisfies

$$E|_{\pi^{-1}(b)} \cong L \otimes (L^* \otimes \det(E)|_{\pi^{-1}(b)})$$

with

$$L \in \text{Pic}^h(T^*), \qquad h \geq 0, \qquad L \ncong L^* \otimes \det(E)|_{\pi^{-1}(b)},$$

then E has an elementary modification at b by λ for every $\lambda \in \text{Pic}^r(T^*)$ with $r \geq h$ [Mor03, Section 4.1.3]. Two elementary modifications by λ corresponding to maps

$$g_1 : E|_{\pi^{-1}(b)} \to \lambda, \qquad g_2 : E|_{\pi^{-1}(b)} \to \lambda$$

are isomorphic if and only if there is a bundle automorphism φ of E so that $g_1 = g_2 \circ \varphi|_{\pi^{-1}(b)}$.

Finally, for the case of an elementary modification by a degree zero line bundle, the behaviour of the elementary modification depends on whether the initial bundle is *regular*.

Definition 4.9. A rank-2 vector bundle E on X is *regular at* b for some $b \in B$ if $E|_{\pi^{-1}(b)}$ is semi-stable and not isomorphic to $\lambda \oplus \lambda$ for any $\lambda \in \text{Pic}(T)$. Moreover, E is *regular* if it is regular at b for every $b \in B$.

The regular rank-2 vector bundles with fixed irreducible spectral curve $\overline{C}$ are classified via the following result:

Proposition 4.10 (Brînzănescu–Moraru [BM06]). *Let $\overline{C}$ be an irreducible bisection of $J(X)$ with normalisation C. If we set W to be the normalisation of $X \times_B C$ with induced projections $\gamma : W \to X$ and $\psi : W \to C$, then:*

i. There is a line bundle L on W such that $\gamma_*(L)$ is a regular rank-2 bundle on X with spectral curve $\overline{C}$.

ii. If $L_1, L_2 \in \mathrm{Pic}(W)$ are such that $\gamma_*(L_1)$ and $\gamma_*(L_2)$ both have spectral curve $\overline{C}$, then $L_1 \otimes L_2^{-1} = \psi^*(\lambda)$ for some $\lambda \in \mathrm{Pic}(C)$, and $\gamma_*(L_1) \cong \gamma_*(L_2)$ if and only if $L_1 \cong L_2$.

Using the notation of the above Proposition, we also have that if $L \in \mathrm{Pic}(W)$ is such that $\gamma_*(L)$ has spectral curve $\overline{C}$ and $\det(\gamma_*(L)) = \delta$, then for any $\lambda \in \mathrm{Pic}(C)$,

$$\det(\gamma_*(L \otimes \psi^*(\lambda))) \cong \delta \otimes \eta_n(\lambda),$$

where $\eta : C \to B$ is the ramified covering induced from $J(X)$, and $\eta_n : \mathrm{Pic}(C) \to \mathrm{Pic}(B)$ is the norm homomorphism of η, which is the group homomorphism defined by

$$\eta_n(\mathcal{O}_C(p)) = \mathcal{O}_B(\eta(p))$$

for all $p \in C$. Because of this, the regular rank-2 bundles on X with determinant δ and spectral curve $\overline{C}$ are of the form $\gamma_*(L \otimes \psi^*(\lambda))$, where $\lambda \in \mathrm{Prym}(C/B) := \ker(\eta_n)$. (See [BM06, Theorem 4.5] for more details.)

An elementary modification of a vector bundle E at b by a degree zero bundle λ exists if and only if $E|_{\pi^{-1}(b)}$ is an extension of λ by another degree-zero line bundle λ'. If E is regular at b, there is a unique surjection from $E|_{\pi^{-1}(b)}$ to λ up to composing with an automorphism of λ, so there is a unique elementary modification. If E admits an elementary modification by λ at b but E is not regular at b, then $E|_{\pi^{-1}(b)} \cong \lambda \oplus \lambda$. In this case, the surjections from $E|_{\pi^{-1}(b)}$ to λ are parameterised by $\mathbb{C}^2$, and since constant multiples of a surjection induce the same elementary modification, the elementary modifications of E by λ at b are parameterised by $\mathbb{P}^1$.

Lemma 4.11. *Let E be a rank-2 vector bundle with spectral curve $\overline{C}$, and let E' be an elementary modification*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \iota_* \lambda \longrightarrow 0, \tag{4.1}$$

where $\iota : \pi^{-1}(b) \to X$ is the inclusion map for some $b \in B$ and $\lambda \in \mathrm{Pic}^0(T)$. Then E' is regular at b if and only if E is.

Proof. If b is of the form $\pi(c)$ where c is a smooth point of $\overline{C}$, then every vector bundle with spectral curve $\overline{C}$ is regular at b [Mor03], so we can restrict to the case where $\pi^{-1}(b)$ contains a singular point of $\overline{C}$. In this case, a vector bundle V with spectral curve $\overline{C}$ has that $V|_{\pi^{-1}(b)}$ is an extension of λ by itself, and

$$V \text{ is regular at } b \Leftrightarrow h^1(T, V|_{\pi^{-1}(b)} \otimes \lambda^{-1}) = 1. \tag{4.2}$$

Let $L \in \mathrm{Pic}^0(X)$ be a bundle with constant factor of automorphy so that $L|_{\pi^{-1}(b)} = \lambda^{-1}$. Then taking the tensor product of the exact sequence (4.1) with L gives

$$0 \longrightarrow E' \otimes L \longrightarrow E \otimes L \longrightarrow \iota_* \mathcal{O}_T \longrightarrow 0,$$

and pushing forward by π gives the long exact sequence

$$\begin{array}{c} 0 \longrightarrow \pi_*(E' \otimes L) \longrightarrow \pi_*(E \otimes L) \longrightarrow (\pi \circ \iota)_* \mathcal{O}_T \\ \swarrow \\ R^1\pi_*(E' \otimes L) \longrightarrow R^1\pi_*(E \otimes L) \longrightarrow R^1(\pi \circ \iota)_* \mathcal{O}_T \longrightarrow 0. \end{array}$$

If V is any vector bundle on X so that $H^0(T, V|_{\pi^{-1}(p)}) \neq 0$ for at most finitely many $p \in B$, then $\pi_* V$ is zero, and $R^1\pi_* V$ is a torsion sheaf with stalks $(R^1\pi_* V)_p \cong H^1(T, V|_{\pi^{-1}(p)})$. (For details see [BM06].) Furthermore, it is easy to check that since $\pi \circ \iota : \pi^{-1}(b) \to B$ is a constant map, $(\pi \circ \iota)_* \mathcal{O}_T \cong R^1(\pi \circ \iota)_* \mathcal{O}_T \cong \mathbb{C}_b$, where $\mathbb{C}_b$ is the skyscraper sheaf supported at b. From this, the above exact sequence reduces to

$$0 \longrightarrow \mathbb{C}_b \longrightarrow R^1\pi_*(E' \otimes L) \longrightarrow R^1\pi_*(E \otimes L) \longrightarrow \mathbb{C}_b \longrightarrow 0.$$

If we now consider the exact sequence induced from the stalks at b, we have

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(T, E'|_{\pi^{-1}(b)} \otimes \lambda^{-1}) \longrightarrow H^1(T, E|_{\pi^{-1}(b)} \otimes \lambda^{-1}) \longrightarrow \mathbb{C} \longrightarrow 0,$$

so $h^1(T, E'|_{\pi^{-1}(b)} \otimes \lambda^{-1}) = h^1(T, E|_{\pi^{-1}(b)} \otimes \lambda^{-1})$. Together with (4.2), this implies that E' is regular at b if and only if E is. ☐

Pairing this lemma with Proposition 4.10 leads to the following result:

Proposition 4.12. *If $\overline{C}$ is an irreducible bisection of $J(X)$, then every vector bundle with spectral curve $\overline{C}$ is regular.*

Proof. Suppose that E is a bundle with spectral curve $\overline{C}$. By [BM06, Theorem 4.1], there is a vector bundle E_0 with spectral curve $\overline{C}$ which is regular. E_0 is an elementary modification of the pushforward of a line bundle on W by a degree-zero bundle, so by Lemma 4.11 we can assume that $E_0 = \gamma_*(L_0)$ for some $L_0 \in \text{Pic}(W)$. We also have that there is some $L_1 \in \text{Pic}(W)$ so that E is an elementary modification of $\gamma_*(L_1)$ by a degree-zero bundle. Since $\gamma_*(L_0)$ and $\gamma_*(L_1)$ have the same spectral curve, there is a line bundle $H \in \text{Pic}(C)$ so that $L_0 \otimes L_1^{-1} \cong \psi^*(H)$, where $\psi : W \to C$ is the map induced from the fibred product [BM06, Theorem 4.5]. Choose effective divisors D_0, D_1 on C so that $H = \mathcal{O}_C(D_0 - D_1)$. Since for any $c \in C$, the pushforward $\gamma_*(L \otimes \psi^*(\mathcal{O}_C(-c)))$ is an elementary modification of $\gamma_*(L)$ by a degree-zero bundle, there are sequences of elementary modifications by degree-zero bundles taking $\gamma_*(L_1 \otimes \psi^*(\mathcal{O}_C(-D_1)))$ to E and E_0 to $\gamma_*(L_0 \otimes \psi^*(\mathcal{O}_C(-D_0)))$, respectively. Since E_0 is regular, we have that $\gamma_*(L_0 \otimes \psi^* \mathcal{O}_C(-D_0))) \cong \gamma_*(L_1 \otimes \psi^*(\mathcal{O}_C(-D_1)))$ is regular by Lemma 4.11, and we similarly get that E is regular since there is a chain of elementary modifications taking $\gamma_*(L_1 \otimes \psi^*(\mathcal{O}_C(-D_1)))$ to E. ☐

Since all of the rank-2 bundles with irreducible spectral curve $\overline{C}$ can be expressed as the pushforward of a line bundle on the normalisation of $X \times_B C$, the bundles with determinant δ and spectral curve $\overline{C}$ are parameterised by $\text{Prym}(C/B)$ [BM06, Theorem 4.5].

In the case of a rank-2 vector bundle E with smooth and irreducible spectral curve C, we have $g(C) = 4\Delta(E) + 1$ [BM05a, Lemma 3.10]. Similar computations give

$$p_a(S_{\mathcal{E}}) = 4\Delta(\mathcal{E}) + 1 \tag{4.3}$$

for any rank-2 stably irreducible sheaf $\mathcal{E}$.

Proposition 4.13. *If E is a rank-2 stably irreducible sheaf, and the spectral curve S_E contains no jumps, then S_E is smooth.*

Remark 4.14. For the case of $\Delta(2, \delta, c) = \frac{1}{4}$ with X a Kodaira surface, this result is given in [AMT12, Corollary 4.4].

Proof. Let E be a rank-2 stably irreducible sheaf in $\mathcal{M}_{2,\delta,c}(X)$, and suppose for a contradiction that S_E consists of a singular irreducible bisection with no jumps. Let C be the normalisation of S_E. Then $g(C) \leq 4\Delta(E)$ by (4.3), so $\dim(\mathcal{G}^{-1}(S_E)) = \dim \mathrm{Prym}(C/B) \leq 4\Delta(E) - 1$. Now take E' to be any sheaf in $\mathcal{M}_{2,\delta,c}(X)$ whose spectral curve is smooth. Since the arithmetic genus of a spectral curve depends only on $\Delta(2, c_1(E), c(E))$, we have $g(S_{E'}) = \rho_a(S_E) = 4\Delta(E) + 1$ from [BM05a, Lemma 3.10]. We assumed $S_{E'}$ is smooth, so we have $\dim(\mathcal{G}^{-1}(S_{E'})) = \dim \mathrm{Prym}(S_{E'}/B) = 4\Delta(E)$. Since a general stably irreducible sheaf has smooth spectral curve, the fibres of the graph map $\mathcal{G} : \mathcal{M}_{2,\delta,c}(X) \to \mathbb{P}_{\delta,c}$ have dimension $4\Delta(E)$ outside a proper Zariski-closed subset of $\mathbb{P}_{\delta,c}$. This is a contradiction since the fibre dimension of a holomorphic map is upper semi-continuous, so S_E is smooth when it has no jumps. □

Chapter 5

Moduli Spaces of Stably Irreducible Sheaves on Kodaira Surfaces

The study of holomorphic symplectic manifolds began with Bogomolov's classification of compact Kähler manifolds with trivial canonical bundle. These manifolds decompose up to finite étale cover as a product of a complex torus, irreducible Calabi-Yau manifolds, and irreducible holomorphic symplectic manifolds [Bea11, Bog74]. It is generally very difficult to construct compact examples of holomorphic symplectic manifolds; nearly all constructions make use of the fact that the Hilbert scheme (or Douady space) of points over a holomorphic symplectic surface is holomorphic symplectic [Bea83], as is a smooth and compact moduli space of stable sheaves with fixed Chern character on a hyperkähler surface [Muk84].

By the Enriques-Kodaira classification, all compact holomorphic symplectic surfaces are complex tori, K3 surfaces, or primary Kodaira surfaces. Each of these holomorphic symplectic surfaces generates an infinite family of holomorphic symplectic manifolds via its Hilbert schemes (or Douady spaces) of points [Bea83]. These give rise to generalised Kummer varieties in the case of complex tori, and Bogomolov-Guan manifolds in the case of primary Kodaira surfaces [Bog96, Gua95]. For K3 surfaces and complex tori it has been shown that the moduli spaces of stable sheaves with fixed Chern character are deformation equivalent to the product of a Hilbert scheme of points with the Picard group of the surface [O'G97, Yos01] whenever they are smooth and compact. It is an open question whether this result also holds for primary Kodaira surfaces.

In the case of primary Kodaira surfaces, Toma showed that the moduli space of stable sheaves with fixed determinant and Chern character is holomorphic symplectic whenever it is smooth and compact, and determined that a sufficient condition to guarantee smoothness and compactness of the moduli space is for it to be stably irreducible [Tom01]. Aprodu, Moraru, and Toma studied the two-dimensional moduli spaces of rank-2 stably irreducible sheaves over primary Kodaira surfaces, and determined that they are also primary Kodaira surfaces [AMT12]. In higher dimensions, it is not yet known whether these moduli spaces are always deformation equivalent to Douady spaces of points over primary Kodaira surfaces.

In this chapter, we determine that there are compact moduli spaces of stably irreducible sheaves on Kodaira surfaces of dimension $2n$ for every n. In addition, we show that these moduli spaces are non-Kähler and have no simply connected components. Douady spaces of points on Kodaira surfaces are the only other known examples of compact holomorphic symplectic manifolds with these properties. An interesting question is to determine whether these moduli spaces are deformation equivalent to Douady spaces of points on Kodaira surfaces or form a new class of examples. Towards answering this question, we analyse a natural fibration on these spaces, which is described in detail for dimensions 4 and 6 in section 5.3.

Consider a general compact complex surface X with Gauduchon metric g, and consider the moduli space $\mathcal{M}^g_{r,\delta,c}(X)$ of g-stable coherent sheaves with rank r, determinant δ, and second Chern class c on X. In his paper [Tom01], Toma gives a sufficient condition for this moduli space to be smooth and compact:

$$\begin{array}{l}\text{Every } g\text{-semi-stable vector bundle } E \text{ with}\\ \operatorname{rank}(E) = r, c_1(E) = c_1(\delta), c_2(E) \leq c\\ \text{is } g\text{-stable.}\end{array} \qquad (*)$$

When X has odd first Betti number, this criterion is equivalent to requiring that every bundle E with rank r, $c_1(E) = c_1(\delta)$ and $c_2(E) \leq c$ is irreducible. In this case, the compactification of the moduli space of stable bundles with rank r, determinant δ, and second Chern class c is isomorphic to the moduli space of stably irreducible torsion-free sheaves. Using Brînzănescu's sufficient conditions for a sheaf to be irreducible [Brî96], we find a range of invariants for which $(*)$ is satisfied when X is a primary Kodaira surface. In particular, we show that the moduli spaces of rank-two sheaves which are smooth and compact can be of any even dimension.

In section 5.2, we review the construction of Brînzănescu and Moraru [BM06] of the fibres of the graph map above spectral curves without jumps, and describe the fibres of the graph map above spectral curves with exactly one jump. Since spectral curves with k jumps can only occur when the moduli space has dimension at least $4k$, understanding these cases allows us to describe all of the fibres of the graph map when the dimension of the moduli space is less than 8. In order to look at the fibres above spectral curves with one jump, we use elementary modifications to parameterise the locally free sheaves, and the structure of the multiplicity one Quot scheme to parameterise the non-locally free sheaves. We also determine which non-locally free sheaves can occur as limit points of vector bundles in the same fibre.

In section 5.3, we use results from section 5.2 to prove the main result of the chapter:

Theorem 5.1. *Let X be a primary Kodaira surface, and let $(\delta, c) \in \mathrm{Pic}(X) \times \mathbb{Z}$ be such that $\mathcal{M}_{2,\delta,c}(X)$ has positive dimension and contains stably irreducible vector bundles. Then $\mathcal{M}_{2,\delta,c}(X)$ is a non-Kähler manifold with no simply connected components.*

In this section we also describe the fibration structure of moduli spaces with dimension at most 6 in more detail using the results from section 5.2, as for these dimensions there are no spectral curves with more than one jump.

The remainder of Section 5.3 discusses comparisons between moduli spaces $\mathcal{M}_{2,\delta,c}(X)$ and Douady spaces $X^{[n]}$, as well as the graph map corresponding to moduli of stable rank-2 sheaves on an elliptically fibred Abelian surface. Any moduli space of stable sheaves on an elliptically fibred Abelian surface is birational to a Hilbert scheme of points, and the birational map can be constructed via allowable elementary modifications [Fri98, Chapter 8]. A similar situation does not occur in the Kodaira surface case as the general bundle does not have allowable elementary modifications. We conclude with a discussion of possible avenues to reconcile this discrepancy, including an analysis of moduli spaces of vector bundles on a product of elliptic curves $C_1 \times C_2$ where different choices of elliptic fibration structure give a description of the moduli space both in terms of a graph map and the birational map to $\mathrm{Pic}^0(C_1 \times C_2) \times (C_1 \times C_2)^{[n]}$.

5.1 The space of graphs

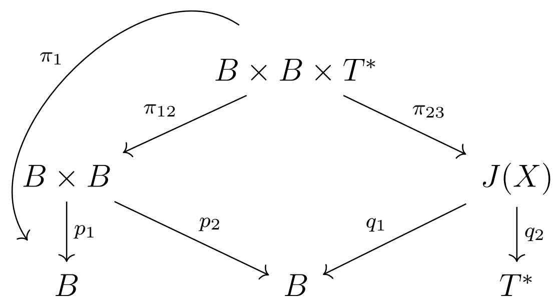
As shown in [Tom01, Remark 4.5], the moduli spaces of sheaves which are compact holomorphic symplectic manifolds are precisely those which are stably irreducible, and there are numerous examples of such moduli spaces by Proposition 3.8. This section looks at the graph map and space of graphs as defined in Chapter 4 in the particular case of Kodaira surfaces, where we can obtain explicit descriptions.

Let $\mathcal{M}_{2,\delta,c}(X)$ be a stably irreducible moduli space of sheaves over a Kodaira surface $\pi : X \to B$. In this case, we have the following result:

Proposition 5.2 (Brînzănescu–Moraru [BM05b])**.** *The graph map $\mathcal{G} : \mathcal{M}_{2,\delta,c}(X) \to \mathbb{P}_{\delta,c}$ is surjective whenever $t(2, c_1(\delta)) > 0$ and $c < 0$, where $t(2, c_1(\delta))$ is as defined in Proposition 3.7. In particular, the graph map is a Lagrangian fibration with respect to the holomorphic symplectic structure on $\mathcal{M}_{2,\delta,c}(X)$.*

Since in the stably irreducible case the graph map is a Lagrangian fibration of $\mathcal{M}_{2,\delta,c}(X)$ over $\mathbb{P}_{\delta,c}$, understanding the base and fibres of the fibration will allow us to determine topological properties of $\mathcal{M}_{2,\delta,c}(X)$ in section 5.3. In the remainder of this section, we investigate the structure of the $\mathbb{P}_{\delta,c}$ by analysing the divisors of the form $A_\delta + \rho^*\mathfrak{b}$ for $\mathfrak{b} \in \mathrm{Pic}^c(B)$.

The following commutative diagram will be helpful to reference for the statement and proof of the following proposition.



Proposition 5.3. *In the case that* $\Delta(2, \delta, c) > 0$, $\mathbb{P}_{\delta,c} \cong \mathbb{P}(E_{\delta,c})$, *where*

$$E_{\delta,c} := (\pi_1)_* \left(\pi_{12}^* \mathcal{P}_c(b_0) \otimes \pi_{23}^* \left(\mathcal{O}_{J(X)} \left(\mathcal{S}_0 + \mathcal{S}_\delta \right) \right) \right),$$

$\mathcal{P}_c(b_0)$ *is the Poincaré bundle* $\mathcal{P}_c = \mathcal{O}_{B \times B}(\Delta + (c-1)B \times \{b_0\} - \{b_0\} \times B)$ *of degree* c *line bundles on* B *with base point* $b_0 \in B$, *and the projections* π_1, π_{ij} *are as in the above commutative diagram.*

Proof. Since $\pi_1 = p_1 \circ \pi_{12}$, we have

$$E_{\delta,c} = (p_1)_* \left(\mathcal{P}_c(b_0) \otimes (\pi_{12})_* \left(\pi_{23}^* \mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta) \right) \right)$$

by the projection formula. Since $q_1 \circ \pi_{23} = p_2 \circ \pi_{12}$, we can apply the base change theorem to obtain

$$\begin{aligned} E_{\delta,c} &\cong (p_1)_* \left(\mathcal{P}_c(b_0) \otimes p_2^* \left((q_1)_* (\mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta) \right) \right) \\ &= (p_1)_* \left(\mathcal{P}_c(b_0) \otimes p_2^* V_\delta \right). \end{aligned}$$

For any $b \in B$, $(\mathcal{P}_c(b_0) \otimes p_2^* V_\delta)\,|_{\{b\} \times B} \cong \mathcal{O}_B(b + (r-1)b_0) \otimes V_\delta$, so since $\Delta(2, \delta, c) > 0$, $\lambda_b \otimes V_\delta$ has positive degree, $E_{\delta,c}$ is locally free, and the fibre above b in $E_{\delta,c}$ is indeed

$$H^0(B, \mathcal{O}_B(b + (r-1)b_0) \otimes V_\delta).$$

From this we conclude that $\mathbb{P}(E_{\delta,c}) \cong \mathbb{P}_{\delta,c}$. $\square$

Lemma 5.4. *For any choice of $\delta \in \mathrm{Pic}(X)$, $c \in \mathbb{Z}$, the bundle $E_{\delta,c}$ has no global sections.*

Proof. Note that

$$\begin{aligned} H^0(B, E_{\delta,c}) &= H^0(B \times B \times T^*, \pi_{12}^*\mathcal{P}_c(b_0) \otimes \pi_{23}^*\mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta)) \\ &= H^0(J(X), (\pi_{23})_*(\pi_{12}^*\mathcal{P}_c(b_0) \otimes \pi_{23}^*\mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta))). \end{aligned}$$

Using the projection formula, we have

$$H^0(B, E_{\delta,c}) = H^0(J(X), (\pi_{23})_*\pi_{12}^*\mathcal{P}_c(b_0) \otimes \mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta)).$$

From the base change theorem, $(\pi_{23})_*\pi_{12}^*\mathcal{P}_c(b_0) \cong q_1^*(p_2)_*\mathcal{P}_c$. Since the restriction to any fibre of p_2 is a degree-0 line bundle which is trivial only for $p_2^{-1}(b_0)$, $(p_2)_*\mathcal{P}_c = 0$. From this we conclude that $H^0(B, E_{\delta,c}) = 0$. □

For the next proof, we will employ the notation of [CC93] of $E_r(b_0) := (p_1)_*(\mathcal{P}_r(b_0))$, which is a stable bundle of degree -1 and rank r for all $r > 0$.

Proposition 5.5. *Let $\delta \in \mathrm{Pic}(X)$ and $c \in \mathbb{Z}$ be such that $\Delta(2, \delta, c) \geq 0$.*

i. If $\Delta(2, \delta, c) = 0$, then $\mathbb{P}_{\delta,c}$ consists of 2 points.

ii. If $\Delta(2, \delta, c) > 0$ and $4\Delta(2, \delta, c)$ is odd, then $\mathbb{P}_{\delta,c} \cong \mathbb{P}(E_{\delta,c})$, where $E_{\delta,c}$ is a stable bundle of rank $4\Delta(2, \delta, c)$ and degree -2.

iii. If $\Delta(2, \delta, c) > 0$ and $4\Delta(2, \delta, c)$ is even, then $\mathbb{P}_{\delta,c} \cong \mathbb{P}(E_{\delta,c})$, where $E_{\delta,c}$ is the direct sum of two stable bundles, each of rank $2\Delta(2, \delta, c)$ and degree -1.

Proof. We begin by constructing a long exact sequence involving the $E_{\delta,c}$ which will be helpful in later computations. Note first that there is a natural exact sequence

$$0 \longrightarrow \mathcal{P}_{c-1}(b_0) \longrightarrow \mathcal{P}_c(b_0) \longrightarrow \mathcal{O}_{B\times\{b_0\}} \longrightarrow 0 \qquad (5.1)$$

relating Poincaré bundles of adjacent degrees [CC93]. Pulling back by π_{12} and twisting by the line bundle $\mathcal{L}_\delta := \pi_{23}^*\mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta)$ gives a new short exact sequence

$$0 \longrightarrow \pi_{12}^*\mathcal{P}_{c-1} \otimes \mathcal{L}_\delta \longrightarrow \pi_{12}^*\mathcal{P}_c \otimes \mathcal{L}_\delta \longrightarrow \mathcal{O}_{B\times\{b_0\}\times T^*} \otimes \mathcal{L}_\delta \longrightarrow 0.$$

If we pushforward by π_{12}, we get

$$\begin{array}{c} 0 \longrightarrow \mathcal{P}_{c-1}(b_0) \otimes (\pi_{12})_*\mathcal{L}_\delta \longrightarrow \mathcal{P}_c(b_0) \otimes (\pi_{12})_*\mathcal{L}_\delta \\ \swarrow \\ (\pi_{12})_*(\mathcal{O}_{B\times\{b_0\}\times T^*} \otimes \mathcal{L}_\delta) \longrightarrow \mathcal{P}_{c-1}(b_0) \otimes R^1(\pi_{12})_*\mathcal{L}_\delta \end{array}$$

from the projection formula. By the base change theorem,

$$R^1(\pi_{12})_*\mathcal{L}_\delta \cong p_2^*R^1(q_1)_*\mathcal{O}_{J(X)}(\mathcal{S}_0 + \mathcal{S}_\delta) = 0,$$

so the previous exact sequence is short exact. Finally, we can pushforward by p_1 to obtain the long exact sequence

$$\begin{array}{c} 0 \longrightarrow (p_1)_*\left(\mathcal{P}_{c-1}(b_0) \otimes p_2^*V_\delta\right) \longrightarrow E_{\delta,c} \longrightarrow (\pi_1)_*\left(\mathcal{O}_{B\times\{b_0\}\times T^*} \otimes \mathcal{L}_\delta\right) \\ \swarrow \\ R^1(p_1)_*(\mathcal{P}_{c-1}(b_0) \otimes p_2^*V_\delta) \longrightarrow R^1(p_1)_*(\mathcal{P}_c(b_0) \otimes p_2^*V_\delta) \longrightarrow \cdots. \end{array}$$

Note that $(\mathcal{P}_c(b_0) \otimes V_\delta)|_{\{b\}\times B}$ is semistable of positive degree when $\Delta(2,\delta,c) > 0$, and any rank-2 semistable vector bundle of positive degree on a genus one curve has trivial first cohomology, so we have $R^1(p_1)_*\left(\mathcal{P}_c(b_0) \otimes p_2^*V_\delta\right) = 0$. We also have

$$(\pi_1)_*\left(\mathcal{O}_{B\times\{b_0\}\times T^*} \otimes \mathcal{L}_\delta\right) \cong (q_1)_*(\mathcal{O}_{B\times T^*}(B\times\{0\} + B\times\{\delta_{b_0}\})) = (q_1)_*q_2^*\mathcal{O}_{T^*}(0 + \delta_{b_0}),$$

so

$$(\pi_1)_*\left(\mathcal{O}_{B\times\{b_0\}\times T^*} \otimes \mathcal{L}_\delta)\right) \cong \mathcal{O}_B \oplus \mathcal{O}_B.$$

This gives the exact sequence

$$\begin{array}{c} 0 \longrightarrow (p_1)_*(\mathcal{P}_{c-1}(b_0) \otimes p_2^*V_\delta) \longrightarrow E_{\delta,c} \\ \swarrow \\ \mathcal{O}_B \oplus \mathcal{O}_B \longrightarrow R^1(p_1)_*(\mathcal{P}_{c-1}(b_0) \otimes p_2^*V_\delta) \longrightarrow 0. \end{array} \tag{5.2}$$

We now consider the case of $\Delta(2,\delta,c)=0$. The exact sequence (5.2) for $E_{\delta,c+1}$ is then

$$0 \longrightarrow E_{\delta,c+1} \longrightarrow \mathcal{O}_B \oplus \mathcal{O}_B \longrightarrow R^1(p_1)_*(\mathcal{P}_c(b_0)\otimes p_2^*V_\delta) \longrightarrow 0,$$

where $R^1(p_1)_*(\mathcal{P}_c(b_0)\otimes p_2^*V_\delta)$ is a torsion sheaf whose stalk at $b\in B$ is given by

$$\left(R^1(p_1)_*(\mathcal{P}_c(b_0)\otimes p_2^*V_\delta)\right)_b = H^0(B,\mathcal{O}_B(b+(c-1)b_0)\otimes V_\delta).$$

Suppose for a contradiction that V_δ is an indecomposable bundle. Then

$$R^1(p_1)_*(\mathcal{P}_c(b_0)\otimes p_2^*V_\delta) = \mathbb{C}_b$$

for some $b\in B$, and by [Boo21] we have $E_{\delta,c+1}\cong \mathcal{O}_B\oplus\mathcal{O}_B(-b)$, which has a non-zero section. This contradicts Lemma 5.4, so V_δ is decomposable. Thus $V_\delta \cong \lambda\otimes(L\oplus L^{-1})$ for some $\lambda\in \mathrm{Pic}^{-c}(B)$ and $L\in\mathrm{Pic}^0(B)$.

From the fact that $\mathcal{M}_{2,\delta,c}(X)$ is finite when $\Delta(2,\delta,c)=0$, there must be a line bundle $\lambda'\in\mathrm{Pic}^c(B)$ such that $H^0(B,\lambda'\otimes V_\delta)=\mathbb{C}$. This implies that $L\not\cong L^{-1}$, so $\mathbb{P}_{\delta,c}$ consists of two points corresponding to $\lambda'=\lambda^{-1}\otimes L$ and $\lambda'=\lambda^{-1}\otimes L^{-1}$.

In the case where $\Delta(2,\delta,c)>0$ and $4\Delta(2,\delta,c)$ is odd, the fact that V_δ is semistable implies that $V_\delta\cong\lambda\otimes\mathcal{F}_p$, where $\lambda\in\mathrm{Pic}^{(-c_1^2-2)/4}(B)$, and $\mathcal{F}_p$ is the unique non-trivial extension of $\mathcal{O}_B(p)$ by $\mathcal{O}_B$ with $p\in B$. Thus $\mathcal{P}_c(b_0)\otimes p_2^*V_\delta$ fits into the exact sequence

$$0 \longrightarrow \mathcal{P}_c(b_0)\otimes p_2^*\lambda \longrightarrow \mathcal{P}_c(b_0)\otimes p_2^*V_\delta \longrightarrow \mathcal{P}_c(b_0)\otimes p_2^*(\lambda\otimes\mathcal{O}_B(p)) \longrightarrow 0.$$

Set $r=2\Delta(2,\delta,c)-\frac{1}{2}$. Then there exist points $b_1,b_2\in B$ and line bundles $\lambda_1,\lambda_2\in \mathrm{Pic}^0(B)$ such that $\mathcal{P}_c(b_0)\otimes p_2^*\lambda = \mathcal{P}_r(b_1)\otimes p_1^*\lambda_1$ and $\mathcal{P}_c(b_0)\otimes p_2^*(\lambda(p)) = \mathcal{P}_{r+1}(b_2)\otimes p_1^*\lambda_2$. Pushing forward by p_1 gives the exact sequence

$$0 \longrightarrow (p_1)_*(\mathcal{P}_r(b_1))\otimes\lambda_1 \longrightarrow E_{\delta,c}\otimes\lambda_2 \longrightarrow E_{r+1}(b_2)\otimes\lambda_2 \longrightarrow R^1(p_1)_*(\mathcal{P}_r(b_1))\otimes\lambda_1 \longrightarrow 0,$$

since we showed previously that $R^1(p_1)_*(\mathcal{P}_c(b_0)\otimes p_2^*V_\delta)=0$.

If $\Delta(2,\delta,c) = \frac{1}{4}$ and therefore $r = 0$, $(p_1)_*(\mathcal{P}_r(b_1)) = 0$ and $R^1(p_1)_*(\mathcal{P}_r(b_1)) = \mathbb{C}_{b_1}$. Therefore when $\Delta(2,\delta,c) = \frac{1}{4}$, we have

$$E_{\delta,c} \cong E_1(b_2) \otimes \mathcal{O}_B(-b_1) \otimes \lambda_2 \cong \mathcal{O}_B(-b_1 - b_2) \otimes \lambda_2$$

by [CC93]. Thus $E_{\delta,c}$ is some line bundle of degree -2 on B.

If instead $\Delta(2,\delta,c) > \frac{1}{4}$, $(p_1)_*(\mathcal{P}_r(b_1)) = E_r(b_1)$ and $R^1(p_1)_*(\mathcal{P}_r(b_1)) = 0$, so $E_{\delta,c}$ is an extension of $E_{r+1}(b_2) \otimes \lambda_2 \cong E_{r+1}(b_2')$ by $E_r(b_1) \otimes \lambda_1 \cong E_r(b_1')$.

The extension class corresponding to $E_{\delta,c}$ is an element of

$$\mathrm{Ext}^1_{\mathcal{O}_B}(E_{r+1}(b_2'), E_r(b_1')) = H^1(B, E_{r+1}(b_2')^\vee \otimes E_r(b_1')).$$

The tensor product of a stable bundle of rank $r+1$ with a stable bundle of rank r is another stable bundle of rank $r(r+1)$ [Ati57, Lemma 28], so in particular there is some point b_3' such that $E_{r+1}(b_2')^\vee \otimes E_r(b_1') = E_{r(r+1)}(b_3')$. As shown in [CC93], $E_{r(r+1}(b_3')$ is an extension

$$0 \longrightarrow E_{r(r+1)}(b_3') \longrightarrow E_{r(r+1)+1}(b_3') \longrightarrow \mathcal{O}_B \longrightarrow 0,$$

and the corresponding long exact sequence in cohomology induces an isomorphism

$$H^0(B, \mathcal{O}_B) \cong H^1(B, E_{r(r+1)}(b_3')).$$

Similarly, the extension class corresponding to $\mathcal{P}_c(b_0) \otimes p_2^* V_\delta$ is a non-zero element of

$$\mathrm{Ext}^1_{\mathcal{O}_{B\times B}}(\mathcal{P}_c(b_0) \otimes p_2^*\lambda(p), \mathcal{P}_c(b_0) \otimes p_2^*\lambda) = H^1(B \times B, p_2^*(\mathcal{O}_B(-p))).$$

Since p_1 maps a surface to a curve, its Leray spectral sequence degenerates at page 2, giving

$$H^1(B \times B, p_2^*(\mathcal{O}_B(-p))) = H^1(B, (p_1)_*(p_2^*(\mathcal{O}_B(-p))) \oplus H^0(B, R^1(p_1)_*(p_2^*(\mathcal{O}_B(-p))).$$

The base change theorem gives that $R^i(p_1)_*(p_2^*(\mathcal{O}_B(-p)))$ is the trivial bundle on B of rank $h^i(B, \mathcal{O}_B(-p))$, so $H^1(B \times B, p_2^*(\mathcal{O}_B(-p))) = H^0(B, \mathcal{O}_B)$. We now have an isomorphism mapping the extension class of $\mathcal{P}_c(b_0) \otimes p_2^* V_\delta$ to the extension class of $E_{\delta,c}$ via pushforward by p_1, so $E_{\delta,c}$ is the unique non-split extension of $E_{r+1}(b_2')$ by $E_{r+1}(b_1')$. We now show that this extension is a stable bundle. Suppose for a contradiction that $\mathcal{F}$ is a non-trivial subbundle of $E_{\delta,c}$ with $\mu(\mathcal{F}) \geq \mu(E_{\delta,c})$. Without loss of generality, we can assume that $\mathcal{F}$ is stable. If $\mathcal{F}$ has degree -2, then $\mu(\mathcal{F}) < \mu(E_{\delta,c})$, so $\mathcal{F}$ can only be a destabilising bundle of $E_{\delta,c}$ if $\deg(\mathcal{F}) \geq -1$. No subbundle of $E_r(b_1')$ is destabilising, so there must be a non-zero morphism $f : \mathcal{F} \to E_{r+1}(b_2')$ given by the inclusion of $\mathcal{F}$ into $E_{\delta,c}$ followed by projection to $E_{r+1}(b_2')$. If f is surjective, $E_{r+1}(b_2')$ is a quotient of $\mathcal{F}$ and $\mu(\mathcal{F}) \leq \mu(E_{r+1}(b_2'))$. Then $\mathcal{F}$ must be of the form $E_k(q)$ for some $0 < k \leq r+1$ and $q \in B$. Since f is surjective, this is only possible if $\mathcal{F} = E_{r+1}(b_2')$. Since the only endomorphisms of $E_{r+1}(b_2')$ are constant multiples of the identity, this would induce a splitting of $E_{\delta,c}$, so f cannot be surjective. Since f is not surjective, then $\mu(\mathcal{F}) \leq \mu(\mathrm{im}(f)) < \mu(E_{r+1}(b_2'))$ by indecomposability of $\mathcal{F}$ and stability of $\mathcal{E}_{r+1}(b_2')$. We now have $\deg(\mathcal{F}) \geq -1$ and $\mu(\mathcal{F}) < \frac{-1}{r+1}$, so $\mu(\mathcal{F}) \leq \frac{-1}{r} < \mu(E_{\delta,c})$. This implies that $\mathcal{F}$ cannot be a destabilising bundle for $E_{\delta,c}$, so $E_{\delta,c}$ is stable.

Finally, we consider the case of $\Delta(2,\delta,c) > 0$ with $4\Delta(2,\delta,c)$ even. As shown in the case where $\Delta(2,\delta,c) = 0$, $V_\delta = \lambda \otimes (L \oplus L^{-1})$ where $\lambda \in \mathrm{Pic}^{-\frac{c_1^2(\delta)}{4}}(B)$ and $L \in \mathrm{Pic}^0(B)$. From this we conclude that

$$E_{\delta,c} \cong (p_1)_*(\mathcal{P}_c(b_0) \otimes \lambda \otimes L) \oplus (p_1)_*(\mathcal{P}_c(b_0) \otimes \lambda \otimes L^{-1}).$$

Set $r = 2\Delta(2,\delta,c)$. Then there are $b_1, b_2 \in B$ and $\lambda_1, \lambda_2 \in \mathrm{Pic}^0(B)$ such that $\mathcal{P}_c(b_0) \otimes \lambda \otimes L \cong \mathcal{P}_r(b_1) \otimes p_1^*\lambda_1$ and $\mathcal{P}_c(b_0) \otimes \lambda \otimes L^{-1} \cong \mathcal{P}_r(b_2) p_1^* \otimes \lambda_2$, so $E_{\delta,c} \cong E_r(b_1') \oplus E_r(b_2')$ for some $b_1', b_2' \in B$. $\square$

Remark 5.6. Recall that the natural Lagrangian fibration on the Douady space $X^{[n]}$ has base $\mathrm{Sym}^n(B) \cong \mathbb{P}(E_n(u))$ for some $u \in B$ [CC93], so in particular $\mathbb{P}_{\delta,c}$ can only be isomorphic to $\mathrm{Sym}^n(B)$ when $\Delta(2,\delta,c) = \frac{1}{4}$.

Remark 5.7. We can write the space of graphs $\mathbb{P}_{\delta,c}$ as the union

$$\mathbb{P}_{\delta,c} = \bigcup_{k=0}^{\lfloor 2\Delta(2,\delta,c) \rfloor} \mathbb{P}^k_{\delta,c},$$

where

$$\mathbb{P}^k_{\delta,c} := \{S \in \mathbb{P}_{\delta,c} : S = C + \sum_{i=1}^{k} \{b_i\} \times T^*, C \text{ a bisection of } J(X)\}.$$

Note that $\bigcup_{i \geq k} \mathbb{P}^i_{\delta,c}$ is a closed subset of codimension k in $\mathbb{P}_{\delta,c}$, and since for any spectral curve $C + \sum_{i=1}^{k}\{b_i\} \times T^* \in \mathbb{P}^k_{\delta,c}$ we have $C \in \mathbb{P}^0_{\delta,c-k}$, there is an isomorphism

$$\mathbb{P}^k_{\delta,c} \cong \mathbb{P}^0_{\delta,c-k} \times \mathrm{Hilb}^k(B)$$

for every integer $0 \leq k \leq 2\Delta(2,\delta,c)$.

In the next section, we compute the fibres of the graph map in order to study the structure of $\mathcal{M}_{2,\delta,c}(X)$.

5.2 Fibres of the graph map

In this section, we parameterise the space $\mathcal{G}^{-1}(G)$ of torsion-free sheaves on X with determinant δ and spectral curve G, where G is of the form $\overline{C} + \sum_{i=1}^{k}\{b_i\} \times T^*$ with $b_i \in B$ for each i, and where $\overline{C}$ is an irreducible bisection of $J(X)$ with normalisation C.

In the case where $G = \overline{C}$, every vector bundle E with spectral curve $\overline{C}$ can be described as an elementary modification

$$0 \longrightarrow E \longrightarrow \gamma_* L \longrightarrow \iota_* \lambda \longrightarrow 0 \tag{5.3}$$

for some $b \in B$, $\lambda \in \mathrm{Pic}^0(\pi^{-1}(b))$, $L \in \mathrm{Pic}(W)$, where $\iota : \pi^{-1}(b) \to X$ is the inclusion, W is the normalisation of $X \times_B C$, and $\gamma : W \to X$ the map coming from the fibred product [AT03]. Since $\gamma_* L|_b$ is a regular bundle by Proposition 4.12, every elementary modification of the form (5.3) is itself the pushforward of a line bundle on W [AT03, Remarque 5]. We then have that $\mathcal{G}^{-1}(\overline{C}) \cong \mathrm{Prym}(C/B)$ [BM06, Theorem 4.5]. Note that each connected component of $\text{Prym}(C/B)$ has dimension $g(C) - 1$, and when C is an unramified cover of degree 2, $\text{Prym}(C/B)$ is the two-element group.

For the next simplest case, where $G = \overline{C} + \{b\} \times T^*$ and $\overline{C} \cong C$, we can compute the fibres of the graph map as in the following propositions:

Proposition 5.8. *Take $\delta \in \text{Pic}(X)$, $b \in B$, and let $C \subset J(X)$ be a smooth and irreducible ι_δ-invariant bisection. Then the torsion-free sheaves with determinant δ and spectral curve $C + \{b\} \times T^*$ are parameterised by a union of two $\mathbb{P}^1$-bundles over $\text{Prym}(C/B) \times T$ that intersect along $|C \cap \{b\} \times T^*|$ sections.*

Remark 5.9. Since C is a bisection of $\text{pr}_1 : B \times T^* \to B$, $C \cap \{b\} \times T^*$ will always contain one or two points.

Proof. The fibre of the graph map above a spectral curve of this type can be decomposed into two irreducible components, with one component containing the non-locally free sheaves, and the other component containing the vector bundles. If E is a vector bundle with determinant δ and spectral curve $C + \{b\} \times T^*$, then $E|_{\pi^{-1}(b)}$ is a degree zero vector bundle so that $h^1(T, E|_{\pi^{-1}(b)} \otimes \lambda) = 1$ for every $\lambda \in T^*$ by Proposition 4.8, so there is a line bundle $L \in \text{Pic}^1(T)$ such that $E|_{\pi^{-1}(b)} \cong L \oplus (\delta_b \otimes L^{-1})$. There is a unique choice of allowable elementary modification $\tilde{E}$ given by

$$0 \longrightarrow \tilde{E} \longrightarrow E \longrightarrow j_*(\delta_b \otimes L^{-1}) \longrightarrow 0 \ ,$$

where $j : \pi^{-1}(b) \to X$ is the inclusion map. In particular, $\tilde{E}$ has $\det(\tilde{E}) = \det(E) \otimes \mathcal{O}_X(-\pi^{-1}(b))$, $c_2(\tilde{E}) = c_2(E) - 1$, and spectral curve C. We then have a well-defined projection $E \mapsto (\tilde{E}, L) \in \mathcal{G}^{-1}(C) \times \text{Pic}^1(T) = \text{Prym}(C/B) \times T$. To determine the fibres of this map, we notice that for any choice of bundle $\tilde{E}$ with determinant $\delta \otimes \pi^*\mathcal{O}_B(b)$ and spectral curve C, $L \in \text{Pic}^1(T)$, and surjective map $f : \tilde{E} \to j_*L$, there is a vector bundle E with determinant δ and spectral curve $C + \{b\} \times T^*$ such that $E \otimes \pi^*\mathcal{O}_B(-b) = \ker(f)$.

Since C is smooth and irreducible, either

$$\tilde{E}|_{\pi^{-1}(b)} \cong \lambda \oplus (\delta|_{\pi^{-1}(b)} \otimes \lambda^{-1})$$

for some $\lambda \in \mathrm{Pic}^0(T)$ with $\lambda^{\otimes 2} \neq \delta|_{\pi^{-1}(b)}$, or

$$\tilde{E}|_{\pi^{-1}(b)} \cong \lambda_0 \otimes A,$$

where A is the unique extension of $\mathcal{O}_T$ by $\mathcal{O}_T$ and $\lambda_0 \in \mathrm{Pic}^0(T)$ such that $\lambda_0^{\otimes 2} = \delta|_{\pi^{-1}(b)}$. In both cases, $\mathrm{Hom}(\tilde{E}, j_*L) \cong \mathbb{C}^2$, and the non-surjective maps correspond to $|\{b\} \times T^* \cap C|$ 1-dimensional subspaces. Thus the vector bundles with determinant δ and spectral curve $C + \{b\} \times T^*$ are parameterised by a fibre bundle with base $\mathrm{Prym}(C/B) \times T$ and fibre $\mathbb{C}$ or $\mathbb{C}^*$.

If $\mathcal{E}$ is a non-locally free sheaf with determinant δ and spectral curve $C + \{b\} \times T^*$, then since the double dual of a torsion-free sheaf on a surface is locally free, $\mathcal{E}^{\vee\vee}$ is a vector bundle with determinant δ and spectral curve C. Since $\mathcal{E}$ has exactly one singularity, $\mathcal{E}^{\vee\vee}/\mathcal{E}$ is a torsion sheaf supported at a point $x \in \pi^{-1}(b)$. This gives a well-defined projection $\mathcal{E} \mapsto (\mathcal{E}^{\vee\vee}, \mathrm{supp}(\mathcal{E}^{\vee\vee}/\mathcal{E})) \in \mathrm{Prym}(C/B) \times T$. Since for any vector bundle E with spectral curve C and determinant δ the sheaves $\mathcal{E}$ with $\mathcal{E}^{\vee\vee} = E$ which are singular at a point are parameterised by $\mathrm{Quot}(E, 1) = \mathbb{P}(E)$ with the projection map sending $\mathcal{E}$ to $\mathrm{supp}(E/\mathcal{E})$ [EL99], the non-locally free sheaves with determinant δ and spectral curve $\{b\} \times T^* + C$ are parameterised by a $\mathbb{P}^1$-bundle with base $\mathrm{Prym}(C/B) \times T$.

As the union of these two components is the fibre of a proper holomorphic map, the union of the components must be compact, and therefore the closure of the locally free component intersects with the non-locally free component along $|C \cap \{b\} \times T^*|$ sections. □

In the previous proposition, we showed that for a spectral curve with a single jump, the fibre of the graph map decomposes into two irreducible components. The following result allows us to explicitly compute the intersection locus of these two components.

Proposition 5.10. *In the context of Proposition 5.8, the intersection of the irreducible components of the fibre consists of sheaves of the form* $\ker(f \oplus g)$ *for* E *a vector bundle with spectral curve* C *and determinant* $\delta \otimes \pi^*\mathcal{O}_B(b)$*,* $f : E \to j_*\lambda$ *non-zero such that* $(b, \lambda) \in C$*, and* $g : E \to \mathcal{O}_x$ *with* $x \in \pi^{-1}(b)$ *such that* $g \circ \ker(f) \neq 0$*. Furthermore, these sheaves are determined up to isomorphism by* (E, λ, x)*.*

Proof. Since the non-locally free component in Proposition 5.8 is compact, we can compute the intersection by finding the limits of families of deformations within the locally free component which are not locally free. Every vector bundle in the locally free component is given by an elementary modification

$$0 \longrightarrow \tilde{E} \longrightarrow E \longrightarrow j_*L \longrightarrow 0\ ,$$

where E is a vector bundle with spectral curve C and determinant $\delta \otimes \pi^*\mathcal{O}_B(b)$, and $L \in \mathrm{Pic}^1(B)$. Fix a choice of E and L. For any λ such that $(b, \lambda) \in C$, there is a surjective sheaf map $\alpha : E \to j_*\lambda$, which is unique up to multiplication by a scalar. Let p be the unique point in T such that L is an extension in $\mathrm{Ext}^1(\mathcal{O}_p, \lambda)$, and set $x = j(p)$. If we now choose sheaf maps $h : E \to j_*L$ and $\beta : E \to \mathcal{O}_x$ so that h is surjective and $\beta \circ \ker(\alpha) \neq 0$, we now have that for any map $\eta : L \to \mathcal{O}_p$ that there is a unique $t \in \mathbb{C}$ satisfying $t(\eta \circ h) = \beta$. Using these data, we can construct a deformation over $\mathrm{Ext}^1(\mathcal{O}_p, \lambda)$ whose non-zero elements are given as follows:

Given an extension

$$0 \longrightarrow \lambda \xrightarrow{\ \varphi\ } L \xrightarrow{\ \psi\ } \mathcal{O}_p \longrightarrow 0$$

corresponding to a non-zero $s \in \mathrm{Ext}^1(\mathcal{O}_p, \lambda)$, define the maps $f_s = \varphi \circ \alpha$ and $g_s = th$, where t is chosen so $t(\psi \circ h) = \beta$, and set $\tilde{E}_s = \ker(f_s + g_s)$. Note that any other extension corresponding to s will be given by maps $z\varphi$ and $\frac{1}{z}\psi$ for some $z \in \mathbb{C}^*$, giving $f_s = z\varphi \circ \alpha$ and $g_s = tzh$, so $(f_s + g_s)$ is unique up to multiplication by a scalar, and $\tilde{E}_s$ is well-defined. As s goes to zero, these maps become $f_0 = \iota_1 \circ \alpha$ and $g_0 = \iota_2 \circ \beta$, where ι_i are the co-product maps for $j_*\lambda \oplus \mathcal{O}_x$. This gives that $\tilde{E}_0$ is of the desired form.

We now show that $\tilde{E}_0$ is independent of the choice of β. Clearly, $\ker(f_0 + g_0) = \ker(\alpha) \cap \ker(\beta)$. Since β vanishes away from x, $\tilde{E}_0$ may only depend on β in the fibre above x. The linear maps $\alpha|_x$ and $\beta|_x$ both have rank one, so either $\ker(\alpha|_x) = \ker(\beta|_x)$ or $\ker(\alpha|_x) \cap \ker(\beta|_x) = 0$. But $\ker(\alpha|_x) \neq \ker(\beta|_x)$, since $\beta \circ \ker\alpha \neq 0$, meaning that all the sections of $\tilde{E}_0$ vanish at x. Thus $\tilde{E}_0$ does not depend on β. □

5.3 Applications

In this section, we use Proposition 5.5 and the results of section 5.2 to prove some results about the fundamental groups of moduli spaces of rank-2 stably irreducible sheaves, as well as compute explicit data about the graph map fibration in the cases where the dimension of the moduli space is at most 6. For this section, the invariants δ, c are assumed to be such that a rank-2 sheaf $\mathcal{E}$ is stably irreducible whenever $\det(\mathcal{E}) = \delta$ and $c_2(\mathcal{E}) = c$.

5.3.1 The topology of the moduli spaces

The case of $\Delta(2, \delta, c) = \frac{1}{4}$ was previously studied in [AMT12] leading to the following result:

Proposition 5.11 (Aprodu–Moraru–Toma)**.** *Let $\delta \in \mathrm{Pic}(X)$ and $c \in \mathbb{Z}$ be such that $\mathcal{M}_{2,\delta,c}(X)$ is 2-dimensional and $t(2, \delta) > \frac{1}{4}$. Then $\mathcal{M}_{2,\delta,c}(X)$ is a primary Kodaira surface with the same base and fibre as X, and their Néron–Severi groups satisfy the relation*

$$\mathrm{ord}(Tors(NS(X)))|\, \mathrm{ord}(Tors(NS(\mathcal{M}_{2,\delta,c}(X)))).$$

Proposition 5.12. *Let δ, c be such that $\Delta(2, \delta, c) < t(2, \delta)$.*

i. If $\Delta(2, \delta, c) = 0$, then $\mathcal{M}_{2,\delta,c}(X)$ consists of four points.

ii. If $\Delta(2, \delta, c) > 0$, then the induced map of fundamental groups

$$\pi_1(\mathcal{G}) : \pi_1(\mathcal{M}_{2,\delta,c}(X)) \to \pi_1(\mathbb{P}_{\delta,c}) \cong \mathbb{Z}^2$$

is surjective.

In particular, $\mathcal{M}_{2,\delta,c}(X)$ is not simply connected when $\Delta(2, \delta, c) > 0$.

Proof. For the case of $\Delta(2,\delta,c) = 0$, every spectral curve is smooth by Proposition 4.8, and the genus formula (4.3) gives that each spectral curve C in $\mathbb{P}_{\delta,c}$ is an unramified double cover of B. From this we can conclude that $\mathcal{G}^{-1}(C) \cong \text{Prym}(C/B)$ is the group with two elements, giving the desired result.

The case of $\Delta(2,\delta,c) = \frac{1}{4}$ is a direct corollary of Proposition 5.11, so for the remainder of the proof we assume $\Delta(2,\delta,c) \geq \frac{1}{2}$.

Recall that given any fibre bundle $F \hookrightarrow Y \to Z$, there is an induced exact sequence

$$\pi_1(F) \to \pi_1(Y) \to \pi_1(Z) \to \pi_0(F) \tag{5.4}$$

of the homotopy groups [BT82, Section 17].

Using (5.4) we see that whenever (δ, c) are such that $\frac{1}{2} \leq \Delta(2,\delta,c) < t(2,\delta)$,we have

$$\pi_1(\mathbb{P}_{\delta,c}) = \pi_1(B) = \mathbb{Z}^2$$

since $\mathbb{P}_{\delta,c}$ is a holomorphic fibre bundle with connected and simply connected fibres. In particular this means that for any section $\sigma : B \to \mathbb{P}_{\delta,c}$ and any element $[\gamma] \in \pi_1(\mathbb{P}_{\delta,c})$, there is a representative of $[\gamma]$ contained in $\sigma(B)$. Let E be a regular rank-2 vector bundle in $\mathcal{M}_{2,\delta,c-1}(X)$ with spectral curve C, and take the section $\sigma_E : B \to \mathbb{P}_{\delta,c}$ given by $b \mapsto C + \{b\} \times T^*$. We will show that for any loop $\gamma \in \sigma_E(B)$, there is a loop in $\mathcal{M}_{2,\delta,c}(X)$ which maps to γ, demonstrating that $\pi_1(\mathcal{G})$ is a surjection. Consider the subset $\text{Quot}(E,1) \subseteq \mathcal{M}_{2,\delta,c}(X)$ consisting of non-locally free sheaves whose double dual is E and which have one singularity counting multiplicity. Since $\text{Quot}(E,1) \cong \mathbb{P}(E)$, the map $\mathcal{G}|_{\text{Quot}(E,1)}$ is a fibration over $\sigma_E(B)$. Applying (5.4) again gives the exact sequence

$$\pi_1(\mathbb{P}(E|_{\pi^{-1}(b)})) \longrightarrow \pi_1(\text{Quot}(E,1)) \longrightarrow \pi_1(\sigma_E(B)) \longrightarrow 0.$$

Since every element $[\gamma] \in \pi_1(\mathbb{P}_{\delta,c})$ can be represented by a loop in $\sigma_E(B)$, factoring through inclusion into $\mathcal{M}_{2,\delta,c}(X)$ gives that the map

$$\pi_1(\mathcal{G}) : \pi_1(\mathcal{M}_{2,\delta,c}(X)) \to \pi_1(\mathbb{P}_{\delta,c})$$

is surjective. □

Remark 5.13. By using [Ara11, Lemma 3.5] we can improve the above result to get a right-exact sequence

$$\mathbb{Z}^{8\Delta(2,\delta,c)} \longrightarrow \pi_1(\mathcal{M}_{2,\delta,c}(X)) \longrightarrow \pi_1(\mathbb{P}_{\delta,c}) \longrightarrow 0$$

when $\Delta(2,\delta,c) \leq \frac{3}{4}$. This result may also work in higher dimensions, but has not yet been verified.

Proposition 5.14. *Let $\delta \in \mathrm{Pic}(X)$ and $c \in \mathbb{Z}$ be such that $0 < \Delta(2,\delta,c) < t(2,\delta)$, where $\Delta(2,\delta,c)$ and $t(2,\delta)$ are defined as in Proposition 3.7. Then $\mathcal{M}_{2,\delta,c}(X)$ does not admit a Kähler structure.*

Proof. The case of $\Delta(2,\delta,c) = \frac{1}{4}$ follows immediately from Proposition 5.11. For the case of $\Delta(2,\delta,c) \geq \frac{1}{2}$, let E be a regular vector bundle in $\mathcal{M}_{2,\delta,c-1}(X)$. Then $\mathrm{Quot}(E,1) \cong \mathbb{P}(E)$ is a complex submanifold of $\mathcal{M}_{2,\delta,c}(X)$. Since $\mathbb{P}(E)$ is a fibre bundle with simply connected fibre, $b_1(\mathbb{P}(E)) \cong b_1(X) = 3$, so $\mathbb{P}(E)$ does not admit any Kähler structure. Since any complex submanifold of a Kähler manifold has an induced Kähler structure, $\mathcal{M}_{2,\delta,c}(X)$ can not admit a Kähler structure. □

Corollary 5.14.1. *When $0 < \Delta(2,\delta,c) < t(2,\delta)$, the moduli space $\mathcal{M}_{2,\delta,c}(X)$ is a non-Kähler compact holomorphic symplectic manifold which is not deformation equivalent to a Bogomolov-Guan manifold.*

Proof. Since Bogomolov-Guan manifolds are simply connected, this follows immediately from Propositions 5.12 and 5.14. □

In particular, the above result implies that either $\mathcal{M}_{2,\delta,c}(X)$ is deformation equivalent to a Douady space of points on a Kodaira surface, or $\mathcal{M}_{2,\delta,c}(X)$ belongs to a deformation class separate from other known examples of compact holomorphic symplectic manifolds.

5.3.2 4- and 6-dimensional moduli spaces

In the case where $\Delta(2,\delta,c) \in \{\frac{1}{2},\frac{3}{4}\}$, Proposition 4.8 gives that spectral curves in $\mathbb{P}_{\delta,c}$ contain at most one jump counting multiplicity. In these cases, we can describe the fibres of the graph map above all spectral curves using Proposition 5.8 and [BM06, Theorem 4.5].

Proposition 5.15. *In the case that $\Delta(2,\delta,c) = \frac{1}{2}$ and $t(2,\delta) > \frac{1}{2}$, the space of graphs $\mathbb{P}_{\delta,c}$ is a ruled surface with base B, and $\mathbb{P}^1_{\delta,c}$ is a bisection of $\mathbb{P}_{\delta,c} \to B$. The fibres of $\mathcal{G}$ above points in $\mathbb{P}^0_{\delta,c}$ are 2-dimensional Prym varieties, and the fibres above points in $\mathbb{P}^1_{\delta,c}$ are given by a union of two ruled surfaces with base $T \times \{1,-1\}$, which intersect along two sections.*

Remark 5.16. The set $\{1,-1\}$ in the above statement corresponds to the Prym variety $\mathrm{Prym}(C/B)$, where C is a bisection of $J(X)$ such that $C + \{b\} \times T^* \in \mathbb{P}_{\delta,c}$ for some $b \in B$.

Proof. For these invariants, since $\Delta(2,\delta,c) = \frac{1}{2} > 0$, Proposition 5.5 gives that $\mathbb{P}_{\delta,c}$ is a ruled surface with base B, and as in Remark 5.7, the graphs with jumps are parameterised by $\mathbb{P}^0_{\delta,c-1} \times \mathrm{Hilb}^1(B) = \mathbb{P}_{\delta,c-1} \times B$. Since we have $\Delta(2,\delta,c-1) = 0$, the space of graphs $\mathbb{P}_{\delta,c-1}$ is a two point set by Proposition 5.5. Using the genus formula (4.3), we see that the spectral curves in this scenario can be either a genus 3 curve C or a genus 1 curve C' plus a jump of length one at some $b \in B$. This immediately gives that the fibres $\mathrm{Prym}(C/B)$ above spectral curves C are 2-dimensional. For the fibres above graphs with a jump, Proposition 5.8 gives that both the non-locally free component of the fibre and the closure of the locally free component are ruled surfaces with base $T \times \mathrm{Prym}(C'/B) \cong T \times \{1,-1\}$, and they intersect along $|C' \cap \{b\} \times T^*|$ sections. As any map from a genus 1 curve to B is an unramified covering map, $|C' \cap \{b\} \times T^*| = 2$ for all $b \in B$. This implies that the locally free and non-locally free components will intersect along two sections of the ruled surfaces, regardless of the position of the jump. □

Remark 5.17. Note that in the 4-dimensional case the singular fibres of the graph map are similar to those in the natural Lagrangian fibration on $X^{[2]}$ as discussed in Section 2.2.

Proposition 5.18. *In the case that $\Delta(2,\delta,c) = \frac{3}{4}$, the space of graphs $\mathbb{P}_{\delta,c}$ is a $\mathbb{P}^2$-bundle with base B, and $\mathbb{P}^1_{\delta,c}$ is isomorphic to $B \times B$. The fibres of $\mathcal{G}$ above points in $\mathbb{P}^0_{\delta,c}$ are*

3-dimensional Prym varieties, and the fibres above spectral curves $C + \{b\} \times T^* \in \mathbb{P}^1_{\delta,c}$ are given by a union of two $\mathbb{P}^1$-bundles with base $T \times T$, which intersect along $|C \cap \{b\} \times T^*|$ sections.

Proof. This case is analogous to Proposition 5.15. Proposition 5.5 gives that $\mathbb{P}_{\delta,c}$ is a $\mathbb{P}^2$-bundle with base B, and the spectral curves with a jump are parameterised by $\mathbb{P}_{\delta,c-1} \times B$. Since $\Delta(2,\delta,c-1) = \frac{1}{4}$, we have $\mathbb{P}_{\delta,c-1} \cong B$. The genus formula (4.3) gives that spectral curves can be either a genus 4 curve C, or a genus 2 curve C' plus a jump of length one at some $b \in B$. When C is genus 4, $\mathrm{Prym}(C/B)$ is 3-dimensional. For the fibres above graphs with a jump, Proposition 5.8 gives that both the non-locally free component of the fibre and the closure of the locally free component are $\mathbb{P}^1$-bundles with base $T \times \mathrm{Prym}(C'/B) \cong T \times T$, and they intersect along $|C' \cap \{b\} \times T^*|$ sections. □

Remark 5.19. For the case of Proposition 5.18, since $C' \to B$ is a degree 2 map from a genus 2 curve to a genus 1 curve, the map has ramification at two points. Thus $|C' \cap \{b\} \times T^*| = 1$ if b is the image of a ramification point of $C' \to B$, and $|C' \cap \{b\} \times T^*| = 2$ otherwise.

Remark 5.20. Note that in this case all of the singular fibres of the graph map have a similar complexity, as the singular fibres correspond after allowable elementary modifications to moduli spaces of sheaves of dimensions $6 - 4k$, for some $k \in \mathbb{Z}_{>0}$, of which 2 is the only non-negative value. This contrasts with the Douady space $X^{[3]}$, where the fibres above points of the form $3p \in \mathrm{Sym}^3(B)$ have significantly different behaviour to the singular fibres above points of the form $2p+q \in \mathrm{Sym}^3(B)$. This can be seen by comparing punctual Hilbert schemes of 2 and 3 points as in [Bri77].

5.3.3 Higher dimensions

In general, when $\Delta(2,\delta,c) > 0$, the generic spectral curve is a smooth curve C of genus $4\Delta(2,\delta,c)+1$, and the fibre above C is given by $\mathrm{Prym}(C/B)$. For a spectral curve

$$S = C + \sum_{i=1}^{k} \ell_i \{b_i\} \times T^*,$$

the non-locally free component of $\mathcal{G}^{-1}(S)$ can be found by describing $\mathrm{Quot}(E,h)$ for all vector bundles E in $\mathcal{M}_{2,\delta,c-h}(X)$ with spectral curve

$$S_E = S - \sum_{i=1}^{k} \nu_i\{b_i\} \times T^*$$

for $\{\nu_i\}$ with $0 \leq \nu_i \leq \ell_i$ and $\sum_{i=1}^{k} \nu_i = h$.

The vector bundles with spectral curve S can be described by parameterising the sequences of elementary modifications taking a vector bundle with spectral curve C to one with spectral curve S. This process is described in detail in [Mor03, Section 4] for Hopf surfaces, and the method for Kodaira surfaces is similar. Because of this, the fibres of the graph map above spectral curves with jumps must be computed inductively using information about moduli spaces of lower dimensions, so an understanding of the fibration structure of $\mathcal{M}_{2,\delta,c}(X)$ requires a description of the fibration structure of $\mathcal{M}_{2,\delta,c-k}(X)$ for all $0 \leq k \leq 2\Delta(2,\delta,c)$.

Chapter 6

Higgs Bundles and Vafa–Witten Pairs on Non-Kähler Elliptic Surfaces

This chapter focuses on regularity conditions for moduli spaces of vector bundles on elliptic surfaces, and the relation of this topic to the existence question for Vafa–Witten pairs. When constructing moduli spaces of stable sheaves for a complex manifold, we want to restrict to the case where every vector bundle E represented by the moduli space is *good*, meaning that $H^2(X, \mathcal{E}nd_0(E)) = 0$. Indeed, when all the stable bundles are good, the locally-free locus of the moduli space is guaranteed to be smooth, as the Zariski tangent space is then $H^1(X, \mathcal{E}nd_0(E)) \simeq \mathbb{C}^{-\chi(\mathcal{E}nd_0(E))}$ for any stable bundle E, since

$$\chi(\mathcal{E}nd_0(E)) = h^0(X, \mathcal{E}nd_0(E)) - h^1(X, \mathcal{E}nd_0(E)) + h^2(X, \mathcal{E}nd_0(E)) = -h^1(X, \mathcal{E}nd_0(E)).$$

To see which bundles are good, we can compute the space of Vafa–Witten pairs (E, ϕ): for any vector bundle E that is not good, there will be a non-trivial Higgs field $\phi \in H^0(X, \mathcal{E}nd_0(E) \otimes K_X) \simeq H^2(X, \mathcal{E}nd_0(E))^*$ corresponding to the obstruction class to deformations of E. Thus we are interested in choices of Chern character v so that no vector bundle E with $\mathrm{ch}(E) = v$ admits a non-trivial K_X-Higgs field.

Additionally, if $\pi : X \to B$ is a principal elliptic surface we show that there is a natural correspondence between Vafa–Witten pairs and Higgs bundles, using the fact that the cotangent bundle of such a surface is a non-trivial extension of $\mathcal{O}_X$ by the canonical bundle (Proposition 2.3).

This chapter begins by studying stable V-pairs on non-Kähler elliptic surfaces for a general vector bundle V, before focusing on the case where V is a pullback bundle. We then apply these results to study Vafa–Witten pairs.

For nearly* all non-Kähler elliptic surfaces with Kodaira dimension 1, we show that for any positive-dimensional moduli space of rank-2 vector bundles $\mathcal{M}_{2,\delta,c}(X)$ that contains a reducible bundle, there is a Vafa–Witten pair (E, ϕ) such that E is represented in $\mathcal{M}_{2,\delta,c}(X)$. (Here, the Kodaira dimension is given by the lowest possible degree d such that $H^0(X, K_X^n)$ is bounded above by a polynomial of degree d in n, where the constant function 0 is assigned degree $-\infty$.) This implies that, for these surfaces, smooth moduli spaces of rank-2 vector bundles occur only in the stably irreducible range.

In the process of proving the above result, we also classify the Vafa–Witten pairs involving a reducible vector bundle that is regular in the sense of Definition 4.9 on its generic fibre in terms of the sections of a line bundle on the base of X.

6.1 A necessary condition for trace-free Higgs fields

Let X be a compact complex manifold with Gauduchon metric g and V be a fixed holomorphic vector bundle on X. In this section, we derive some facts about V-pairs (E, ϕ) where E is a holomorphic vector bundle on X and $\phi \in H^0(X, \mathcal{E}nd E \otimes V)$.

We first consider the case where V is a line bundle.

Proposition 6.1. *Let (X, g) be a compact Gauduchon manifold. If V is a line bundle on X and $(E, \phi : E \to E \otimes V)$ is a g-stable V-pair with $\phi \neq 0$, then* $\deg V \geq 0$.

*The methods used in this chapter cannot fully account for the case where X has base $\mathbb{P}^1$ or when the genus of B is one with all multiple fibres of order 2.

Proof. We first note that the Higgs field $\phi : E \to E \otimes V$ on E induces a Higgs field $\phi' := \phi \otimes \mathrm{Id}_V : E \otimes V \to (E \otimes V) \otimes V$ on $E \otimes V$. Moreover, since V is a line bundle, P is a ϕ-invariant subsheaf of E if and only if $P \otimes V$ is a ϕ'-invariant subsheaf of $E \otimes V$. Therefore, (E, ϕ) is stable if and only if $(E \otimes V, \phi')$ is stable. Also note that $\ker \phi$ is a ϕ-invariant subsheaf of E and $\operatorname{im} \phi$ is a ϕ'-invariant subsheaf of $E \otimes V$. Moreover, $\operatorname{rank}(E) = \operatorname{rank}(E \otimes V)$ so that

$$\mu(E \otimes V) = \mu(E) + \deg(V).$$

Suppose that $\mu(\operatorname{im} \phi) = \mu(E \otimes V)$. By stability of $(E \otimes V, \phi')$, we must then have

$$\operatorname{rank}(\operatorname{im} \phi) = \operatorname{rank}(E \otimes V) = \operatorname{rank}(E).$$

However, $\operatorname{im} \phi \simeq E/ \ker \phi$ and $\ker \phi$ is a torsion-free subsheaf of E if ϕ is not injective, in which case $\operatorname{rank}(\operatorname{im} \phi) < \operatorname{rank}(E)$. Hence, $\ker \phi = 0$, implying that $\operatorname{im} \phi \simeq E$. Thus, $\mu(E) = \mu(\operatorname{im} \phi)$ and $\deg V = 0$ in this case.

Let us now assume that $\mu(\operatorname{im} \phi) \neq \mu(E \otimes V)$. Then, $\operatorname{im} \phi$ is a non-zero torsion-free subsheaf of $E \otimes V$ since $\operatorname{im} \phi \neq 0$ by assumption. If $\operatorname{rank}(\operatorname{im} \phi) = \operatorname{rank}(E \otimes V)$, then $\mu(\operatorname{im} \phi) \leq \mu(E \otimes V)$ so that $\mu(\operatorname{im} \phi) < \mu(E \otimes V)$. Otherwise, $\operatorname{im} \phi$ is a ϕ'-invariant proper subsheaf of $E \otimes V$, which means that $\mu(\operatorname{im} \phi) < \mu(E \otimes V)$ by stability of $(E \otimes V, \phi')$. Thus, $\mu(\operatorname{im} \phi) < \mu(E \otimes V)$ in both cases. Consider the exact sequence

$$0 \longrightarrow \ker \phi \longrightarrow E \longrightarrow E/ \ker \phi \simeq \operatorname{im} \phi \longrightarrow 0 \ .$$

If $\ker \phi = 0$, then $E \simeq \operatorname{im} \phi$ and $\mu(E) = \mu(\operatorname{im} \phi)$. Hence,

$$\mu(E) = \mu(\operatorname{im} \phi) < \mu(E \otimes V) = \mu(E) + \deg V,$$

implying that $\deg V > 0$. If instead $\ker \phi \neq 0$, then it is a proper ϕ-invariant subsheaf of E with $\operatorname{rank}(\ker \phi) < \operatorname{rank}(E)$. Indeed, $\ker \phi \neq E$ and $\operatorname{im} \phi \neq 0$ since $\phi \neq 0$. And if $\operatorname{rank}(\ker \phi) = \operatorname{rank} E$, then $\operatorname{im} \phi \simeq E/ \ker \phi$ is a non-zero torsion subsheaf of E, which is impossible. Thus, $\operatorname{rank}(\ker \phi) < \operatorname{rank}(E)$ and $\mu(\ker \phi) < \mu(E)$ by stability of (E, ϕ). Moreover, given the above exact sequence, we obtain $\mu(E) < \mu(\operatorname{im} \phi)$, implying again that $\mu(E) < \mu(E \otimes V)$ and $\deg V > 0$. Putting all cases together gives $\deg V \geq 0$. □

Remark 6.2. This is a result similar to what is known about (co)-Higgs bundles on curves: if X is a curve, then stable (co)-Higgs bundles with non-zero Higgs field exist on X if and only if $\mathcal{T}_X^*$ (resp. $\mathcal{T}_X$) has non-negative degree [Hit87, Ray11].

In the case where V is the trivial line bundle on X, we can say more about g-stable V-pairs:

Proposition 6.3. *Let (X, g) be a compact Gauduchon manifold and (E, ϕ) be a g-stable $\mathcal{O}_X$-pair. Then, $\phi = \lambda Id_E$ for some $\lambda \in \mathbb{C}$.*

Remark 6.4. This proposition gives a slight generalisation of [Bis11, Theorem 2.1].

Proof. Suppose that $\phi \neq 0$ so that $\operatorname{im}\phi \neq 0$. If $\ker\phi = 0$, then $E \simeq \operatorname{im}\phi$ and $\operatorname{im}\phi$ is a subbundle of E with $\operatorname{rank}(\operatorname{im}\phi) = \operatorname{rank}(E)$, implying that ϕ is an automorphism. Let us assume instead that $\ker\phi \neq 0$ so that $\ker\phi$ is a torsion-free subsheaf of E and $\operatorname{rank}(\operatorname{im}\phi) = \operatorname{rank}(E/\ker\phi) < \operatorname{rank}(E)$. Moreover, since $\operatorname{im}\phi \neq 0$, it is a torsion-free subsheaf of E, implying that $\operatorname{rank}(\ker\phi) < \operatorname{rank}(E)$. Hence, $\ker\phi$ is a ϕ-invariant proper subsheaf of E that fits into the exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow E \longrightarrow E/\ker\phi \simeq \operatorname{im}\phi \longrightarrow 0 \ .$$

By stability of (E, ϕ), we have $\mu(\ker\phi) < \mu(E)$, and so $\mu(E) < \mu(\operatorname{im}\phi)$. On the other hand, $\operatorname{im}\phi$ is a non-zero ϕ-invariant subsheaf of E. Moreover, if $\operatorname{rank}(\operatorname{im}\phi) = \operatorname{rank}(E)$, then $\mu(\operatorname{im}\phi) \leq \mu(E)$; and if $\operatorname{rank}(\operatorname{im}\phi) < \operatorname{rank}(E)$, then $\operatorname{im}\phi$ is a proper subsheaf of E, implying that $\mu(\operatorname{im}\phi) < \mu(E)$ by stability of (E, ϕ). Hence, $\mu(E) < \mu(\operatorname{im}\phi) \leq \mu(E)$, leading to a contradiction. Thus, $\ker\phi = 0$ and ϕ is an automorphism of E.

Since $\mathbb{C}$ is algebraically closed and $\phi \neq 0$, there exists $\lambda \in \mathbb{C}^*$ such that $\ker(\phi - \lambda \mathrm{Id}_E) \neq 0$. Moreover, (E, ϕ) is stable if and only if $(E, \phi' = \phi - \lambda\mathrm{Id}_E)$ is stable (because any subsheaf P of E is ϕ-invariant if and only if it is ϕ'-invariant). Therefore, as before, ϕ' is an automorphism if $\phi' \neq 0$. However, ϕ' cannot be an automorphism since $\ker(\phi - \lambda\mathrm{Id}_E) \neq 0$. Hence, $\phi' = 0$ implying that $\phi = \lambda\mathrm{Id}_E$ for some $\lambda \in \mathbb{C}^*$. □

Finally, if V is an extension of vector bundles, we have the following:

Proposition 6.5. *Let (X,g) be a compact Gauduchon manifold. Suppose that V is an extension of holomorphic vector bundles*

$$0 \longrightarrow V_1 \xrightarrow{\iota} V \xrightarrow{p} V_2 \longrightarrow 0$$

on X. If X has no non-trivial g-stable V_1-pairs or V_2-pairs, then it has no non-trivial g-stable V-pairs.

Proof. Suppose that every trace-free g-stable V_i-pair $(E, \varphi_j : E \to E \otimes V_i)$ has $\varphi_j = 0$, where $j = 1, 2$. Let $\phi : E \to E \otimes V$ be a trace-free stable Higgs field, and set $\varphi_2 := (\mathrm{Id}_E \otimes p) \circ \phi$. Suppose P_2 is a φ_2-invariant proper subsheaf of E. Then $\varphi_2(P_2) \subset P_2 \otimes V_2$, so $\phi(P_2) \subset P_2 \otimes V$, meaning that P_2 is also ϕ-invariant. By stability of (E, ϕ), we have $\mu(P_2) < \mu(E)$ so that (E, φ_2) is also stable. Since φ_2 is clearly trace-free, $\varphi_2 = 0$, implying that $\phi = \varphi_1 \otimes \iota$ for some trace-free $\varphi_1 : E \to E \otimes V_1$, where ι is the map appearing in the statement of the proposition. Suppose P_1 is a φ_1-invariant proper subsheaf of E. Then $\varphi_1(P_1) \subset P_1 \otimes V_1 \subset P_1 \otimes V$, so $\mu(P_1) < \mu(E)$ by stability of (E, ϕ). This implies that (E, φ_1) is stable, so $\phi = \varphi_1 \otimes \iota = 0$. □

We now assume that $\pi : X \to B$ is a non-Kähler elliptic surface, and that the Gauduchon metric g on X is normalised so that $\deg_g(\pi^* L) = \deg(L)$ for any $L \in \mathrm{Pic}(B)$.

Let us consider Higgs fields $\phi : E \to E \otimes V$, where $V = \pi^* W$ for some vector bundle W on B.

Proposition 6.6. *Let E be a rank-2 filtrable vector bundle on X with maximal destabilising bundles K_1 and K_2 as in Proposition 3.5. Set $H := \pi_*(\det(E)^{-1} \otimes K_1 \otimes K_2)$.*

(a) Suppose that E is regular on the generic fibre of π. Then,

$$h^0(X, \mathcal{E}nd_0(E) \otimes V) = h^0(B, H \otimes W).$$

In particular, $H^0(X, \mathcal{E}nd_0 E \otimes V) \simeq H^0(X, \mathcal{H}om(E, K_i \otimes V)$ for $i = 1, 2$, implying that K_1 and K_2 are both ϕ-invariant for all $\phi \in H^0(X, \mathcal{E}nd E \otimes V)$. A Higgs field $\phi : E \to E \otimes V$ is thus stable if and only if E is.

(b) *Suppose that E is not regular on the generic fibre of π so that $K = K_1 = K_2$ is its unique maximal destabilising bundle. We have two cases:*

   (i) *If E is an extension of line bundles, then $E \simeq K \otimes \pi^*(F)$ for some rank-2 vector bundle F on B that is an extension of H^{-1} by $\mathcal{O}_B$. Furthermore, $H^0(X, \mathcal{E}nd E \otimes V) = H^0(B, \mathcal{E}nd F \otimes W)$ so the Higgs fields on E twisted by V are precisely the pullbacks of Higgs fields on F twisted by W.*

   (ii) *If E is not an extension of line bundles, then*

$$h^0(X, \mathcal{E}nd_0(E) \otimes V) \geq h^0(X, \mathcal{H}om(E, K \otimes V) \geq h^0(B, H \otimes W).$$

In fact, $H^0(X, \mathcal{E}nd_0 E \otimes V) \simeq H^0(X, \mathcal{H}om(E, K \otimes V)$ if $h^0(B, \pi_*(K^{-1} \otimes E) \otimes W) = h^0(B, W)$, in which case K is ϕ-invariant for all Higgs fields $\phi : E \to E \otimes V$, and Higgs fields are stable if and only if E is.

Remark 6.7. By [BM05b, Proposition 3.4], the bundle $\det(E)^{-1} \otimes K_1 \otimes K_2 = \pi^*(H)$ for some $H \in \text{Pic}(B)$ whenever E is filtrable. Moreover, $h^0(B, \pi_*(K^{-1} \otimes E) \otimes W) = h^0(B, W)$ when $h^0(X, \pi^* H^{-1} \otimes V) = 0$.

Proof. Since E is filtrable with maximal destabilising bundle K_1, it fits into an exact sequence of the form

$$0 \longrightarrow K_1 \xrightarrow{\ i\ } E \xrightarrow{\ p\ } \delta \otimes K_1^{-1} \otimes \mathcal{I}_Z \longrightarrow 0 \ , \tag{6.1}$$

where $\delta = \det(E)$ and Z is a zero-dimensional subset of X.

Let us first assume that E is regular on the generic fibre of π. Then, $\pi_*(K_1^{-1} \otimes E) = \pi_*(K_2^{-1} \otimes E) = \mathcal{O}_B$. To determine the dimension of the space of trace-free Higgs fields on E, we tensor the exact sequence (6.1) by $E^\vee \otimes V = \delta^{-1} \otimes E \otimes V$ and look at cohomology. This gives the left-exact sequence

$$0 \longrightarrow \text{Hom}_{\mathcal{O}_X}(E, K_1 \otimes V) \longrightarrow H^0(X, \mathcal{E}nd(E) \otimes V) \longrightarrow H^0(X, K_1^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z).$$

Note that for any element of $H^0(X, \mathcal{E}nd(E) \otimes V)$ of the form $\text{Id} \otimes s$ with $s \in H^0(X, V)$, its image in $H^0(X, K_1^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z)$ is zero if and only if $s = 0$, so

$$h^0(X, K_1^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z) \geq h^0(X, V) = h^0(B, W).$$

Since $K_1^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z$ is a subsheaf of $K_1^{-1} \otimes E \otimes V$, we also have

$$\begin{aligned} h^0(X, K_1^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z) &\leq h^0(X, K_1^{-1} \otimes E \otimes V) \\ &= h^0(B, \pi_*(K_1^{-1} \otimes E) \otimes W) = h^0(B, W). \end{aligned}$$

Therefore, $H^0(X, V) \simeq H^0(X, K_1^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z)$ and

$$H^0(X, \mathcal{E}nd_0(E) \otimes V) \simeq \mathrm{Hom}_{\mathcal{O}_X}(E, K_1 \otimes V).$$

Note that

$$\begin{aligned} \mathcal{H}om(E, K_1 \otimes V) &\simeq E^\vee \otimes K_1 \otimes V \simeq E \otimes \delta^{-1} \otimes K_1 \otimes V \simeq E \otimes K_2^{-1} \otimes \pi^*(H) \otimes V \\ &\simeq (K_2^{-1} \otimes E) \otimes \pi^*(H \otimes W). \end{aligned}$$

Consequently,

$$\begin{aligned} H^0(X, \mathcal{E}nd_0(E) \otimes V) &\simeq H^0(X, (K_2^{-1} \otimes E) \otimes \pi^*(H \otimes W)) \\ &= H^0(B, \pi_*(K_2^{-1} \otimes E) \otimes H \otimes W) = H^0(B, H \otimes W). \end{aligned}$$

Since every non-trivial Higgs field ϕ on E is of the form

$$\phi = (\iota \otimes \mathrm{Id}_V) \circ \psi - \frac{1}{2}\mathrm{tr}((\iota \otimes \mathrm{Id}_V) \circ \psi)$$

for some $\psi \in \mathrm{Hom}(E, K_1 \otimes V)$, this means that K_1 is always ϕ-invariant; a similar argument shows that K_2 is also ϕ-invariant, implying that (E, ϕ) is stable if and only if E is. This proves (a).

Let us now assume that E is not regular on the generic fibre of π so that $K_1 = K_2$. In other words, $K = K_1 = K_2$ is the unique maximal destabilising bundle of E. If E is an extension of line bundles, then $Z = \emptyset$ and the exact sequence (6.1) becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow K^{-1} \otimes E \longrightarrow \pi^* H^{-1} \longrightarrow 0 \tag{6.2}$$

after tensoring by K^{-1}. Recall that extensions of the form (6.2) are parameterised by $H^1(X, \pi^* H)$, and $H^1(X, \pi^* H) \cong H^0(B, H) \oplus H^1(B, H)$ by the Leray spectral sequence.

Furthermore, any extension whose representative has non-zero first factor in this decomposition will be regular on the generic fibre, so those which are not regular on the generic fibre of π are parameterised by $H^1(B, H)$. This group also parameterises extensions of H^{-1} by $\mathcal{O}_B$ on B, and the pullback of such an extension is not regular on the generic fibre of π. Therefore, every extension of π^*H^{-1} by $\mathcal{O}_X$ that is not regular on the generic fibre of π is a pullback. Let F be the rank-2 bundle on B such that $K \otimes \pi^*F \cong E$. Then,

$$H^0(X, \mathcal{E}nd E \otimes V) = H^0(B, \pi_*(\mathcal{E}nd E) \otimes W) = H^0(B, \mathcal{E}nd F \otimes W),$$

proving (b) (i).

Finally, let us assume E is not regular on the generic fibre of π and is not an extension of line bundles. Then, $Z \neq \emptyset$ and $\pi_*(K^{-1} \otimes E)$ is a rank 2 vector bundle on B given by an extension of the form

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \pi_*(K^{-1} \otimes E) \longrightarrow L \longrightarrow 0 \tag{6.3}$$

with $L = H^{-1} \otimes \pi_*(\mathcal{I}_Z) \in \mathrm{Pic}(B)$. By taking the tensor product of the exact sequence (6.1) with $E^\vee \otimes V = \delta^{-1} \otimes E \otimes V$, we obtain

$$0 \longrightarrow \mathcal{H}om(E, K \otimes V) \xrightarrow{\ i\ } \mathcal{E}nd(E) \otimes V \xrightarrow{\ p\ } K^{-1} \otimes E \otimes V \otimes \mathcal{I}_Z \longrightarrow 0\ ,$$

and as in the previous cases the multiples of the identity in $\mathcal{E}nd(E) \otimes V$ are mapped injectively by p, so the space of trace-free Higgs fields contains a subspace isomorphic to

$$H^0(X, \mathcal{H}om(E, K \otimes V)) \simeq H^0(B, \pi_*(K^{-1} \otimes E) \otimes H \otimes W) = h^0(B, H \otimes W)$$

by (6.3) since $h^0(B, \pi_*(\mathcal{I}_Z)) = h^0(X, \mathcal{I}_Z) = 0$ because $Z \neq \emptyset$. Therefore,

$$h^0(X, \mathcal{E}nd_0(E) \otimes V) \geq h^0(B, \mathcal{H}om(E, K \otimes V)) = h^0(B, H \otimes W).$$

In fact, $H^0(X, \mathcal{E}nd_0 E \otimes V) \simeq H^0(X, \mathcal{H}om(E, K \otimes V)$ whenever $h^0(B, \pi_*(K^{-1} \otimes E) \otimes W) = h^0(B, W)$, in which case K is ϕ-invariant for all Higgs fields $\phi : E \to E \otimes V$, and Higgs fields are stable if and only if E is. $\square$

Remark 6.8. If $\pi : X \to B$ has multiple fibres $\{T_i\}_{i=1}^r$ lying over points $p_i \in B$ with multiplicities m_i, then $K_1 \otimes K_2 \otimes \det(E)^{-1}$ will be of the form

$$\pi^*(H) \otimes \mathcal{O}_X \left(\sum_{i=1}^{r} a_i T_i \right)$$

with $H \in \mathrm{Pic}(B)$ and $0 \leq a_i < m_i$. In this case the results of Proposition 6.6 parts (a) and (b)(ii) hold with H replaced by

$$H \otimes \mathcal{O}_B \left(\sum_{i=1}^{r} \left\lfloor \frac{a_i}{m_i} \right\rfloor p_i \right).$$

6.2 Vafa–Witten pairs in the Kodaira dimension 1 case

Let $\pi : X \to B$ be a non-Kähler principal elliptic surface with base B. Note that for any trace-free stable Higgs bundle (E, ϕ) on X, ϕ is of the form $(\mathrm{Id}_E \otimes i) \circ \varphi$ for some trace-free $\varphi : E \to E \otimes K_X$ with (E, φ) stable by Proposition 6.5, where $i : K_X \to \mathcal{T}_X^*$ is the injection given in Proposition 2.3. One can easily verify that $((\mathrm{Id}_E \otimes i) \circ \varphi) \wedge ((\mathrm{Id}_E \otimes i) \circ \varphi) = 0$ for any $\varphi \in H^0(\mathcal{E}nd_0(E) \otimes K_X)$, so it suffices to study trace-free stable K_X-pairs. As a consequence of this, if B has genus 0 or 1 there will be no non-trivial Higgs bundles on X. For the remainder of the chapter we will thus focus on the case where X has Kodaira dimension 1.

Proposition 6.9. *Let E be a rank-2 filtrable vector bundle with maximal destabilising bundles K_1 and K_2. Set $H := \pi_*(\det(E)^{-1} \otimes K_1 \otimes K_2)$.*

(a) Suppose that E is regular on the generic fibre of π. Then,

$$H^0(X, \mathcal{E}nd_0(E) \otimes K_X) \simeq H^0(B, H \otimes K_B),$$

and a trace-free Higgs field $\phi : E \to E \otimes K_X$ is stable if and only if E is.

(b) Suppose that E is not regular on the generic fibre of π so that $K = K_1 = K_2$ is its unique maximal destabilising bundle. We have two cases:

(i) If E is an extension of line bundles, then $E \simeq K \otimes \pi^*(F)$ for some rank-2 vector bundle F on B that is an extension of H^{-1} by $\mathcal{O}_B$. Furthermore,

$$H^0(X, \mathcal{E}nd E \otimes K_X) = H^0(B, \mathcal{E}nd F \otimes K_B)$$

so that Higgs fields on E are pullbacks of Higgs fields on F.

(ii) If E is not an extension of line bundles, then

$$h^0(X, \mathcal{E}nd_0(E) \otimes K_X) \geq h^0(X, \mathcal{H}om(E, K \otimes K_X)) = h^0(B, H \otimes K_B).$$

In fact, $H^0(X, \mathcal{E}nd_0 E \otimes K_X) \simeq H^0(X, \mathcal{H}om(E, K \otimes K_X))$ whenever $h^0(B, K_B) = h^0(B, \pi_*(K^{-1} \otimes E) \otimes K_B)$, in which case K is ϕ-invariant for all Higgs fields $\phi : E \to E \otimes K_X$, and Higgs fields are stable if and only if E is.

Proof. This is a direct consequence of Proposition 6.6 with $V = K_X = \pi^* K_B$. □

Remark 6.10. To get solutions to the Vafa–Witten equations for surfaces with multiple fibres (i.e. stable V-pairs (E, ϕ) with $V = K_X = \pi^* K_B \otimes \omega_{X/B}$), we set

$$K_1 \otimes K_2 \otimes \det(E)^{-1} = \pi^*(H) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r} a_i T_i\right)$$

as in Remark 6.8 and obtain the results of Proposition 6.9 (a) and (b)(ii) with $H \otimes K_B$ replaced by

$$H \otimes K_B \otimes \mathcal{O}_B\left(\sum_{i=1}^{r} \left\lfloor \frac{m_i + a_i - 1}{m_i} \right\rfloor p_i\right).$$

By using the descriptions of the trace-free Higgs fields in the above cases, we can show that the Chern classes of a filtrable rank-2 bundle with a non-trivial Higgs field are only restricted in that the Chern classes must admit filtrable bundles.

Proposition 6.11. *Let E be a rank-2 filtrable bundle with $c(E) = (1, c_1, c_2)$. Then, there is a stable rank-2 vector bundle F with $\det(F) = \det(E)$ and $c_2(F) = c_2(E)$ such that F has a non-trivial trace-free Higgs field.*

Proof. Since E is filtrable, there are line bundles L_1 and L_2 and a finite set of points Z on X (counting multiplicity) such that E fits into an exact sequence of the form

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \otimes \mathcal{I}_Z \longrightarrow 0 \ .$$

In particular, $\det E = L_1 \otimes L_2$ and $c_2(E) = c_1(L_1) \cdot c_1(L_2) + |Z|$. Moreover, if Σ_1 and Σ_2 are the sections of $J(X)$ corresponding to L_1 and L_2, respectively, then the spectral curve of E is

$$S_E = \left(\Sigma_1 + \Sigma_2 + \sum_{i=1}^{|Z|} \{b_i\} \times T^* \right),$$

where each b_i is $\pi(z_i)$ for some $z_i \in Z$. Recall that $\Sigma_1 \neq \Sigma_2$ if and only if $\Sigma_1 \cdot \Sigma_2 \neq 0$ if and only if $c_1(L_1 \otimes L_2^{-1})^2 \neq 0$. We split the proof into three cases:

(i) $Z \neq \emptyset$,

(ii) $Z = \emptyset$ and $\Sigma_1 \neq \Sigma_2$, and

(iii) $Z = \emptyset$ and $\Sigma_1 = \Sigma_2$.

We first consider the case where $Z \neq \emptyset$. Let L_a be a line bundle on X given by a constant factor of automorphy a such that

$$-1 < \deg(L_1) - \deg(L_2) + 2\deg(L_a) < 0.$$

Since the degree function maps line bundles with constant factor of automorphy surjectively onto $\mathbb{R}$, such an L_a always exists. Note that $\deg(L_1) - \deg(L_2) + 2\deg(L_a)$ is not an integer. Therefore, $L_1 \otimes L_2^{-1} \otimes L_a^2$ is not the pullback of a line bundle on B so that its restriction to at least one fibre of π is not trivial. In other words, $(L_1 \otimes L_a)|_{\pi^{-1}(b)} \not\cong (L_2 \otimes L_a^{-1})|_{\pi^{-1}(b)}$ for some point $b \in B$. Define

$$F' := (L_1 \otimes L_a \otimes \pi^*(\mathcal{O}_B(b))) \oplus (L_2 \otimes L_a^{-1}).$$

Then, $\det F' = \det E(\pi^{-1}(b))$ and $c_2(F') = c_1(L_1) \cdot c_1(L_2)$ because $c_1(\mathcal{O}_X(\pi^{-1}(b)))$ is a torsion element of $H^2(X, \mathbb{Z})$. Let λ be any line bundle on $\pi^{-1}(b)$ such that $\deg(\lambda) = |Z| > 0$, and define F to be the elementary modification

$$0 \longrightarrow F \longrightarrow F' \longrightarrow \iota_{b,*}\lambda \longrightarrow 0 \ ,$$

where ι_b is the inclusion of $\pi^{-1}(b)$ into X. Then $\det(F) = \det(E)$, $c_2(F) = c_2(E)$, and F has maximal destabilising bundles $L_1 \otimes L_a$ and $L_2 \otimes L_a^{-1} \otimes \pi^*(\mathcal{O}_B(-b))$. Therefore, F fits into the exact sequences

$$0 \longrightarrow L_1 \otimes L_a \longrightarrow F \longrightarrow L_2 \otimes L_a^{-1} \otimes \mathcal{I}_Y \longrightarrow 0,$$

$$0 \longrightarrow L_2 \otimes L_a^{-1} \otimes \pi^*(\mathcal{O}_B(-b)) \longrightarrow F \longrightarrow L_1 \otimes L_a \otimes \pi^*(\mathcal{O}_B(b)) \otimes \mathcal{I}_Y \longrightarrow 0$$

for some finite set of points Y on X (counting multiplicity) with $|Y| = |Z| > 0$. By assumption,

$$\deg(L_2 \otimes L_a^{-1} \otimes \pi^*(\mathcal{O}_B(-b))) < \deg(L_1 \otimes L_a) < \deg(L_2 \otimes L_a^{-1})$$

so that

$$\mu(L_1 \otimes L_a) = \deg(L_1 \otimes L_a) < \frac{1}{2}(\deg L_1 + \deg L_2) = \mu(F)$$

and

$$\mu(L_2 \otimes L_a^{-1} \otimes \pi^*(\mathcal{O}_B(-b))) = \deg(L_2 \otimes L_a^{-1} \otimes \pi^*(\mathcal{O}_B(-b))) < \frac{1}{2}(\deg L_1 + \deg L_2) = \mu(F).$$

Hence, F is a stable bundle. Finally, by Proposition 6.9 the trace-free Higgs fields on F are parameterised by $H^0(B, K_B(-b))$ and by Riemann-Roch,

$$h^0(B, K_B(-b)) = h^0(B, \mathcal{O}_B(b)) + (g-2) = 1 + (g-2) = g - 1 \geq 1,$$

since $g = g(B) \geq 2$, so F has a non-zero trace-free Higgs field, proving case (i).

Let us now assume that $Z = \emptyset$ and that the spectral curve of E has non-trivial self-intersection. Then, $c_2(E) = c_1(L_1) \cdot c_1(L_2)$ and $c_1(L_1 \otimes L_2^{-1})^2 \neq 0$. We again choose L_a to be a line bundle on X given by a constant factor of automorphy a such that

$$-1 < \deg(L_1) - \deg(L_2) + 2\deg(L_a) < 0.$$

Extensions

$$0 \longrightarrow L_1 \otimes L_a \longrightarrow F \longrightarrow L_2 \otimes L_a^{-1} \longrightarrow 0$$

are parameterised by

$$H^1(X, \mathcal{H}om(L_2 \otimes L_a^{-1}, L_1 \otimes L_a)) \cong H^0(B, R^1\pi_*(L_2^{-1} \otimes L_1 \otimes L_a^2)),$$

since $\pi_*(L_2^{-1} \otimes L_1 \otimes L_a^2) = 0$, where $R^1\pi_*(L_2^{-1} \otimes L_a \otimes L_a^2)$ is a torsion sheaf supported on points $b \in B$ such that

$$\Sigma_{L_1 \otimes L_a} \cap \Sigma_{L_2 \otimes L_a^{-1}} \cap \pi^{-1}(b) \neq \emptyset.$$

(By the assumption that $c_1(L_2^{-1} \otimes L_1)^2 \neq 0$, there is at least one such point in the support.) In addition, for any choice of section $s \in H^0(B, R^1\pi_*(L_2^{-1} \otimes L_1 \otimes L_a^2))$, the corresponding extension has maximal destabilising bundles $L_1 \otimes L_a$ and $L_2 \otimes L_a^{-1} \otimes \pi^*\mathcal{O}_B(-D_s)$, where D_s is the divisor on which s is supported. Therefore, if we choose s so that it is supported on a single point b, the corresponding extension F will have maximal destabilising bundles $L_1 \otimes L_a$ and $L_2 \otimes L_a^{-1} \otimes \pi^*\mathcal{O}_B(-b)$. The bundle F is stable and has the same determinant and second Chern class as E. Furthermore, the trace-free Higgs fields on F are parameterised by $H^0(B, K_B(-b)) \neq 0$.

Finally, let us assume that $Z = \emptyset$ and $\Sigma_1 = \Sigma_2$. Therefore, $L_1 \simeq L_2 \otimes \pi^*H$ for some line bundle $H \in \mathrm{Pic}(B)$ so that $\det(E) = L_1^2 \otimes \pi^*H^{-1}$ and $c_2(E) = c_1(L_1)^2$. Let V be a rank-2 stable bundle on B with determinant H^{-1} and set $F = L_1 \otimes \pi^*V$. Then, F is a stable bundle on X with $\det(F) = \det(E)$ and $c_2(F) = c_2(E)$. Since V is a stable bundle on B, it is simple, implying that $h^0(B, \mathcal{E}nd_0 V) = 0$. By Riemann-Roch, we then have $h^0(B, \mathcal{E}nd_0 V \otimes K_B) = h^1(B, \mathcal{E}nd_0 V) = 3(1-g) > 0$. Hence, V must have a non-trivial trace-free Higgs field, which lifts to a non-trivial trace-free Higgs field on F, proving (iii). □

Remark 6.12. The above result also holds for elliptic surfaces $\pi : X \to B$ where B has genus $g \geq 1$ and π has multiple fibres, although an assumption that there is a multiple fibre of order $m \geq 3$ is required for the proof of case (iii) when B has genus one.

Finally, we consider the case where E is not filtrable.

Proposition 6.13. *If E is a rank-2 non-filtrable bundle on X, then E has a non-trivial trace-free Higgs field if and only if the spectral curve of E is a bisection whose normalisation is an unramified cover of B.*

Proof. Suppose E is a rank-2 non-filtrable bundle on X. This means, in particular, that E is stable and so $h^0(X, \mathcal{E}nd_0(E)) = 0$. Moreover, on the general fibre $\pi^{-1}(b)$ of π,

$$\mathcal{E}nd_0(E) \otimes K_X|_{\pi^{-1}(b)} = \mathcal{O}_{\pi^{-1}(b)} \oplus \lambda \oplus \lambda^{-1}$$

with $\lambda \in \mathrm{Pic}^0(\pi^{-1}(b))$ such that $\lambda^2 \neq \mathcal{O}_{\pi^{-1}(b)}$. Consequently, $\pi_*(\mathcal{E}nd_0(E) \otimes K_X)$ is a rank 1 torsion free sheaf on B. In other words,

$$\pi_*(\mathcal{E}nd_0(E) \otimes K_X) = L$$

for some $L \in \mathrm{Pic}(B)$. Hence, $H^0(X, \mathcal{E}nd_0(E) \otimes K_X) = H^0(B, L)$ and $\pi_*(\mathcal{E}nd_0(E)) = L \otimes K_B^{-1}$, implying that $H^0(B, \mathcal{H}om(K_B, L)) = 0$.

Note that E has non-trivial trace-free Higgs fields if and only if $h^0(B, L) \neq 0$. In fact, we show that $h^0(B, L) \neq 0$ if and only if $K_B^{-1} \otimes L$ is a non-trivial half-period of $\mathrm{Pic}^0(B)$. (Such elements exist since $g = g(B) \geq 2$.) Indeed, if $h^0(B, L) \neq 0$, then E admits a non-trivial trace-free Higgs field ϕ. Since $\phi \neq 0$ and E is non-filtrable, ϕ is injective, so there is an elementary modification

$$0 \longrightarrow E \xrightarrow{\ \phi\ } E \otimes K_X \longrightarrow \mathcal{Q} \longrightarrow 0 \ ,$$

where $\mathcal{Q}$ is a torsion sheaf supported on an effective divisor D linearly equivalent to $\pi^* L$, and $\mathcal{Q}|_{\pi^{-1}(b)} = E|_{\pi^{-1}(b)}$ for every $b \in B$ with $\pi^{-1}(b) \in D$. The determinant relations for elementary modifications then tell us that $\det E \simeq \det(E \otimes K_X)(-2D)$ as the restriction of $\mathcal{Q}$ to D has rank 2. Given that $\mathcal{O}_X(D) = \pi^* L$, we obtain

$$\det(E) \otimes \pi^* L^2 = \det(E \otimes K_X) = \det(E) \otimes K_X^2,$$

implying that $K_B \otimes L^{-1} = \lambda_0$ for some $\lambda_0 \in \mathrm{Pic}^0(B)$ such that $\lambda_0^2 = \mathcal{O}_B$. Since $H^0(B, \mathcal{H}om(K_B, L)) = 0$, we must have $\lambda_0 \neq \mathcal{O}_B$. Conversely, if $K_B \otimes L^{-1} \cong \lambda_0$ is a non-trivial half-period of $\mathrm{Pic}^0(B)$, Riemann-Roch gives $h^0(B,L) = h^0(B,L) - h^0(B, K_B \otimes L^{-1}) = g-1 > 0$ since $g = g(B) \geq 2$.

The proof therefore boils down to showing that $K_X \otimes L^{-1}$ is a non-trivial half-period of $\mathrm{Pic}^0(B)$ if and only if the spectral curve of E is a bisection $\overline{C}$ of $J(X)$ whose normalisation C is an unramified cover of B. To do this, we first describe how the spectral curve of E is related to $2c_1(K_X \otimes L^{-1})$.

Given that $\pi_*(\mathcal{E}nd_0 E \otimes K_X) = L$, the pullback $\pi^* L$ injects into $\mathcal{E}nd_0 E \otimes K_X$ and we have an exact sequence

$$0 \longrightarrow \pi^* L \longrightarrow \mathcal{E}nd_0 E \otimes K_X \longrightarrow \mathcal{F} \longrightarrow 0$$

with $\mathcal{F}$ a rank-2 sheaf on X whose torsion-free component is non-filtrable. Taking the long exact sequence induced by pushforward gives

$$0 \longrightarrow \pi_*(\mathcal{F}) \longrightarrow R^1\pi_*(\pi^* L) \longrightarrow R^1\pi_*(\mathcal{E}nd_0(E) \otimes K_X) \longrightarrow R^1\pi_*(\mathcal{F}) \longrightarrow 0\ .$$

Notice that $R^1\pi_*(\pi^* L) = L$. Moreover, since the torsion-free part of $\mathcal{F}$ is non-filtrable, $\pi_*(\mathcal{F})$ and $R^1\pi_*(\mathcal{F})$ are both torsion sheaves on B. And as the torsion sheaf $\pi_*(\mathcal{F})$ injects into L, we must in fact have $\pi_*(\mathcal{F}) = 0$, giving us the short exact sequence

$$0 \longrightarrow L \longrightarrow R^1\pi_*(\mathcal{E}nd_0 E \otimes K_X) \longrightarrow R^1\pi_*(\mathcal{F}) \longrightarrow 0\ .$$

Furthermore, since $\mathcal{F}$ is the quotient of $\mathcal{E}nd_0(E) \otimes K_X$ by a pullback line bundle, its Chern character is given by $2 + (c_1^2(E) - 4c_2(E))w_X = 2 - 8\Delta(E)w_X$, where w_X is the generator of $H^4(X, \mathbb{Z})$. If we now consider Grothendieck-Riemann-Roch for $\pi_!(\mathcal{F})$, we have $\pi_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(X)) = -8\Delta(E)w_B$, where B is the generator of $H^2(B, \mathbb{Z})$, and

$$\mathrm{ch}(\pi_!(\mathcal{F})) \cdot \mathrm{td}\, B = (\mathrm{ch}(0) - \mathrm{ch}(R^1\pi_*\mathcal{F}) \cdot (1 + (1-g)w_B) = -\mathrm{ch}(R^1\pi_*\mathcal{F}),$$

implying that $c_1(R^1\pi_*\mathcal{F}) = 8\Delta(E)$.

By relative Serre duality,

$$\begin{aligned} R^1\pi_*(\mathcal{E}nd_0 E \otimes K_X)^\vee &\simeq \pi_*(\mathcal{E}nd_0 E \otimes K_X^{-1} \otimes \omega_{X/B}) \\ &\simeq \pi_*(\mathcal{E}nd_0 E \otimes K_X) \otimes K_B^{-2} = L \otimes K_B^{-2}, \end{aligned}$$

We therefore have

$$R^1\pi_*(\mathcal{E}nd_0 E \otimes K_X) \simeq (K_X^2 \otimes L^{-1}) \oplus \mathcal{S},$$

where $\mathcal{S}$ the torsion component of $R^1\pi_*(\mathcal{E}nd_0(E) \otimes K_X)$. Using Grothendieck-Riemann-Roch again for $\pi_!(\mathcal{E}nd_0 E \otimes K_X)$, we have

$$\mathrm{ch}(\pi_!(\mathcal{E}nd_0 E \otimes K_X)) = \mathrm{ch}(L) - \mathrm{ch}(K_B^2 \otimes L^{-1}) - \mathrm{ch}(\mathcal{S}) = -8\Delta(E) w_B,$$

so in particular

$$c_1(\mathcal{S}) = 8\Delta(E) - 2c_1(K_B \otimes L^{-1}).$$

Note that $\mathcal{S}$ is the skyscraper sheaf whose support is the set of points $b \in B$ where $E|_{\pi^{-1}(b)}$ is not regular, weighted in the following manner: the skyscraper sheaf has rank 2 at points where $E|_{\pi^{-1}(b)} = \lambda_0 \oplus \lambda_0$, and rank $2m$ when the allowable elementary modification of E at b decreases the second Chern class by m.

Suppose that the spectral curve of E has bisection $\overline{C}$. The fibres where E is not regular thus correspond either to a jump or to a singularity of $\overline{C}$. Let C be the normalisation of $\overline{C}$ and R be its ramification divisor over B. Moreover, let $s = \sum_{p \in \overline{C}} \epsilon_p$, where $\epsilon_p \in \mathbb{Z}^{\geq 0}$ is the degree of singularity of p (so that p is a singular point of $\overline{C}$ if and only if $\epsilon_p > 0$). Then, $g(C) = \rho_a(\overline{C}) + s$ and

$$\rho_a(\overline{C}) = 2g - 1 + \frac{1}{2}\deg R + s$$

by the Riemann-Hurwitz formula. However, the self-intersection formula tells us that

$$\rho_a(\overline{C}) = \frac{\overline{C}^2}{2} + 2g - 1.$$

Therefore, $\overline{C}^2 = \deg R + 2s$. Furthermore, since E is non-filtrable, we have $8\Delta(E) = \overline{C}^2 + 2\ell$, where ℓ is the number of points in $\overline{C}$ lying on a vertical component of the spectral curve of E, weighted with multiplicity [BM05a, Theorem 4.1]. Note that ℓ is obtained by considering the pullback of E to the fibred product $X \times_B C$, which is a double cover of X. Since this pullback essentially has twice the number of jumps, we then have

$$c_1(\mathcal{S}) = 2s + \ell,$$

implying that $8\Delta(E) = \deg R + 2s + 2\ell = \deg R + \ell + c_1(\mathcal{S})$ and

$$\deg R + \ell = 2c_1(K_B \otimes L^{-1}). \tag{6.4}$$

We are now in a position to finish the proof. Suppose that $K_B \otimes L^{-1}$ is a non-trivial half-period of $\mathrm{Pic}^0(B)$. Then, $c_1(K_B \otimes L^{-1}) = 0$ and $\deg R + \ell = 0$ by equation (6.4). Since $\deg R$ and ℓ are both non-negative integers, this means that the spectral curve of E is a bisection whose normalisation is unramified. Conversely, if the spectral curve of E is a bisection whose normalisation is unramified, then $\deg R = \ell = 0$ and $c_1(K_X \otimes L^{-1}) = 0$ so that $K_X \otimes L^{-1} = \lambda_0$ for some $\lambda_0 \in \mathrm{Pic}^0(B)$. But L maps into the torsion-free component $K_X^2 \otimes L^{-1}$ of $R^1\pi_*(\mathcal{E}nd_0(E) \otimes K_X)$, which means that $h^0(B, \lambda_0^2) = h^0(B, \mathcal{H}om(L, K_X^2 \otimes L^{-1})) \neq 0$, implying that $\lambda_0^2 = \mathcal{O}_B$. □

Remark 6.14. If the Chern classes of E are in the stably irreducible range and $\Delta(E) > 0$, then E is regular away from jumps by Proposition 4.13, so E does not admit non-zero trace-free Higgs fields by the above proposition.

We end the chapter with an application of our analysis of Higgs bundles on non-Kähler elliptic surfaces to the smoothness of moduli spaces of bundles on them.

Theorem 6.15. *Let $\pi : X \to B$ be a non-Kähler elliptic surface with base curve B of genus at least two. Let $\mathcal{M}_{2,\delta,c}(X)$ be the moduli space of rank-2 stable bundles on X with determinant δ and second Chern class c. If a non-empty moduli space $\mathcal{M}_{2,\delta,c}(X)$ has $\Delta(2, \delta, c) > 0$, then every bundle is good if and only if the moduli space is stably irreducible. Equivalently, when $\Delta(2, \delta, c) > 0$, a non-empty moduli space $\mathcal{M}_{2,\delta,c}(X)$ is smooth (as a ringed space) of dimension $8\Delta(2, \delta, c)$ if and only if it is stably irreducible.*

Proof. By a standard deformation theory argument, the Zariski tangent space of $\mathcal{M}_{2,\delta,c}(X)$ at a point corresponding a vector bundle E is isomorphic to $H^1(X, \mathcal{E}nd_0(E))$. Since E is by assumption stable we have

$$\begin{aligned} h^1(X, \mathcal{E}nd_0(E)) &= h^2(X, \mathcal{E}nd_0(E)) - \chi(X, \mathcal{E}nd_0(E)) \\ &= h^0(X, \mathcal{E}nd_0(E) \otimes K_X) - \chi(X, \mathcal{E}nd_0(E)), \end{aligned}$$

so $\mathcal{M}_{2,\delta,c}(X)$ is singular at E if and only if E has a non-zero trace-free Higgs field. The result then follows immediately from Proposition 6.11 and Remark 6.14. □

Remark 6.16. As discussed in Section 3.2, X admits moduli spaces of sheaves in the stably irreducible range if there is a line bundle $\delta \in \operatorname{Pic}(X)$ with $e_\delta < 4t(2, \delta)$. Since $e_\delta \equiv c_1(\delta)^2/2 \mod 2$, it is sufficient to find an elliptic surface X with base curve B of genus g and a line bundle $\delta \in \operatorname{Pic}(X)$ such that $\nu(X) \geq g + 1$. This is equivalent to finding a genus g curve B and an elliptic curve T such that there is a non-constant map from B to T and every non-constant map from B to T has degree at least $g + 1$. One example satisfying this is when B is the hyperelliptic curve of genus 2 given on an affine piece by

$$V(45x^6 - 297x^5 + 845x^4 - 1306x^3 + 1073x^2 - 360x - 45y^2) \subset B$$

and T is the elliptic curve given on an affine piece by

$$V(1125x^4 - 4777x^3 + 6212x^2 - 2560x - 1125y^2) \subset T.$$

(These curves can be found using [Sha01, Lemma 5.1] with $a = 4, c = 5$, which shows that B has a maximal cover of degree 3 over T. Any other map factors through an isogeny between elliptic curves in the decomposition of the Jacobian of B, so we can check that this is the minimal degree by verifying that the two elliptic curves in the decomposition have no isogenies. A simple way to check this is to note that the two elliptic curves have a different number of $\mathbb{F}_7$-points, meaning they are not isogenous.)

Chapter 7

Future Questions

In this chapter, we consider several problems extending from topics related to the thesis. One obvious example is the fact that while the majority of the results in the thesis deal only with the case of rank-2 sheaves, one expects that many of these results should still hold in higher rank, though the methods would likely involve more combinatorial complexity. The remaining problems are broken down by subject.

Moduli spaces of sheaves on Kodaira surfaces: The main problem remaining after the results in Chapter 5 is to determine when the smooth and compact moduli spaces of sheaves on Kodaira surfaces are deformation equivalent to Douady spaces of points. One way to do this would be to compute holomorphic or topological invariants of the moduli spaces. In the case of a Lagrangian fibration $f : M \to P$ with both M and P Kähler, a result from [SV21] (due to Matsushita [Mat05] in the projective case) gives an isomorphism between $R^i\pi_*\mathcal{O}_M$ and Ω^i_P for integers i, from which the cohomology of $\mathcal{O}_M$ can be computed from the Hodge numbers of P via the Leray spectral sequence. The proof of the above result uses the Kähler condition mainly to show the isomorphism away from the singular fibres of the Lagrangian fibration, so if $\mathcal{M}_{2,\delta,c}(X)$ has a Kähler metric away from the singular fibres of its Lagrangian fibration, the above result may still hold in this case. Under these hypotheses, the Leray spectral sequence would degenerate at the second page, giving

$$H^i(\mathcal{O}_{\mathcal{M}_{2,\delta,c}(X)}) = \begin{cases} \mathbb{C}, & i = 0, \\ \mathbb{C}^2, & 0 < i < 8\Delta(2,\delta,c), \\ \mathbb{C}, & i = 8\Delta(2,\delta,c). \end{cases}$$

With respect to the fundamental group, in addition to determining whether the result from Remark 5.13 that

$$\mathbb{Z}^{8\Delta(2,\delta,c)} \longrightarrow \pi_1(\mathcal{M}_{2,\delta,c}(X)) \longrightarrow \pi_1(\mathbb{P}_{\delta,c}) \longrightarrow 0$$

is a right exact sequence of groups extends to higher dimensions, the bounds on the number of generators for the fundamental group may also be improved by studying when loops in the smooth fibres are homotopy equivalent to loops in a singular fibre. This would naturally generalise the case of elliptic surfaces with singular fibres but no multiple fibres, where the fundamental group is entirely determined by the base as all loops in smooth fibres are homotopy equivalent to loops inside the simply connected singular fibres. Proving such a result for $\mathcal{M}_{2,\delta,c}(X)$ would improve the bounds on the number of generators of the fundamental group to

$$\mathbb{Z}^{8\Delta(2,\delta,c)-2} \longrightarrow \pi_1(\mathcal{M}_{2,\delta,c}(X)) \longrightarrow \pi_1(\mathbb{P}_{2,\delta}) \longrightarrow 0$$

for $\Delta(2,\delta,c) \geq \frac{1}{2}$. In the $\Delta(2,\delta,c) = \frac{1}{2}$ case, this would imply that the fundamental group of $\mathcal{M}_{2,\delta,c}(X)$ has at most four generators, which is exactly the number of generators for the fundamental group of $X^{[2]}$.

Another avenue to determine when the moduli spaces are deformation equivalent to Douady spaces of points is to perform further comparisons of the Lagrangian fibration of $\mathcal{M}_{2,\delta,c}(X)$ with the natural Lagrangian fibration on $X^{[n]}$. While the Lagrangian fibrations for these two families both have base a $\mathbb{P}^n$-bundle over B, they are never isomorphic. At this point, it is unclear if this difference in base for the Lagrangian fibrations is sufficient to force the $\mathcal{M}_{2,\delta,c}(X)$ to be distinct from Douady spaces or if it can be reconciled to get spaces which are still deformation equivalent, perhaps by composing with a finite cover. This contrasts with the case of an elliptically fibred Abelian surface, where for appropriately chosen invariants ($\gcd(c_1 \cdot f, r) = 1$ with f a fibre of the elliptic fibration) every stable bundle is uniquely determined by its allowable elementary modifications up to twisting by a line bundle [Fri98, Chapter 8, Proposition 9]. This fact directly gives the correspondence between moduli spaces of stable bundles and Hilbert schemes of points in this case. One way in which we can investigate this difference in the future is to look at moduli spaces of stable sheaves on a product $E_1 \times E_2$ of elliptic curves and its spectral construction with respect to both natural elliptic fibrations. If the rank r and first Chern class c_1 are chosen so that $\gcd(r, c_1 \cdot f_1) = 1$ and $r|(c_1 \cdot f_2)$ with f_1 and f_2 general fibres of the two fibrations, the spectral constructions will immediately give the Hilbert scheme structure for the first fibration and behaviour similar to the Kodaira case for the second fibration. This question will be analysed in a future paper.

As simply connected holomorphic symplectic manifolds are particularly important in the field, another interesting problem related to these moduli spaces is whether there is a method to "reduce" a stably irreducible moduli space $\mathcal{M}_{2,\delta,c}(X)$ to obtain an associated holomorphic symplectic manifold which has smaller fundamental group. Such a construction would be analogous to the relationship between a Douady space of points on a Kodaira surface and the corresponding Bogomolov–Guan manifold. In the Bogomolov–Guan construction, one chooses a fibre Y of the map $\Sigma \circ \pi^{[n]} : X^{[n]} \to B$ induced by the Lagrangian fibration $\pi^{[n]} : X^{[n]} \to B^{[n]}$ and the $\mathbb{P}^n$-bundle structure of $\Sigma : B^{[n]} \to B$. The fibre Y is a hypersurface of $X^{[n]}$ and the leaf space of the corresponding co-isotropic reduction is a holomorphic symplectic orbifold. One can then desingularise the orbifold to obtain a Bogomolov–Guan manifold. In analogy to this construction, one could construct a hypersurface Z of $\mathcal{M}_{2,\delta,c}(X)$ by taking a fibre of the map $\mathcal{M}_{2,\delta,c}(X) \to B$ induced by the graph map and the $\mathbb{P}^n$-bundle structure on $\mathbb{P}_{\delta,c}$. The leaf space of the co-isotropic foliation on Z will again be a holomorphic symplectic orbifold [Saw09], and we expect that its desingularisation will have smaller fundamental group than $\mathcal{M}_{2,\delta,c}(X)$. This construction would likely give yet another example of a family of non-Kähler holomorphic symplectic manifolds whose deformation type we could study.

Vafa–Witten pairs on surfaces: In Chapter 6, we provide a variety of examples of surfaces where the only smooth moduli spaces of stable bundles are the stably irreducible ones. The natural question raised by this is which other complex surfaces also have this property? Interesting candidates for which to check this property would include the remaining cases of non-Kähler elliptic surfaces (i.e. those with Kodaira dimension 1 and base $\mathbb{P}^1$, as well as those with base of genus 1 and all multiple fibres of order 2), as well as elliptic surfaces with Kodaira dimension 1 that are Kähler but not algebraic.

Co-Higgs bundles and holomorphic Poisson geometry: One problem in [BM22] that is not discussed in this thesis is the classification of rank-2 co-Higgs bundles on Hopf surfaces. A *co-Higgs bundle* on a complex manifold X is a V-pair with $V \simeq \mathcal{T}_X$. The methods used [BM22] to study co-Higgs bundles on Hopf surfaces are directly analogous to the methods used in Chapter 6 to study Vafa–Witten pairs and Higgs bundles on elliptic surfaces with positive-degree canonical bundle. Co-Higgs bundles have close ties to holomorphic Poisson geometry. Indeed, given a holomorphic vector bundle $E \to X$, any holomorphic Poisson structure on $\mathbb{P}(E)$ with co-isotropic fibres induces a non-trivial co-Higgs bundle (E, ϕ) on X. This process can be reversed provided that the co-Higgs bundle satisfies some additional integrability conditions ([Pol97, Ray11] in the rank-2 case and [Mat20] for higher ranks). Some open problems in this area involve studying the impacts of stability of a co-Higgs bundle (E, ϕ) on the symplectic leaf structure of the induced holomorphic Poisson structure on $\mathbb{P}(E)$ and finding examples of co-Higgs bundles of rank at least 3 on manifolds other than $\mathbb{P}^1$ which satisfy the strong integrability conditions of [Mat20]. An ongoing project with Brady Ali Medina and Ruxandra Moraru is to use the results of [BM22] classifying rank-2 co-Higgs bundles on Hopf surfaces to study holomorphic Poisson structures on $\mathbb{P}^1$-bundles over Hopf surfaces and to construct examples of rank-3 co-Higgs bundles on Hopf surfaces and Hirzebruch surfaces which satisfy the strong integrability conditions of [Mat20].

References

[AB96] Marian Aprodu and Vasile Brînzănescu. Fibrés vectoriels de rang 2 sur les surfaces réglées. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(6):627–630, 1996.

[AB97] Marian Aprodu and Vasile Brînzănescu. Stable rank-2 vector bundles over ruled surfaces. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(3):295–300, 1997.

[ABT02] Marian Aprodu, Vasile Brînzănescu, and Matei Toma. Holomorphic vector bundles on primary Kodaira surfaces. *Math. Z.*, 242(1):63–73, 2002.

[AMT12] Marian Aprodu, Ruxandra Moraru, and Matei Toma. Two-dimensional moduli spaces of vector bundles over Kodaira surfaces. *Adv. Math.*, 231(3-4):1202–1215, 2012.

[Ara11] Donu Arapura. Homomorphisms between Kähler groups. In *Topology of algebraic varieties and singularities*, volume 538 of *Contemp. Math.*, pages 95–111. Amer. Math. Soc., Providence, RI, 2011.

[AT03] Marian Aprodu and Matei Toma. Une note sur les fibrés holomorphes non-filtrables. *C. R. Math. Acad. Sci. Paris*, 336(7):581–584, 2003.

[Ati57] M. F. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452, 1957.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[Bea11] Arnaud Beauville. Holomorphic symplectic geometry: a problem list. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 49–63. Springer, Heidelberg, 2011.

[BH89] Peter J. Braam and Jacques Hurtubise. Instantons on Hopf surfaces and monopoles on solid tori. *J. Reine Angew. Math.*, 400:146–172, 1989.

[BHPV03] Wolf Barth, Klaus Hulek, Chris Peters, and Antonius van de Ven. *Compact Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2003.

[Bis94] I. Biswas. A remark on a deformation theory of Green and Lazarsfeld. *J. Reine Angew. Math.*, 449:103–124, 1994.

[Bis11] Indranil Biswas. Stable Higgs bundles on compact Gauduchon manifolds. *C. R. Math. Acad. Sci. Paris*, 349(1-2):71–74, 2011.

[BM05a] Vasile Brînzănescu and Ruxandra Moraru. Holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces. *Ann. Inst. Fourier (Grenoble)*, 55(5):1659–1683, 2005.

[BM05b] Vasile Brînzănescu and Ruxandra Moraru. Stable bundles on non-Kähler elliptic surfaces. *Comm. Math. Phys.*, 254(3):565–580, 2005.

[BM06] Vasile Brînzănescu and Ruxandra Moraru. Twisted Fourier-Mukai transforms and bundles on non-Kähler elliptic surfaces. *Math. Res. Lett.*, 13(4):501–514, 2006.

[BM22] Eric Boulter and Ruxandra Moraru. (Co)-Higgs bundles on principal elliptic surfaces, 2022. Preprint. arXiv:2210.09839.

[Bog74] F. A. Bogomolov. The decomposition of Kähler manifolds with a trivial canonical class. *Mat. Sb. (N.S.)*, 93(135):573–575, 630, 1974.

[Bog96] Fedor A. Bogomolov. On Guan's examples of simply connected non-Kähler compact complex manifolds. *Amer. J. Math.*, 118(5):1037–1046, 1996.

[Boo21] David Boozer. Moduli spaces of Hecke modifications for rational and elliptic curves. *Algebr. Geom. Topol.*, 21(2):543–600, 2021.

[Bot95] Francesco Bottacin. Poisson structures on moduli spaces of sheaves over Poisson surfaces. *Invent. Math.*, 121(2):421–436, 1995.

[Bot00] Francesco Bottacin. A generalization of Higgs bundles to higher dimensional varieties. *Math. Z.*, 233(2):219–250, 2000.

[Bou21] Eric Boulter. Moduli spaces of stably irreducible sheaves on Kodaira surfaces, 2021. Preprint: arXiv:2112.00058.

[Bri77] Joël Briançon. Description de $\mathrm{Hilb}^n C\{x, y\}$. *Invent. Math.*, 41(1):45–89, 1977.

[Brî96] Vasile Brînzănescu. *Holomorphic vector bundles over compact complex surfaces*, volume 1624 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[Buc88] N. P. Buchdahl. Hermitian-Einstein connections and stable vector bundles over compact complex surfaces. *Math. Ann.*, 280(4):625–648, 1988.

[CC93] F. Catanese and C. Ciliberto. Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$. *J. Algebraic Geom.*, 2(3):389–411, 1993.

[dCM00] Mark Andrea A. de Cataldo and Luca Migliorini. The Douady space of a complex surface. *Adv. Math.*, 151(2):283–312, 2000.

[Don85] S. K. Donaldson. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. London Math. Soc. (3)*, 50(1):1–26, 1985.

[Don87] S. K. Donaldson. Infinite determinants, stable bundles and curvature. *Duke Math. J.*, 54(1):231–247, 1987.

[EF82] G. Elencwajg and O. Forster. Vector bundles on manifolds without divisors and a theorem on deformations. *Ann. Inst. Fourier (Grenoble)*, 32(4):25–51 (1983), 1982.

[EL99] Geir Ellingsrud and Manfred Lehn. Irreducibility of the punctual quotient scheme of a surface. *Ark. Mat.*, 37(2):245–254, 1999.

[FK74] O. Forster and K. Knorr. Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen. *Math. Ann.*, 209:291–346, 1974.

[FM88a] Robert Friedman and John W. Morgan. On the diffeomorphism types of certain algebraic surfaces. I. *J. Differential Geom.*, 27(2):297–369, 1988.

[FM88b] Robert Friedman and John W. Morgan. On the diffeomorphism types of certain algebraic surfaces. II. *J. Differential Geom.*, 27(3):371–398, 1988.

[Fri98] Robert Friedman. *Algebraic surfaces and holomorphic vector bundles.* Universitext. Springer-Verlag, New York, 1998.

[Gau84] Paul Gauduchon. La 1-forme de torsion d'une variété hermitienne compacte. *Math. Ann.*, 267(4):495–518, 1984.

[GD60] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, 4:228, 1960.

[GD61] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, 11:167, 1961.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[Gie77] D. Gieseker. On the moduli of vector bundles on an algebraic surface. *Ann. of Math. (2)*, 106(1):45–60, 1977.

[Göt90] Lothar Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. *Math. Ann.*, 286(1-3):193–207, 1990.

[Gro57] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[Gua95] Daniel Guan. Examples of compact holomorphic symplectic manifolds which are not Kählerian. II. *Invent. Math.*, 121(1):135–145, 1995.

[Hit87] Nigel Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54(1):91–114, 1987.

[Kob82] Shoshichi Kobayashi. Curvature and stability of vector bundles. *Proc. Japan Acad. Ser. A Math. Sci.*, 58(4):158–162, 1982.

[Leh11] Christian Lehn. *Symplectic Lagrangian Fibrations.* PhD thesis, Mainz, 2011.

[LT95] Martin Lübke and Andrei Teleman. *The Kobayashi-Hitchin correspondence.* World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[LT06] M. Lübke and A. Teleman. The universal Kobayashi-Hitchin correspondence on Hermitian manifolds. *Mem. Amer. Math. Soc.*, 183(863):vi+97, 2006.

[LY87] Jun Li and Shing-Tung Yau. Hermitian-Yang-Mills connection on non-Kähler manifolds. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 560–573. World Sci. Publishing, Singapore, 1987.

[Mat05] Daisuke Matsushita. Higher direct images of dualizing sheaves of Lagrangian fibrations. *Amer. J. Math.*, 127(2):243–259, 2005.

[Mat20] Mykola Matviichuk. *Quadratic Poisson Brackets and co-Higgs Fields.* PhD thesis, University of Toronto, 2020.

[Mor03] Ruxandra Moraru. Integrable systems associated to a Hopf surface. *Canad. J. Math.*, 55(3):609–635, 2003.

[Muk84] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface. *Invent. Math.*, 77(1):101–116, 1984.

[Mum63] David Mumford. Projective invariants of projective structures and applications. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 526–530. Inst. Mittag-Leffler, Djursholm, 1963.

[Nak99] Hiraku Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[NS65] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)*, 82:540–567, 1965.

[O'G97] Kieran G. O'Grady. The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface. *J. Algebraic Geom.*, 6(4):599–644, 1997.

[O'G99] Kieran G. O'Grady. Desingularized moduli spaces of sheaves on a $K3$. *J. Reine Angew. Math.*, 512:49–117, 1999.

[O'G03] Kieran G. O'Grady. A new six-dimensional irreducible symplectic variety. *J. Algebraic Geom.*, 12(3):435–505, 2003.

[Oka50] Kiyoshi Oka. Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques. *Bull. Soc. Math. France*, 78:1–27, 1950.

[Pol97] Alexander Polishchuk. Algebraic geometry of Poisson brackets. *J. Math. Sci. (New York)*, 84(5):1413—-1444, 1997.

[Ray11] Steve Rayan. *Geometry of Co-Higgs Bundles*. PhD thesis, University of Oxford, 2011.

[Saw09] Justin Sawon. Foliations on hypersurfaces in holomorphic symplectic manifolds. *Int. Math. Res. Not. IMRN*, 23:4496–4545, 2009.

[Sha01] T. Shaska. Curves of genus 2 with (N, N) decomposable Jacobians. *J. Symbolic Comput.*, 31(5):603–617, 2001.

[Sim92] Carlos T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, 75:5–95, 1992.

[SV21] Andrey Soldatenkov and Misha Verbitsky. The Moser isotopy for holomorphic symplectic and C-symplectic structures, 2021. Preprint. arXiv:2109.00935.

[Tak72] Fumio Takemoto. Stable vector bundles on algebraic surfaces. *Nagoya Math. J.*, 47:29–48, 1972.

[Tan17] Yuuji Tanaka. On the singular sets of solutions to the Kapustin-Witten equations and the Vafa-Witten ones on compact Kähler surfaces. *Geom. Dedicata*, 199:177–187, 2017.

[Tel98] Andrei Teleman. Moduli spaces of stable bundles on non-Kählerian elliptic fibre bundles over curves. *Exposition. Math.*, 16(3):193–248, 1998.

[Tom01] Matei Toma. Compact moduli spaces of stable sheaves over non-algebraic surfaces. *Doc. Math.*, 6:11–29, 2001.

[TT20] Yuuji Tanaka and Richard P. Thomas. Vafa-Witten invariants for projective surfaces I: stable case. *J. Algebraic Geom.*, 29(4):603–668, 2020.

[UY86] Karen Uhlenbeck and Shing-Tung Yau. On the existence of hermitian-yang-mills connections in stable vector bundles. *Communications on Pure and Applied Mathematics*, 39(S1):S257–S293, 1986.

[VW94] Cumrun Vafa and Edward Witten. A strong coupling test of S-duality. *Nuclear Phys. B*, 431(1-2):3–77, 1994.

[Yos01] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.*, 321(4):817–884, 2001.

APPENDICES

Appendix A

Homological Algebra and Deformation Theory of Coherent Sheaves

This appendix begins with a brief overview of basic sheaf theory, primarily to fix notation. Much of this material is taken from [GH94] and [GD60, GD61].

Let X be a topological space. A *pre-sheaf of sets (or groups, rings, etc.)* on X is a contravariant functor $\mathcal{F} : \mathfrak{Top}(X) \to \mathfrak{Set}$ (or $\mathfrak{Gp}, \mathfrak{Ring}$, etc.), where $\mathfrak{Top}(X)$ is the category whose objects are open subsets of X and whose morphisms are inclusions. We refer to the elements of $\mathcal{F}(U)$ as *sections* of $\mathcal{F}$ over U We represent the image of $U \subseteq U'$ under $\mathcal{F}$ by $\cdot|_U$. The pre-sheaf $\mathcal{F}$ is a *sheaf* if it satisfies the additional condition that

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram for any open cover $\{U_i\}_{i\in I}$ of U, where the arrows at each step are the restriction maps. Alternatively, $\mathcal{F}$ is a sheaf if for any open set U, any open cover $\{U_i\}_{i\in I}$ of U, and any choice of sections $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, there is a unique element $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$.

Example A.1. Given a set S (or a group, ring, vector space, etc.), the sheaf $\underline{S}$ of locally constant functions to S has $\underline{S}(U)$ equal to the locally constant functions from U to S; i.e. functions $f : U \to S$ such that f is constant on every connected component of U.

More generally, if P is any property of functions which is local on the source, we can define the sheaf of functions to S with property P. A specific example of this, which appears frequently in this thesis, is the following:

Example A.2. If X is a complex manifold, the sheaf of holomorphic functions $\mathcal{O}_X$ is a sheaf of rings given by $\mathcal{O}_X(U) = \{f : U \to \mathbb{C} | f \text{ is holomorphic}\}$. This sheaf is called the *structure sheaf* of X.

Definition A.1. Given a pre-sheaf $\mathcal{F}$ on X and a point $x \in X$, the *stalk* $\mathcal{F}_x$ is the set

$$\mathcal{F}_x := \{(s, U) : x \in U, s \in \mathcal{F}(U)\}/\sim,$$

where $(s, U) \sim (t, V)$ if $s|_{U \cap V} = t|_{U \cap V}$.

Given a pre-sheaf $\mathcal{F}$, there is a unique sheaf $\mathcal{F}^\dagger$ up to isomorphism satisfying $\mathcal{F}_x \simeq \mathcal{F}^\dagger_x$ for all points $x \in X$, called the *sheafification* of $\mathcal{F}$.

Definition A.2. If X is a topological space and $\mathcal{O}$ is a sheaf of rings on X, a *sheaf of $\mathcal{O}$-modules* (or an *$\mathcal{O}$-module*) is a sheaf of Abelian groups $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-module, and $\cdot|_U : \mathcal{F}(U') \to \mathcal{F}(U)$ is a map of $\mathcal{O}(U')$-modules.

The category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules inherits many of the useful categorical properties which correspond to the category $R - \mathfrak{Mod}$ for a ring R. The precise statement of this fact is that $\mathfrak{Mod}(X)$ is an *Abelian category.*

Definition A.3. A category $\mathcal{C}$ is *Abelian* if it has the following properties:

- it contains a zero object $\mathbf{0}$ such that for any object $C \in \mathcal{C}$, there are unique maps $\mathbf{0} \to C$ and $C \to \mathbf{0}$;

- it is closed under kernels, cokernels, finite products, and finite coproducts;
- any finite product is equal to the corresponding coproduct;
- For any two objects $A, B \in \mathcal{C}$, the Hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has an Abelian group structure with identity $A \to \mathbf{0} \to B$;
- Composition of maps $\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ is bilinear with respect to the Abelian group structures on the Hom-sets.
- For any objects A, B and any map $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $\ker(\mathrm{coker}(\varphi)) \simeq \mathrm{coker}(\ker(\varphi))$.

Example A.3. Let $\pi : E \to X$ be a holomorphic vector bundle. Then there is a sheaf of $\mathcal{O}_X$-modules also denoted E such that $E(U)$ is the set of holomorphic sections of E on U. Conversely, if a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *locally-free*, i.e. if there is an open cover $\{U_i\}$ and an integer r so that $\mathcal{F}(U_i) = \mathcal{O}_X^r(U_i)$ for all i, then the relative affine space $\mathrm{Spec}(\mathrm{Sym}(\mathcal{F}^\vee))$ is a holomorphic vector bundle. (In the sequel, we will not distinguish between a vector bundle and its sheaf of sections unless clarification is specifically necessary.)

Definition A.4. An $\mathcal{O}$-module $\mathcal{F}$ is *coherent* if for any map $f : \mathcal{F} \to \mathcal{O}^r$ with r an integer, $\ker(f)$ is finitely generated.

The structure sheaf $\mathcal{O}_X$ of a complex manifold X is always a coherent $\mathcal{O}_X$-module by a result of Oka [Oka50], as is any finite-rank holomorphic vector bundle. In particular, the category $\mathrm{Coh}(X)$ of coherent $\mathcal{O}_X$-modules is an Abelian subcategory of $\mathfrak{Mod}(X)$ containing all finite-rank vector bundles.

Example A.4. Some important classes of coherent sheaves that are not locally free include ideal sheaves and structure sheaves of closed subspaces. Given a closed embedding $\iota : Z \to X$, the ideal sheaf $\mathcal{I}_Z$ consisting of holomorphic functions that vanish along $\iota(Z)$ is a coherent $\mathcal{O}_X$-module if and only if Z is a complex analytic subspace. We also have the sheaf $\iota_*\mathcal{O}_Z$ given by $(\iota_*\mathcal{O}_Z)(U) := \mathcal{O}_Z(U \cap Z)$. These sheaves fit into the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Z \longrightarrow 0.$$

The benefit of working with coherent sheaves rather than holomorphic vector bundles lies in the fact that coherent sheaves are amenable to techniques of homological algebra. These techniques come up frequently when attempting to classify coherent sheaves.

A.1 Some left- and right-exact functors on coherent sheaves

Let $\mathcal{C}, \mathcal{D}$ be two Abelian categories, let $f : \mathcal{C} \to \mathcal{D}$ be a covariant functor, and $g : \mathcal{C} \to \mathcal{D}$ a contravariant functor. We say that f is *exact* if for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{C}$

$$0 \longrightarrow f(A) \longrightarrow f(B) \longrightarrow f(C) \longrightarrow 0$$

is also a short exact sequence. Similarly, g is exact if

$$0 \longrightarrow g(C) \longrightarrow g(B) \longrightarrow g(A) \longrightarrow 0$$

is a short exact sequence.

Many important invariants of coherent sheaves are derived by looking at the failure of a functor being exact. We will be particularly interested in those functors which are left- or right-exact.

Definition A.5. Let $\mathcal{C}, \mathcal{D}$ be two Abelian categories, $f : \mathcal{C} \to \mathcal{D}$ an additive covariant functor, and $g : \mathcal{C} \to \mathcal{D}$ an additive contravariant functor. (By additive here we mean that the functor preserves finite products.) The functor f is *left-exact* if for any short-exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{C}$,

$$0 \longrightarrow f(A) \longrightarrow f(B) \longrightarrow f(C)$$

is exact, and *right-exact* if

$$f(A) \longrightarrow f(B) \longrightarrow f(C) \longrightarrow 0$$

is exact. Similarly, g is *left-exact* if

$$0 \longrightarrow g(C) \longrightarrow g(B) \longrightarrow g(A)$$

is exact, and *right-exact* if

$$g(C) \longrightarrow g(B) \longrightarrow g(A) \longrightarrow 0$$

is exact.

Some important examples of such functors appear below:

Example A.5 (Homomorphism sheaf)**.** Given a coherent sheaf $\mathcal{F}$ on X, the left-exact functor $\mathcal{H}om(\mathcal{F}, -)$ from $\mathrm{Coh}(X)$ to itself is given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U)),$$

where $Hom_{\mathcal{O}_X(U)}(\mathcal{F}(U), -) : \mathcal{O}_X(U){-}\mathfrak{Mod} \to \mathfrak{Ab}$ is the homomorphism functor on $\mathcal{O}_X(U)$-modules. If $\mathcal{F}$ is locally free, then $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mathcal{F}^\vee \otimes \mathcal{G}$.

Example A.6 (Tensor product)**.** Given a coherent sheaf $\mathcal{F}$ on X, the right-exact functor $\mathcal{F} \otimes_{\mathcal{O}_X} -$ from $\mathrm{Coh}(X)$ to itself is defined at stalks by $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ for all points $x \in X$.

Example A.7 (Pushforward)**.** Given a holomorphic map $\varphi : X \to Y$ between two complex manifolds, the *pushforward* $\varphi_* : \mathfrak{Mod}(X) \to \mathfrak{Mod}(Y)$ of φ is a left-exact functor defined by $(\varphi_*\mathcal{F})(U) := \mathcal{F}(\varphi^{-1}(U))$ for any open set $U \subseteq Y$. If φ is proper, then φ_* restricts to a left-exact functor from $\mathrm{Coh}(X)$ to $\mathrm{Coh}(Y)$.

When X is compact, the *global sections functor* $\Gamma : \mathfrak{Mod}(X) \to \mathbb{C} - \mathfrak{Mod}$, given by $\Gamma(\mathcal{F}) := \mathcal{F}(X)$, is a special case of the pushforward where Y is a point. This functor is typically referred to as $H^0(X, -)$, as the derived functor of Γ is naturally isomorphic to the Čech cohomology functor.

Example A.8 (Homomorphism group)**.** Given a coherent sheaf $\mathcal{F}$ on X, the left-exact functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$ from $\mathrm{Coh}(X)$ to $\mathfrak{Ab}$ is given by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$. This is equal to the Hom functor associated to $\mathrm{Coh}(X)$ as an Abelian category.

Example A.9 (Pullback)**.** Given a holomorphic map $\varphi : X \to Y$ between two complex manifolds, the *pullback* $\varphi^* : \mathfrak{Mod}(Y) \to \mathfrak{Mod}(X)$ is a right-exact functor defined at stalks by $(\varphi^*\mathcal{F})_x := \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,\varphi(x)}} \mathcal{F}_{\varphi(X)}$.

When $\varphi : X \to Y$ is a locally closed embedding, $\varphi^*\mathcal{F}$ is often denoted as the restriction $\mathcal{F}|_X$.

A.2 Derived functors on coherent sheaves

Derived functors offer a way of measuring the failure of a left- or right-exact functor to be exact. Thus one can obtain better information from the set of derived functors than from the original left- or right-exact functor.

Definition A.6. Let $\mathcal{A}, \mathcal{B}$ be Abelian categories, and let $f : \mathcal{A} \to \mathcal{B}$ be a covariant left-exact functor. A *δ-functor for f* is a collection of functors $g_i : \mathcal{A} \to \mathcal{B}$ for $i \in \mathbb{Z}^+$ such that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$, there is a long exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & f(A) & \longrightarrow & f(B) & \longrightarrow & f(C) & \\
& \dots & \longrightarrow & & \longrightarrow & \dots & \\
& g_i(A) & \longrightarrow & g_i(B) & \longrightarrow & g_i(C) & \\
& g_{i+1}(A) & \longrightarrow & g_{i+1}(B) & \longrightarrow & g_{i+1}(C) & \longrightarrow \dots .
\end{array}$$

If f is instead right-exact, a δ-functor for f is a collection of functors $g_i : \mathcal{A} \to \mathcal{B}$ for $i \in \mathbb{Z}^+$ such that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

there is a long exact sequence

$$\begin{array}{ccccccc}
\dots \longrightarrow & g_{i+1}f(A) & \longrightarrow & g_{i+1}(B) & \longrightarrow & g_{i+1}(C) & \\
& g_i(A) & \longrightarrow & g_i(B) & \longrightarrow & g_i(C) & \\
& \dots & \longrightarrow & & & \dots & \\
& f(A) & \longrightarrow & f(B) & \longrightarrow & f(C) & \longrightarrow 0.
\end{array}$$

The δ-functors are defined similarly in the contravariant case. A δ-functor g_i for f is universal if for any natural transformation $\alpha : f \to f'$ and any δ-functor g'_i for f', there is a morphism of δ-functors sending g_i to g'_i. If f is left-exact and admits a universal δ-functor, we call it the *right-derived functor* $\{R^i f\}_{i \in \mathbb{Z}^+}$. Similarly, if f is right exact, we call a universal δ-functor the *left-derived functor* $\{L_i f\}_{i \in \mathbb{Z}^+}$. (Note that the $R^i f$ are constructed using a cohomology theory and are denoted with a superscript, whereas the $L_i f$ are constructed using a homology theory and are denoted with a subscript.)

Derived functors do not necessarily exist in general, but they can be constructed when the category has enough *acyclic objects*. In order to construct derived functors, we first need to consider the idea of acyclic objects.

Definition A.7. Given Abelian categories $\mathcal{A}$ and $\mathcal{B}$ and a covariant left-exact functor $f : \mathcal{A} \to \mathcal{B}$, an object $A \in \mathcal{A}$ is *f-acyclic* if $R^1 f(A) = 0$, meaning that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow f(A) \longrightarrow f(B) \longrightarrow f(C) \longrightarrow 0$$

is also exact. If f is instead right-exact, $C \in \mathcal{A}$ is *f-acyclic* if $L_1 f(C) = 0$, meaning that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

$$0 \longrightarrow f(A) \longrightarrow f(B) \longrightarrow f(C) \longrightarrow 0$$

is also exact. An f-acyclic object is defined similarly when f is a contravariant left- or right-exact functor. The Abelian category $\mathcal{A}$ has *enough f-acyclic objects* if every object $A \in \mathcal{A}$ admits a *resolution by f-acyclic objects*; if f is either covariant and left-exact or contravariant and right-exact, there is a (possibly infinite) exact sequence

$$0 \longrightarrow A \xrightarrow{p_0} \mathcal{I}_0 \xrightarrow{p_1} \mathcal{I}_1 \xrightarrow{p_2} \dots$$

with each $\mathcal{I}_i$ f-acyclic, and if f is covariant and right-exact or contravariant and left-exact, there is a (possibly infinite) exact sequence

$$\dots \xrightarrow{p_2} \mathcal{I}_1 \xrightarrow{p_1} \mathcal{I}_0 \xrightarrow{p_0} A \longrightarrow 0$$

with each $\mathcal{I}_i$ f-acyclic.

Let $f : \mathcal{A} \to \mathcal{B}$ be a covariant left-exact functor such that $\mathcal{A}$ has enough f-acyclic objects. For any $A \in \mathcal{A}$, choose an f-acyclic resolution $\mathcal{I}_\bullet$ for A and take the induced complex

$$0 \longrightarrow \mathcal{I}_0 \xrightarrow{p_1} \mathcal{I}_1 \xrightarrow{p_2} \dots .$$

The *right-derived functors* $R^i f$ for $i \in \mathbb{Z}^+$ can then be computed as

$$R^i f(A) = H^i(f(\mathcal{I}_\bullet)) = \ker(f(p_{i+1}))/\operatorname{im}(f(p_i)).$$

If f is contravariant or left-exact, it can be similarly constructed via the (co)homology of an f-acyclic resolution.

Remark A.8. While acyclic resolutions are highly non-unique, the derived functors are independent of a choice of resolution.

Example A.10 (Higher direct image functors)**.** The right derived functors $R^i\varphi_*$ of φ_* are known as the higher direct image functors. Using the fact that flasque sheaves (sheaves whose restriction maps are surjective) are φ_*-acyclic, one can show that the higher direct image sheaves of a sheaf $\mathcal{F}$ are the sheafification of the pre-sheaf

$$H^i(U, \mathcal{F}(\varphi^{-1}(U))).$$

If $\mathcal{F}$ is a coherent sheaf and φ is a proper map, $R^i\varphi_*\mathcal{F}$ is coherent for all $i \in \mathbb{Z}_{>0}$. In the case that φ is a constant map, $R^i\varphi_*(\mathcal{F})$ is the sheaf cohomology $H^i(X, \mathcal{F})$.

Example A.11 (Ext sheaves)**.** Given a coherent $\mathcal{O}_X$-module $\mathcal{F}$, the right derived functors $R^i\mathcal{H}om(\mathcal{F}, -)$ of $\mathcal{H}om(\mathcal{F}, -)$ are the Ext sheaves $\mathcal{E}xt^i(\mathcal{F}, -)$. The sheaves $\mathcal{E}xt^i(\mathcal{F}, \mathcal{G})$ are defined with respect to an injective resolution of $\mathcal{E}xt^i(\mathcal{F}, -)$, but they can also be computed using locally free resolutions of $\mathcal{E}xt^i(-, \mathcal{G})$ if $\mathcal{G}$ is coherent. Since $\mathrm{Coh}(X)$ has enough locally-free sheaves when X is a compact complex surface, we can always work with the locally-free resolutions in this case.

These two families of derived functors will be particularly useful in the sequel.

A.2.1 Useful results for derived functors on coherent sheaves

Many of the results in this thesis depend on computations of Ext sheaves and higher direct image functors, so in this section we detail some useful theorems which will aid in these computations. The first of these involves the composition of two derived functors.

Proposition A.9 (Grothendieck spectral sequence)**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories, and suppose that $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$ are left-exact functors. If F and G both admit right-derived functors and G maps G-acyclic objects to F-acyclic objects, then for any $A \in \mathcal{A}$ there is a spectral sequence with second page given by*

$$E_2^{p,q} = R^pF(R^qG(A))$$

which converges to $R^{p+q}(F \circ G)(A)$.

Particularly important examples of this result are the Leray spectral sequence for the higher direct image functors $R^i(f \circ g)_*$ and the local-to-global Ext sequence for

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) := R^i(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = R^i(H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}))).$$

Example A.12 (The Leray spectral sequence of a principal elliptic surface [Tel98])**.** Let $\pi : X \to B$ be a non-Kähler principal elliptic surface with fibre T. As shown in Proposition 2.2, X can be uniquely expressed as $\Theta^*/\langle\tau\rangle$, where $\Theta \in \mathrm{Pic}^d(B)$ is a line bundle of degree $d > 0$, Θ^* is the complement of the zero section in the total space of Θ, and τ is a complex number with $|\tau| > 1$ so that $\mathbb{C}^*/\langle\tau\rangle \simeq T$. In this case, we have that $R^i\pi_*\underline{\mathbb{Z}}$ is the locally constant sheaf with group $H^i(T, \underline{Z})$ for all i, so the spectral sequence corresponding to $H^i(X, \underline{\mathbb{Z}}) = R^i(H^0(B, \pi_*\underline{\mathbb{Z}}))$ has second page

$$\begin{tikzcd}
H^0(B, H^2(T, \mathbb{Z})) \arrow[drr, "d_{02}"] & H^1(B, H^2(T, \mathbb{Z})) & H^2(B, H^2(T, \mathbb{Z})) \\
H^0(B, H^1(T, \mathbb{Z})) \arrow[drr, "d_{01}"] & H^1(B, H^1(T, \mathbb{Z})) & H^2(B, H^1(T, \mathbb{Z})) \\
H^0(B, H^0(T, \mathbb{Z})) & H^1(B, H^0(T, \mathbb{Z})) & H^2(B, H^0(T, \mathbb{Z}))
\end{tikzcd}$$

where, up to choosing bases for

$$H^0(B, H^2(T, \mathbb{Z})) \simeq \mathbb{Z}, \qquad H^0(B, H^1(T, \mathbb{Z})) \simeq \mathbb{Z}^2,$$
$$H^2(B, H^1(T, \mathbb{Z})) \simeq \mathbb{Z}^2, \qquad H^2(B, H^0(T, \mathbb{Z})) \simeq \mathbb{Z},$$

d_{01} is given by the matrix $\begin{bmatrix} d & 0 \end{bmatrix}$ and d_{02} is given by the matrix $\begin{bmatrix} d \\ 1 \end{bmatrix}$ with d the degree of Θ; all other maps are zero. (d_{01} and d_{02} can be computed using standard results about the Gysin spectral sequence of a circle bundle as discussed in [BHPV03, p.196].) We can now compute that the third page is

$$\begin{array}{ccc} \ker(d_{02}) & H^1(B,\mathbb{Z}) & \mathbb{Z} \\ \\ \ker(d_{01}) & H^1(B,\mathbb{Z}) \oplus H^1(B,\mathbb{Z}) & \operatorname{coker}(d_{02}) \\ \\ \mathbb{Z} & H^1(B,\mathbb{Z}) & \operatorname{coker}(d_{01}) \end{array}$$

with all maps zero. Thus, the spectral sequence degenerates at page three, and since we have $H^1(B, Z) \simeq \mathbb{Z}^{2g}$ with g the genus of B and

$$\ker(d_{02}) \simeq 0, \qquad \operatorname{coker}(d_{02}) \simeq \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z},$$
$$\ker(d_{01}) \simeq \mathbb{Z}, \qquad \operatorname{coker}(d_{01}) \simeq \mathbb{Z}/d\mathbb{Z},$$

the singular cohomology of X is

$$H^0(X,\mathbb{Z}) \simeq \mathbb{Z}, \qquad H^2(X,\mathbb{Z}) \simeq \mathbb{Z}^{4g} \oplus \mathbb{Z}/d\mathbb{Z}, \qquad H^4(X,\mathbb{Z}) \simeq \mathbb{Z}.$$
$$H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g+1}, \qquad H^3(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/d\mathbb{Z},$$

The remaining results concern the higher direct image functors corresponding to flat proper maps of complex manifolds.

Proposition A.10 (Projection formula)**.** *Let X, Y be compact complex manifolds and let $f : X \to Y$ be a flat proper map. If $\mathcal{F} \in \mathrm{Coh}(X)$ and $\mathcal{G} \in \mathrm{Coh}(Y)$, then*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{G}) \simeq \mathcal{G} \otimes R^i f_*\mathcal{F}.$$

Proposition A.11 (Base change theorem)**.** *Let*

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

be a Cartesian diagram (i.e. $W \simeq X \times_Z Y$) with f and g flat and proper. Then

$$f^*(R^i g_*(\mathcal{F})) \simeq R^i g'_*(f'^*(\mathcal{F}))$$

for any $\mathcal{F} \in \mathrm{Coh}(X)$.

Proposition A.12 (Relative Serre duality)**.** *Let X be a compact complex surface, C a compact Riemann surface, and $f : X \to C$ a finite proper map. Then there is a rank-1 sheaf $\omega_{X/C} \in \mathrm{Coh}(X)$, called the* relative dualising sheaf*, such that for any vector bundle E on X,*

$$\left(R^1 f_*(E)\right)^\vee \simeq f_* \left(E^\vee \otimes \omega_{X/C}\right).$$

A.3 Deformation theory for holomorphic vector bundles

This section is based on the treatment in [Brî96, Section 5.1], and discusses the tools by which we can understand questions of dimension and regularity for moduli spaces of vector bundles on complex manifolds.

Definition A.13. Let X be a complex manifold, and let E be a holomorphic vector bundle on X. A *deformation* of E is a triple $(S, \mathcal{E}, \tau)$ consisting of a complex analytic germ S about a point 0, a holomorphic vector bundle $\mathcal{E}$ on $S \times X$, and an isomorphism $\tau : E \to \mathcal{E}|_{0\times X}$. A *morphism of deformations* from $(S, \mathcal{E}, \tau)$ to $(S', \mathcal{E}', \tau')$ is a pair (α, φ) of a holomorphic map $\alpha : S \to S'$ and a holomorphic bundle map $\varphi : \mathcal{E} \to \alpha^*\mathcal{E}'$ such that $\varphi|_{0\times X} \circ \tau = \alpha^*\tau'$.

The main purpose of deformations in this context is to understand what the tangent space of E should be in a moduli space by associating it to the tangent space of an analytic germ. To do this, we want a natural choice of deformation.

Definition A.14. Let X be a complex manifold, and let E be a holomorphic vector bundle on X. A deformation $(S, \mathcal{E}, \tau)$ is *versal* if for any deformation $(S', \mathcal{E}', \tau')$, there is a morphism to $(S, \mathcal{E}, \tau)$, and any pair $(\alpha, \varphi), (\beta, \psi)$ of morphisms to $(S, \mathcal{E}, \tau)$ have $\mathrm{d}\alpha = \mathrm{d}\beta$, where $\mathrm{d}\alpha : TS' \to TS$ is the induced map of tangent bundles. If E admits a versal deformation, it is unique up to unique isomorphism.

By [FK74], there is a versal deformation of E whose tangent space is $H^1(X, \mathcal{E}nd(E))$. However, the surfaces considered in this thesis all have non-compact Picard groups, so in order to obtain compact moduli spaces, we are only interested in those deformations that induce the trivial deformation of the determinant bundle. This can be done by comparing deformations of E to those of its associated $\mathrm{PGL}(r)$-bundle $\mathbb{P}(E)$:

Theorem A.15 (Elencwajg–Forster [EF82])**.** *Let X be a compact complex manifold and E a holomorphic vector bundle on X. Then there is a one-to-one correspondence between deformations of $\mathbb{P}(E)$ and deformations of E which induce the trivial deformation of* $\det(E)$.

When X is compact, the $\mathrm{PGL}(r)$-bundle $\mathbb{P}(E)$ admits a versal deformation with tangent space $H^1(X, \mathcal{E}nd_0(E))$, and the obstruction class of this deformation belongs to $H^2(X, \mathcal{E}nd_0(E))$. If $H^2(X, \mathcal{E}nd_0(E)) = 0$, the deformations of E are unobstructed and we say that E is *good*. In this case, a moduli space of bundles containing E will be smooth at E of dimension $H^1(X, \mathcal{E}nd_0(E))$.