Uniqueness of Low Rank Matrix Completion and Schur Complement

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Abstract

We study the low rank matrix completion problem. We give a sufficient and necessary condition such that the completed matrix is globally unique. We assume the observed entries of the matrix corresponds to a special chordal graph. Under this assumption, the matrix completion problem is either globally unique or it has infinitely many solutions (thus excluding local uniqueness). The proof of the theorems make extensive use of the Schur complement. Algorithmically, we give a polynomial algorithm to decide if a partial matrix can be uniquely recovered if the graph corresponding to the sampled data is chordal with some special structure.

Keywords: Low-rank matrix completion, matrix recovery, chordal graph
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Contents

1 Introduction **2**
1.1 Outline . . . 2

2 Background on LRMC , SDP **2**
2.1 Models . . . 2

3 Uniqueness of the completion by clique intersection **2**

4 Graph Representation of the Problem **6**
4.1 Chordal graphs . . . 7

5 Generalization to Completions of Positive Semidefinite Matrices **8**

6 Conclusion **9**

Acknowledgment **10**

Index **10**

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List of Tables

List of Algorithms

1 Introduction

The problem of low rank matrix completion has many applications to real applications in various scenarios. Given an low rank matrix Z with partially sampled data z, an important problem is under which conditions can we guarantee the exact recovery of the matrix Z. Assume we know that $\operatorname{rank}(Z) = z$, then the uniqueness of the completions with the target rank z ensures the exact recovery of Z. Currently, most of the work studying this problem is based on generic approach. In this paper, we study this problem with no generic assumption. We give sufficient and necessary conditions for the low rank matrix completion to be unique.

1.1 Outline

2 Background on LRMC , SDP

We now introduce the framework for the problem we are considering in this paper. We also include a graph theoretic framework that allows us to explore the facial structure of our problem.

2.1 Models

Suppose that we are given a low rank $m \times n$ real matrix $Z \in \mathbb{R}^{m\times n}$ with $r = \operatorname{rank}(Z)$ where a subset of entries are *sampled*.

The hard nonconvex *low rank matrix completion* problem, **LRMC** , for recovering the low rank matrix Z can be reformulated as follows:

$$(\textbf{LRMC}) \qquad \begin{array}{ll} \min & \operatorname{rank}(L) \\ \text{s.t.} & \mathcal{P}_{\hat{E}}(L) = z \end{array} \tag{2.1}$$

where $\hat{E}$ is the set of indices containing the known (*sampled*) entries of Z, $\mathcal{P}_{\hat{E}}(\cdot) : \mathbb{R}^{m\times n} \to \mathbb{R}^{|\hat{E}|}$ is the projection onto the corresponding entries in $\hat{E}$, and $z = \mathcal{P}_{\hat{E}}(Z)$ is the vector of known entries formed from Z. It is known that if the linear mapping $\mathcal{P}_{\hat{E}}$ satisfies some restricted isometry properties, then Z is the unique matrices satisfying $\operatorname{rank}(L) \leq r, \mathcal{P}_{\hat{E}}(L) = z$.

3 Uniqueness of the completion by clique intersection

In this section, we study the uniqueness of the low rank matrix completion. Suppose we know the target rank, a natural question the readers may ask is which elements can be uniquely determined and which elements can not? Also under which situation the whole low rank matrix L admits a unique completion? In this section, we provide a sufficient condition by exploiting the properties of the intersections of bicliques.

Question 3.1. How many bicliques do we need and what is the relation between the bicliques in order to have a unique completion?

Recall the basic property of Schur complement:

Lemma 3.2. *[2] Consider the partitioned matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

if we assume that $\mathrm{Range}(B) \subseteq \mathrm{Range}(A)$ *and* $\mathrm{Range}(C^T) \subseteq \mathrm{Range}(A^T)$*, then* $M/A = D - CA^\dagger B$ *is well-defined and*

$$\begin{bmatrix} I & 0 \\ -CA^\dagger & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^\dagger B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix}$$

and hence

$$\mathrm{rank}(M) = \mathrm{rank}(A) + \mathrm{rank}(M/A).$$

We first start with a simple case.

Theorem 3.3. *Consider the partitioned matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\mathrm{rank}(M) = r$*, and the blocks* A, B, C *are fixed. Then* M *is unique if and only if* $\mathrm{rank}(A) = r$.

Proof. First assume $\mathrm{rank}(A) = r$. Let $M/A = D - CA^\dagger B$ be the generalized Schur complement, e.g., [3], where $A^\dagger$ denotes the Moore-Penrose generalized inverse. Then we have $\mathrm{rank}(M/A) + \mathrm{rank}(A) = \mathrm{rank}(M)$. Therefore $\mathrm{rank}(M/A) = 0$ and $D = CA^\dagger B$ is unique.

For necessity, assume $\mathrm{rank}(A) < r$, we first assume $\mathrm{Range}(B) \subseteq \mathrm{Range}(A)$ and $\mathrm{Range}(C^T) \subseteq \mathrm{Range}(A^T)$. Then we by Lemma 3.2 we have the following equality

$$\mathrm{rank}(D - CA^\dagger B) + \mathrm{rank}(A) = \mathrm{rank}(M)$$

Since $\mathrm{rank}(A) < r$ and $\mathrm{rank}(M) = r$, we have $\mathrm{rank}(D - CA^\dagger B) = \mathrm{rank}(M) - \mathrm{rank}(A) = \bar{r} > 0$. We can then let $D = CA^\dagger B + E$ where E is an arbitrary matrix of rank $\bar{r}$. Therefore D is not unique.

Now suppose either $\mathrm{Range}(B) \subsetneq \mathrm{Range}(A)$ or $\mathrm{Range}(C^T) \subsetneq \mathrm{Range}(A^T)$. Without loss we can assume $\mathrm{Range}(C^T) \subsetneq \mathrm{Range}(A^T)$. Then we have $\mathrm{Null}(A) \subsetneq \mathrm{Null}(C)$ which means there exists a column vector x such that $Ax = 0, Cx \neq 0$. Now we can add the column vector $\begin{bmatrix} Ax \\ Cx \end{bmatrix} = \begin{bmatrix} 0 \\ Cx \end{bmatrix}$ to any column of $\begin{pmatrix} B \\ D \end{pmatrix}$ without changing the rank of M and we get a different D. So D is not unique. □

We now start with the simple case with only two bicliques.

Corollary 3.4. *Let* $Z, \mathrm{rank}(Z) = r$*, be the following matrix with two intersecting bicliques and corresponding submatrices* X *and* Y *which are fixed,*

$$Z = \left[\begin{array}{c|cc} Z_1 & X_1 & X_2 \\ \hline Y_1 & \multicolumn{1}{|c|}{Q} & X_3 \\ \cline{2-3} Y_2 & Y_3 & \multicolumn{1}{|c}{Z_2} \end{array}\right], \quad X = \left[\begin{array}{cc} X_1 & X_2 \\ \hline Q & \multicolumn{1}{|c}{X_3} \end{array}\right], \quad Y = \left[\begin{array}{cc} Y_1 & \multicolumn{1}{|c}{Q} \\ \cline{2-2} Y_2 & Y_3 \end{array}\right]. \tag{3.1}$$

submatrix Q *is the part that lies in both* X *and* Y*. If* $\mathrm{rank}(Q) = r$*, then* Z *is unique.*

Proof. If Q has rank r, then the top left union of four blocks must also be rank r. Therefore, Theorem 3.3 implies Z_1 is unique. Similarly, Z_2 is unique by looking at the four blocks at the bottom right. $\square$

However, the necessity may not be true. Consider the following example

$$Z = \begin{bmatrix} Z_1 & 6 & 5 & 3 \\ 1 & 2 & 3 & 2 \\ 3 & 4 & 2 & Z_2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 3 \end{bmatrix}, \tag{3.2}$$

Assume $\operatorname{rank}(Z) = 2$ and $\operatorname{rank}(Q) = 1 < \operatorname{rank}(Z)$. However, Z_1 and Z_2 are still unique and by basic linear algbera we have $Z_1 = 4$ and $Z_2 = 1$.

From Theorem 3.3, if we have two bicliques such that their intersection has the target rank, we can now merge these two bicliques into one bigger biclique and recover the corresponding missing entries of Z. We can then use this bigger biclique to merge with other bicliques. This process can carry on until all the missing entries are recovered.

We consider the following case.

Theorem 3.5. *Consider the partitioned matrix*

$$M = \begin{bmatrix} E & F \\ A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* A, B, C, F *are fixed. Then* M *is unique if and only if* $\operatorname{rank}(A) = r$ *and* $\operatorname{rank}(B) = r$*.*

Proof. Suppose $\operatorname{rank}(A) = \operatorname{rank}(B) = r$, then it is obvious that D, E are unique by Theorem 3.3.

For necessity, without loss we assume $\operatorname{rank}(A) = \bar{r} < r$. By a permutation, let

$$M = \begin{bmatrix} A & B \\ E & F \\ C & D \end{bmatrix}.$$

If $\operatorname{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) < r$, then by Therem 3.3, D is not unique so M is not unique.

If $\operatorname{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) = r$, then we have $\operatorname{Range}(B) \subseteq \operatorname{Range}(A)$. Now we partition A, E, C such that

$$M = \begin{bmatrix} A_1 & A_2 & B \\ E_1 & E_2 & F \\ C_1 & C_2 & D \end{bmatrix}$$

where A_1 has full column rank $\bar{r}$. So we have $\operatorname{Range}(A_1) = \operatorname{Range}(A)$.

Let M_1, M_2, M_3, M_4 be the four Schur complements corresponding to E_2, F, C_2, D such that $M_1 = E_2 - E_1 A_1^\dagger A_2$, $M_2 = F - E_1 A_1^\dagger B$, $M_3 = C_2 - C_1 A_1^\dagger A_2$, $M_4 = D - C_1 A_1^\dagger B$. Since $\operatorname{Range}(A_2) \subseteq \operatorname{Range}(A_1)$, $\operatorname{Range}(B) \subseteq \operatorname{Range}(A_1)$ and $\operatorname{Range}(E_1^T) \subseteq \operatorname{Range}(A_1^T)$, $\operatorname{Range}(C_1^T) \subseteq \operatorname{Range}(A_1^T)$, we have

$$\operatorname{rank}(M) = \operatorname{rank}(A_1) + \operatorname{rank}(\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}) = r.$$

Also

$$\operatorname{rank}(\begin{bmatrix} A \\ E \end{bmatrix}) = \operatorname{rank}(A_1) + \operatorname{rank}(M_1) = r.$$

Therefore $\operatorname{rank}(M_1) = \operatorname{rank}(\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}) = r - \bar{r}$ and we have

$$M_4 = M_3 M_1^\dagger M_2. \tag{3.3}$$

Now $M_1 \neq 0$, since $\operatorname{rank}(M_1) = r - \bar{r} > 0$, we can perturb E_2 such that $\bar{E}_2 = E_2 + M_1$ and perturb D such that $\bar{D} = D - \frac{1}{2}M_4$ and the corresponding full perturbed matrix is $\bar{M}$. After similar arguments we can get

$$\begin{aligned} \operatorname{rank}(\bar{M}) &= \operatorname{rank}(A_1) + \operatorname{rank}(\begin{bmatrix} 2M_1 & M_2 \\ M_3 & \frac{1}{2}M_4 \end{bmatrix}) \\ &= \operatorname{rank}(A_1) + \operatorname{rank}(2M_1) + \operatorname{rank}(\frac{1}{2}M_4 - M_3(2M_1)^\dagger M_2) \\ &= \operatorname{rank}(A_1) + \operatorname{rank}(2M_1) \quad \text{(due to (3.3))} \\ &= \bar{r} + r - \bar{r} = r. \end{aligned}$$

Therefore M is not unique. □

The more general case is also true:

Theorem 3.6. *Consider the partitioned matrix*

$$M = \begin{bmatrix} F & H & E \\ A & G & B \\ C & K & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* A, B, C, E, G *are fixed. Then the matrix* M *is unique if and only if* $\operatorname{rank}(A) = r$ *and* $\operatorname{rank}(B) = r$*.*

Proof. Without loss we assume $\operatorname{rank}(A) < r$. Now let $B = [G,\ B]$, $E = [H,\ E]$, the result follows directly from Theorem 3.5. □

Theorem 3.7. *Consider the partitioned matrix*

$$M = \begin{bmatrix} F & H & E \\ A & G & B \\ C & K & D \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\operatorname{rank}(M) = r$*, and the blocks* F, A, G, K, D *are fixed. Then the matrix* M *is unique if and only if* $\operatorname{rank}(A) = \operatorname{rank}(G) = \operatorname{rank}(K) = r$*.*

Proof. If $\operatorname{rank}(A) < r$ or $\operatorname{rank}(K) < r$, then according to Theorem 3.5, M is not unique. Therefore we only need to consider the case when $\operatorname{rank}(G) < r$.

If $\operatorname{rank}(\begin{bmatrix} H \\ G \end{bmatrix}) < r$ or $\operatorname{rank}(\begin{bmatrix} G & B \end{bmatrix}) < r$, then again according to Theorem 3.5, M is not unique. Therefore we consider the case where $\operatorname{rank}(\begin{bmatrix} H \\ G \end{bmatrix}) = r$ and $\operatorname{rank}(\begin{bmatrix} G & B \end{bmatrix}) = r$.

By a permutation, let

$$M = \begin{bmatrix} G & B & A \\ H & E & F \\ K & D & C \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Let $P = \begin{bmatrix} G & B \\ H & E \end{bmatrix}$, since $\text{rank}(G) < r$, by Theorem 3.3, there exists a different $\bar{E}$ and $\bar{P}$ such that $\text{rank}(\bar{P}) = r$, we let $\bar{C} = \begin{bmatrix} K & D \end{bmatrix} \bar{P}^{\dagger} \begin{bmatrix} A \\ F \end{bmatrix})$, since $\text{Range}(\begin{bmatrix} A \\ F \end{bmatrix}) \subseteq \text{Range}(\begin{bmatrix} G \\ H \end{bmatrix})$ and $\text{Range}(\begin{bmatrix} K^T \\ D^T \end{bmatrix}) \subseteq \text{Range}(\begin{bmatrix} G^T \\ B^T \end{bmatrix})$, we have $\text{rank}(\bar{M}) = \text{rank}(\bar{P}) + \text{rank}(\bar{C} - \begin{bmatrix} K & D \end{bmatrix} \bar{P}^{\dagger} \begin{bmatrix} A \\ F \end{bmatrix}) = \text{rank}(\bar{P}) = r$. The corresponding $\bar{M}$ is different from M and the proof is finished. □

Theorem 3.8. *Consider the partitioned matrix*

$$M = \begin{bmatrix} F & H & E \\ A & G & B \\ C & K & D \\ J & I & L \end{bmatrix} \in \mathbb{R}^{m\times n},$$

where $\text{rank}(M) = r$, *and the blocks* F, A, G, K, D, L *are fixed. Then the matrix* M *is unique if and only if* $\text{rank}(A) = \text{rank}(G) = \text{rank}(K) = \text{rank}(D) = r$.

Proof. Direct consequences from Theorem 3.5 and Theorem 3.7. □

The above arguments can be extended into a more general case where we have a stair case of known block matrices. We conclude that the whole matrix is unique if and only if every "corner" matrix has rank r.

Theorem 3.9. *Given a low rank matrix* $Z \in \mathbb{R}^{m\times n}$ *and a partial sampling* $\mathcal{P}_{\hat{E}}(Z) = z$. *If by a permutation there exists a chain of bicliques* $\alpha_1, ..., \alpha_l$ *with the corresponding edge sets* $E_1, \cdots, E_l$. *Assume* $\cup_{i=1}^{l} E_i = \hat{E}$ *and for any* i *we have* $E_i \cap E_j = \emptyset \; \forall j > i+2 \mod l$ *and the union of all the vertices of the bicliques satisfy* $\cup_{i=1}^{l} \alpha_i = \{1, ..., m\} \times \{1, ..., n\}$. *Then the matrix* Z *can be uniquely recovered if and only if* $\text{rank}(X_{\alpha_i \cap \alpha_{i+1}}) = r, i = 1, ..., l-1$

Proof. Suppose the bicliques satisfy $\text{rank}(X_{\alpha_i \cap \alpha_{i+1}}) = r, i = 1, ..., l-1$, then it is abvious to see the uniqueness of the completion by repeatedly using Theorem 3.3, Theorem 3.5, Theorem 3.6 and Corollary 3.4.

For the other direction, suppose one of the bicluqes X_{α_i} satisfies $\text{rank}(X_{\alpha_i}) \neq r$, then one can recover a different matrix $\bar{Z}$ with the same rank and sampled data by reducing the problem to the sample cases as shown in Theorem 3.3 - 3.8. □

4 Graph Representation of the Problem

Our sampling yields elements $b = \mathcal{P}_{\hat{E}}(Z)$. With the matrix Z and the sampled elements we can associate a bipartite graph $G_Z = (U_m, V_n, \hat{E})$, where

$$U_m = \{1, \ldots, m\}, \quad V_n = \{1, \ldots, n\}.$$

For our needs we associate Z with the *undirected graph,* $G = (V, E)$, with node set $V = \{1, \ldots, m, m+1, \ldots, m+n\}$ and edge set E that satisfies

$$\big\{\{ij \in V \times V : i < j \leq m\} \cup \{ij \in V \times V : m+1 \leq i < j \leq m+n\}\big\} \subseteq E \subseteq \{ij \in V \times V : i < j\}.$$

Note that as above, $\bar{E}$ is the set of edges excluding the trivial ones, that is,

$$\bar{E} = E \backslash \Bigg\{\{ij \in V \times V : i \leq j \leq m\} \cup \{ij \in V \times V : m+1 \leq i \leq j \leq m+n\}\Bigg\}.$$

Recall that a *biclique* α in the graph G_Z is a complete bipartite subgraph in G_Z with corresponding complete submatrix $z[\alpha]$. q This corresponds to a nontrivial[1] *clique* in the graph G, a complete subgraph in G. The cliques of interest are $C = \{i_1, \ldots, i_k\}$ with cardinalities

$$|C \cap \{1, \ldots, m\}| = p \neq 0, \quad |C \cap \{m+1, \ldots, m+n\}| = q \neq 0. \tag{4.1}$$

The submatrix $z[\alpha]$ of Z for the corresponding biclique from the clique C is

$$z[\alpha] \equiv X \equiv \{Z_{i(j-m)} : ij \in C\}, \quad \text{sampled } p \times q \text{ rectangular submatrix.} \tag{4.2}$$

These non-trivial cliques in G that correspond to bicliques of G_Z are at the center of our algorithm.

4.1 Chordal graphs

Chordal graph is a special graph with a chordless structure. An undirected graph is chordal if every cycle of length greater than three has a chord.

A clique tree of a graph $G = (V, E)$ is a tree which has the cliques of G as its vertices. A clique tree T has the induced subtree property if for every $v \in V$, the cliques that contain v form a subtree (connected subgraph) of T. Buneman [4, Theorem 2.7] and Gavril [1, Theorem 3] it has been shown that chordal graphs are exactly the graphs for which a clique tree with the induced subtree property exists.

We show the graph $\hat{E}$ in Theorem 3.9 has a unique clique tree with the induced subtree property. Therefore it is chordal. In addition, we show this tree is actually a path.

Theorem 4.1. *Given a graph $\hat{E}$, if by a permutation there exists a chain of bicliques $\alpha_1, ..., \alpha_l$ with the corresponding edge sets $E_1, \cdots, E_l$. Assume $\cup_{i=1}^{l} E_i = \hat{E}$ and for any i we have $E_i \cap E_j = \emptyset \ \forall j > i+2 \mod l$ and the union of all the vertices of the bicliques satisfy $\cup_{i=1}^{l} \alpha_i = \{1, ..., m\} \times \{1, ..., n\}$. Then the graph $\hat{E}$ is chordal and it has a unique clique tree with the induced subtree property. In addition, this tree is a path.*

Proof. Consider the set of bicliques $\alpha_1, \cdots, \alpha_l$, each clique α_i is a node and connect α_i and α_{i+1} by an edge, then it forms a clique tree. For any vertex v in $\hat{E}$, it either belongs to α_i or belongs to $\alpha_i \cap \alpha_i + 1$ therefore the set of cliques containing v is connected. Hence it has the induced subtree property. Therefore by Theorem 3 in [1] it is a chordal graph. Now consider a different clique tree, then there exists α_i which is not connected α_{i+1}, but there exists $v \in \alpha_i \cap \alpha_i + 1$, therefore the induced subtree property is not satisfied. □

[1]For G we have the additional trivial cliques of size k, $C = \{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ and $C = \{j_1, \ldots, j_k\} \subset \{m+1, \ldots, m+n\}$, that are not of interest to our algorithm.

Theorem 4.2. *Given a graph $\hat{E}$, if $\hat{E}$ has a unique clique path with the induced subtree property, then there exists a polynomial algorithm to determine if the correponding low rank matrix completion problem 2.1 has a unique solution.*

Proof. □

Algorithm:
(1) Find a clique path of the graph $\hat{E}$ with the induced subtree property.
(2) Check if the intersection of two neighbouring clique has rank r

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5 Generalization to Completions of Positive Semidefinite Matrices

We recall the following theorem about symmetric matrix:

Theorem 5.1. *Suppose M is symmetric and partitioned as*

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

in which A and C are square. Then $M \succeq 0$ if and only if $A \succeq 0, \mathrm{Range}(B) \subseteq \mathrm{Range}(A)$, and $M/A \succeq 0$.

Theorem 5.2. *Consider the partitioned matrix*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where $\mathrm{rank}(M) = r$ and $M \succeq 0$ in which A, C are square, and the blocks A, B are fixed. Then there exists a unique positive semidefinite matrix M if and only if $\mathrm{rank}(A) = r$.

Proof. First assume $\mathrm{rank}(A) = r$. Let $M/A = C - B^T A^\dagger B$ be the generalized Schur complement, where $A^\dagger$ denotes the Moore-Penrose generalized inverse. The existence of $M \succeq 0$ ensures $\mathrm{Range}(B) \subseteq \mathrm{Range}(A)$ and $A \succeq 0$, therefore M/A is well-defined. We then have $\mathrm{rank}(M/A) + \mathrm{rank}(A) = \mathrm{rank}(M)$. Hence $\mathrm{rank}(M/A) = 0$ and $C = B^T A^\dagger B \succeq 0$ is unique.

For necessity, assume $\mathrm{rank}(A) < r$, the existence of $M \succeq 0$ ensures $\mathrm{Range}(B) \subseteq \mathrm{Range}(A)$ and $A \succeq 0$, therefore M/A is well-defined and

$$\mathrm{rank}(M/A) + \mathrm{rank}(A) = \mathrm{rank}(M)$$

Since $\mathrm{rank}(A) < r$ and $\mathrm{rank}(M) = r$, we have $\mathrm{rank}(C - B^T A^\dagger B) = \mathrm{rank}(M) - \mathrm{rank}(A) = \bar{r} > 0$. We can then let $C = B^T A^\dagger B + E$ where E is an arbitrary positive semidefinite matrix of rank $\bar{r}$. Hence by Theorem 5.1

$$\bar{M} = \begin{bmatrix} A & B \\ B^T & B^T A^\dagger B + E \end{bmatrix} \succeq 0.$$

Therefore M is not unique. □

Theorem 5.3. *Consider the partitioned matrix*

$$M = \begin{bmatrix} A & B & D \\ B^T & C & E \\ D^T & E^T & F \end{bmatrix} \in \mathbb{R}^{m \times n},$$

M is symmetric and positive semidefinite and the diagonal elements of M are all nonzeros. Suppose A, B, C, E, F are fixed and $\operatorname{rank}(M) = r$. *Then M can be uniquely completed if and only if* $\operatorname{rank}(C) = r$.

Proof. The if part is obvious, now we prove the only if part.

Let $H = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ and assume $\operatorname{rank}(\begin{bmatrix} C \end{bmatrix}) < r$ and $\operatorname{rank}(H) = r$.

Let $\tilde{D} = D + X$, then by Schur complement, $X = 0$ is a solution of the equation

$$F - [\tilde{D}^T, E^T] H^\dagger \begin{bmatrix} \tilde{D} \\ E \end{bmatrix} = 0. \tag{5.1}$$

Therefore this equation is homogeneous and we can assume (5.1) has the following form:

$$[X^T, 0](\frac{1}{2} H^\dagger \begin{bmatrix} X \\ 0 \end{bmatrix} + H^\dagger \begin{bmatrix} D \\ E \end{bmatrix}) + ([X^T, 0] \frac{1}{2} H^\dagger + \begin{bmatrix} D^T & E^T \end{bmatrix} H^\dagger) \begin{bmatrix} X \\ 0 \end{bmatrix} \tag{5.2}$$

Let $H^\dagger = \begin{bmatrix} A^\dagger & B^\dagger \\ (B^\dagger)^T & C^\dagger \end{bmatrix}$ and consider $H^\dagger \begin{bmatrix} X \\ 0 \end{bmatrix} + 2H^\dagger \begin{bmatrix} D \\ E \end{bmatrix}$. Clearly we only need to require $A^\dagger X - 2(A^\dagger D + \bar{B}E) = 0$ such that equation (5.2) holds true. Note that $A^\dagger D + \bar{B}E = 0$ implies $D = E = 0$ since $\begin{bmatrix} D & E \end{bmatrix} \in \operatorname{Range}(H) = \operatorname{Range}(H^\dagger)$. But that would imply $F = 0$ since $\operatorname{rank}(H) = \operatorname{rank}(M) = r$ which contradicts our assumption that the diagonal elements are all nonzeros. Therefore Let $2(A^\dagger D + B^\dagger E) = G$, then $G \neq 0$ and $X = AG \neq 0$ is a solution since $\operatorname{Range} G \subseteq \operatorname{Range} A^\dagger$. Now we need to show that $\begin{bmatrix} X \\ 0 \end{bmatrix} \in \operatorname{Range}(H) = \operatorname{Range}(H^\dagger)$. Note this is true if $\operatorname{Range}(\begin{bmatrix} A \\ B^T \end{bmatrix}) \cap \operatorname{Range}(\begin{bmatrix} B \\ C \end{bmatrix}) = \{0\}$. Although this assumption may not hold in general, however, since $\operatorname{rank}(C) < \operatorname{rank}(H)$, we can always perform symmetrical row and column elementary operations (adding a multiple of rows from $[B^E, C, E]$ to $[A, B, D]$ and correspondingly symmetric operations of columns) such that this assumption holds. After it is done and a nonzero X is found, we can do reverse operations such that the fixed elements of M stay the same.

Now consider the case where $\operatorname{rank}(C) < \operatorname{rank}(H) < r$, this can be reduced to the previous case by adding one row and one column to H each time such that $\operatorname{rank}(H)$ is increased by one until $\operatorname{rank}(H) = r$. Note that the rank of C can be increased by at most one in this case since M is a psd matrix. Therefore, we can guarantee $\operatorname{rank}(C) < r$ when $\operatorname{rank}(H)$ hits r.

At last we consider the case where $\operatorname{rank}(C) = \operatorname{rank}(H) < r$. Suppose $\operatorname{rank}(C) = \operatorname{rank}(H) = s < r$. Then we must have $\operatorname{rank}(\begin{bmatrix} C & E \\ E^T & F \end{bmatrix}) > s$ which is reduced to the previous cases. Otherwise, we conclude $\operatorname{rank}(M) = r$ which is a contradiction. □

6 Conclusion

In this paper, we derived sufficient and necessary conditions for generical matrices and positive semidefinite matrices to be uniquely completable.

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7

Index

$A^\dagger$, Moore-Penrose generalized inverse, 3
$G = (V, E)$, undirected graph, 7
$G_Z = (U_m, V_n, \hat{E})$, 6
M/A, generalized Schur complement, 3
$\hat{E}$, indices of sampled entries of Z, 2
$z[\alpha]$, complete submatrix, 7

biclique, 7

clique, 7
complete submatrix, $z[\alpha]$, 7

generalized Schur complement, M/A, 3

indices of sampled entries of Z, $\hat{E}$, 2

Moore-Penrose generalized inverse, $A^\dagger$, 3

undirected graph, $G = (V, E)$, 7