Math 127: Calculus I for Science

Fiona Dunbar

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This set of lecture notes summarizes the content that will be assessed on assignments and tests. The notes include definitions, explanations and examples. The examples are boxed in grey and have typed or written solutions in the space below. Some recommended homework problems are boxed in blue and are included at the end of each chapter (week).

The harder problems boxed in red often combine multiple topics and have a link to a video solution and/or written solution. Some of these problems are resources from the Centre for Education in Mathematics and Computing in the Faculty of Mathematics at the University of Waterloo. In particular, some of these problems are taken from past high school math contests such as the Euclid. The links to the written solutions will bring you to the complete solution set for the contest this problem was taken from. These problems are meant to challenge you and also to strengthen your problem solving skills that will help you to succeed in this course. Try thinking about how you might solve the problem yourself before clicking on the solutions.

The table of contents can be used to navigate to a particular numbered section by clicking on its title. The course formally begins with Week 1, but there is some review material in Chapter 0 that has some pre-calculus topics that will be used throughout the course.

Contents

0 Pre-Calculus Review

0.1 Notation, Absolute Value and Inequalities

0.1.1 Interval Notation: Video Lesson

Let $a, b \in \mathbb{R}$ with $a < b$. Then the intervals represent sets of real numbers as follows:

$(a, b) = \{x \in \mathbb{R} | a < x < b\}$
$(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$
$[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
$[a, \infty) = \{x \in \mathbb{R} | a \leq x\}$
$(-\infty, b] = \{x \in \mathbb{R} | x \leq b\}$
$(-\infty, \infty) = \mathbb{R}$
$(a, a) = \emptyset$, the empty set
$[a, a] = \{a\}$.

0.1.2 Rules for Inequalities: Video Lesson

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$
3. If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$ and $a^2 < b^2$.
4. If $a < b$ and $c > 0$, then $ac < bc$.
5. If $a < b$ and $c < 0$, then $ac > bc$.
6. If $0 < a < b$, then $1/a > 1/b$.

0.1.3 Solving Inequalities

Example: Solve for x and express your answer using interval notation: $x^2 < 4$.

Solution: Rearranging the inequality, we have $x^2 - 4 < 0$
Factoring the left side gives $(x - 2)(x + 2) < 0$
Expression changes sign at $x = 2$ and $x = -2$.
For $x \in (-\infty, -2)$, $x - 2 < 0$, $x + 2 < 0$ so $(x - 2)(x + 2) > 0$
For $x \in (-2, 2)$, $x - 2 < 0$, $x + 2 > 0$ so $(x - 2)(x + 2) < 0$
For $x \in (2, \infty)$, $x - 2 > 0$, $x + 2 > 0$ so $(x - 2)(x + 2) > 0$.
Thus $x^2 - 4 < 0$ for $x \in (-2, 2)$.
Video Solution

Example: Solve for x and express your answer using interval notation: $x^2 + 6 \leq 5x$.
Video Solution

Harder Problem

Determine all values of x for which $0 < \dfrac{x^2 - 11}{x + 1} < 7$.
Video Solution Written Solution

0.1.4 Absolute Value: Video Lesson

The absolute value of a number a, denoted $|a|$, is the distance from a to 0. Since distance is always positive or zero, we have $|a| \geq 0$ for all a. Note that if a is negative, then $-a$ is positive. Thus, we have the following definition:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note: $\sqrt{a}$ is defined to be the positive square root of a only and so we can also write $|a| = \sqrt{a^2}$.
Suppose $a > 0$. Then

1. $|x| = a$ iff $x = \pm a$
2. $|x| < a$ iff $x > -a$ and $x < a$
3. $|x| > a$ iff $x > a$ or $x < -a$

0.1.5 Properties of Absolute Values: Video Lesson

If a and b are real numbers and n is an integer, then

1. $|ab| = |a||b|$
2. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, $|b| \neq 0$
3. $|a^n| = |a|^n$
4. $|a+b| \leq |a| + |b|$ (Triangle Inequality)

Example: Solve $|3x+2| \geq 4$ for x and express your solution in interval notation.

Solution:

$$\begin{aligned} 3x+2 \geq 4 \text{ or } 3x+2 &\leq -4 \\ 3x \geq 2 \text{ or } 3x &\leq -6 \\ x \geq \frac{2}{3} \text{ or } x &\leq -2 \\ x \in (-\infty, -2] \cup &\left[\frac{2}{3}, \infty\right) \end{aligned}$$

Video Solution

Example: Solve $\left|3 - \dfrac{4}{x}\right| < 2$ for x and express your solution in interval notation.
Video Solution

Example: Solve $x^2 < 4$ for x by taking the positive square root of both sides.
Video Solution

0.2 Points and Curves in the Plane

0.2.1 Cartesian Plane: Video Lesson

The Cartesian plane consists of a horizontal axis called the x-axis and vertical axis called the y-axis that intersect at the origin. Any point in the plane can be located by an ordered pair of numbers depicting its distance from this perpendicular pair of axis. If the coordinates of P are (a, b), then a is the x-coordinate of P and is the distance from P to the y-axis and b is the y-coordinate of P and is the distance from P to the x-axis. The **distance** between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_x)$ is $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, the **midpoint** between P_1 and P_2 is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$, and the **slope** of the line segment connecting P_1 and P_2 is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Lines

The **point-slope** form of the equation of a line passing through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.
Two non-vertical lines are **parallel** iff they have the same slope.
Two lines with slopes m_1 and m_2 are **perpendicular** iff $m_2 = -\dfrac{1}{m_1}$.

Example: If $P(1,4)$ and $Q(2,-1)$ are two points in the plane,

(a) Find the distance between the points P and Q.

(b) Find the midpoint between P and Q.

(c) Find the point-slope form of the equation of the line through the points P and Q.

(d) Find the point-slope form of the equation of the line through P that is perpendicular to the line found in part (c).

Solution:

(a) The distance between P and Q is $|PQ| = \sqrt{(2-1)^2 + (-1-4)^2} = \sqrt{1+25} = \sqrt{26}$.

(b) The midpoint between P and Q is $\left(\frac{1+2}{2}, \frac{4+(-1)}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}\right)$.

(c) Now to find the equation of the line through P and Q, we first find the slope m of the line segment PQ. $m = \frac{-1-4}{2-1} = \frac{-5}{1} = -5$.
Using the point $(1,4)$ and the slope $m=-5$, we write the point-slope equation of the line $y-4 = -5(x-1)$.
Note that we could have alternatively used the point $(2,-1)$ to obtain $y-(-1) = -5(x-2)$. Verify that these equations are identical.

(d) A line perpendicular to the line found in (c) has slope equal to $\frac{-1}{-5} = \frac{1}{5}$.
Since the line we are looking for goes through $P(1,4)$, the point-slope form of the equation of the line is $y-1 = \frac{1}{5}(x-4)$.

Video Solution

0.2.2 Circles: Video Lesson

A circle with radius r and centre (h,k) has all its points (x,y) at a distance of r from the point (h,k). Using the formula for distance we find the equation of the circle is

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$
$$(x-h)^2 + (y-k)^2 = r^2$$

Sometimes we need to use some algebra first to recognize the equation of a circle in its standard form. Completing the square is useful here.

$$\begin{aligned} x^2 + bx &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \end{aligned}$$

Example: Show that the graph of $2x^2 + 2y^2 - x + y = 0$ is a circle and find the radius and centre.

0.2.3 Parabolas: Video Lesson

The parabola $y = ax^2$ opens upwards if $a > 0$ and downwards if $a < 0$.
The vertex (the origin in this case) is the point where the parabola changes direction.

If we shift the graph of a parabola up k units and to the right h units, then the equation becomes

$$y - k = a(x-h)^2 \text{ or } y = a(x-h)^2 + k$$

The shifted vertex now has coordinates (h, k).
We may need to complete the square to put the equation of a parabola into this form.

Example: Find the vertex of the parabola $y = 2x^2 - 3x + 1$.

0.2.4 Ellipses and Hyperbolas: Video Lesson

An **ellipse** is a set of points $P(x, y)$ such that the sum of the distances to two fixed points is a constant. The standard equation for an ellipse centred at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers. Note that $x = \pm a$ and $y = \pm b$ represent the x- and y-intercepts.

A **hyperbola** is a set of points such that the difference of distance from two fixed points is a constant. The standard equation for a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Note that $x = \pm a$ represent the x-intercepts but there are no y-intercepts since $x \geq a$ or $x \leq -a$. This gives the two branches of the hyperbola. The asymptotes of the hyperbola are the lines $y = (b/a)x$ and $y = -(b/a)x$.

Example: Sketch the curve $3y^2 + 4x^2 = 12$. Find the point(s), if any, where this curve intersects the line $y - x = 2$.
Solution: We begin by dividing both sides by 12 to get a 1 on the right side of the equation to put it in the form of a hyperbola.

$$\frac{3y^2}{12} + \frac{4x^2}{12} = \frac{12}{12}$$
$$\frac{y^2}{4} + \frac{x^2}{3} = 1$$

To recognize the values of a and b, we write each denominator as a square

$$\frac{x^2}{\sqrt{3}^2} + \frac{y^2}{2^2} = 1$$

Substituting $y = x + 2$ into the equation of the curve, we get

$$\begin{aligned} 3(x+2)^2 + 4x^2 &= 12 \\ 3(x^2 + 4x + 4) + 4x^2 &= 12 \\ 3x^2 + 12x + 12 + 4x^2 &= 12 \\ 7x^2 + 12x &= 0 \\ x(7x + 12) &= 0 \end{aligned}$$

Therefore, we obtain $x = 0$ or $x = -\frac{12}{7}$.

Lyryx HW: 1.2.12, 1.2.14, 1.2.15, 1.1.5, 1.1.6, 1.1.7
Stewart HW: Appendix B #23, 33, 35, 58, Appendix C # 3, 5, 7, 1.1 # 32-35

1 Week 1

1.1 Introduction to Calculus

Calculus helps us to understand how a quantity changes value. The advancement of modern science is due to the development of calculus. Before calculus, math could be used for calculating properties of an object at rest (areas, volume, distance etc). Calculus can be used to find how particles, cells, planets move. It was invented by Leibniz and Newton in the 17th century and is now used in physics, engineering, economics, statistics and medicine.

There are two types of calculus - differential and integral calculus. Differential calculus allows us to compute the rate of change of a quantity. Looking at the graph of this quantity, the derivative gives us the slope of the curve at any point. Integral calculus allows us to determines the quantity when we know its rate of change. An integral allows us to find the area under the curve. We will learn about both branches of calculus in this course. First we will look at how quantities can be measured. We will use functions to describe the relationships between quantities. For example, we might be measuring the number of infected in a population, the amount of morphine in the blood stream, temperature of solution, length of a bridge or the cost of producing materials. We will see that functions are often described relative to time but can also be can be measured relative to an angle, distance, cost, fuel or other amounts. In this course, we will be interested in the relationship between two quantities at a time. We often use x to represent the quantity we are measuring relative to and call it the independent variable. We use y to describe the quantity that is changing depending on the value of x. We call y the dependent variable. We will introduce function notation and definitions in the next section.

1.2 Representing Functions

1.2.1 Introduction to Functions: Video Lesson

A **function** is a rule that assigns to each element $x \in A$ exactly one element called $f(x)$ in a set B. The set A is called the **domain** of the function. The set of all possible values of $f(x)$ as x varies throughout the domain is called the **range** of the function. If x denotes an arbitrary number in the domain of f, then x is called an **independent variable**. An arbitrary number y in the range of f is called a **dependent variable**. If for each value of x there corresponds exactly one value of y, we write $y = f(x)$ and say that y is a function of x.

Example Determine whether y is a function of x and if it is, state the domain and range of the function.

(a) $y^2 + x^2 = 2$

(b) $y = \sqrt{4 - x^2}$

(c) $x - 4 - y^2 = 0$

The **graph** of a function $f(x)$ is the set of points (x, y) in the xy-plane for which $y = f(x)$. We can tell if a curve in the xy-plane is a function or not by looking at its graph.

1.2.2 Vertical Line Test: Video Lesson

A curve in the xy-plane is the graph of a function of x iff no vertical line intersects the curve more than once.

Example

Use the Vertical Line Test to verify your answers in the example above.

1.2.3 Piecewise Defined Functions: Video Lesson

A piecewise function is defined by different formulas in different parts of their domains.

Heaviside Function

A very useful piecewise function for engineering purposes is the Heaviside unit step function $h(t)$. We can think of this as an **off-on** function since it is off (or has a value of 0) for $t < 0$ and turns on (or has a value of 1) for $t > 0$. It is used to model voltage switching on or off in an electrical circuit.

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

1.2.4 Piecewise Function Example

Express $g(x) = \dfrac{|x^2 - 4|}{x}$ as a piecewise function.
Video Solution

1.2.5 Floor and Ceiling Functions

These functions take as input a real number and output an integer.
The **floor** or **greatest integer** function is defined by $f(x) = \lfloor x \rfloor$ and is the greatest integer that is not greater than x. Think of rounding down. Our age is an example of a floor function.
The **ceiling** or **least integer** function is written $g(x) = \lceil x \rceil$ and is defined to be the least integer that is not less than x. Think of rounding up.
We can think of these as piecewise constant functions.

$$\lfloor x \rfloor = \begin{cases} \vdots \\ -2 & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ \vdots \end{cases}$$

1.2.6 Symmetry: Video Lesson

Suppose D represents the domain of a function f.
If f satisfies $f(-x) = f(x)\ \forall x \in D$, then f is an **even** function. The graph of an even function is symmetric about the y-axis. This means if we have the graph of f for $x \ge 0$, we can obtain the entire graph by reflecting this portion about the y-axis.
If f satisfies $f(-x) = -f(x)\ \forall x \in D$, then f is an **odd** function. The graph of an odd function is symmetric about the origin. This means if we have the graph of f for $x \ge 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.
If f satisfies $f(x + p) = f(x)\ \forall x \in D$, then f is a **periodic** function with period equal to p. The graph of f repeats on intervals of length p.

1.2.7 Example

Determine whether each of the following functions is even, odd or neither.

(a) $g(x) = x^4|4x|$ (b) $f(x) = \dfrac{|x^2-4|}{x}$ (c) $h(x) = 2x - x^3$

Solution:
(a)

$$\begin{aligned} g(-x) &= (-x)^4|4(-x)| \\ &= (-1)^4x^4|-4x| \\ &= x^4|4x| \\ &= g(x) \end{aligned}$$

Thus, g is even.

1.3 Sequences

A sequence is an infinite list of numbers. The numbers in the list are called **terms**. In the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

a_1 is the first term, a_2 is the second term and, in general, a_n is the nth term. We can think of a sequence as a function where the domain is the set of positive integers.

Examples of Sequences: Write a formula for a_n for each of the following sequences:

1. the sequence of positive integers: $\{1, 2, 3, \ldots\}$
2. a geometric sequence: $\{2, 4, 8, 16, \ldots\}$
3. an arithmetic sequence: $\{3, 7, 11, 15, 19, \ldots\}$
4. a decreasing sequence: $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\}$

Harder Problem

Euclid 2020 4b *Written Solution*

A geometric sequence has first term 10 and common ratio 12. An arithmetic sequence has first term 10 and common difference d. The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence. Determine all possible values of d.

Monotonic Sequences

A sequence $\{a_n\}$ is **increasing** if $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$

A sequence $\{a_n\}$ is **decreasing** if $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots$

If the inequality signs are strict ($<$ or $>$), then the sequence is **strictly** increasing or decreasing, respectively. A sequence that is either increasing or decreasing is called **monotonic**.

Harder Problem

Video Solution

Show that the sequence $\dfrac{n}{n^2+1}$ is decreasing.

1.4 A Catalog of Essential Functions

Linear Functions

The graph of a linear function is a line, so we can rewrite the point-slope form and obtain the slope-intercept form of the equation of a line $y = f(x) = mx + b$, where m is the slope of the line and b is the y-intercept.

Polynomials

A polynomial P is a function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where n is a non-negative integer and the a_i's are constants called the **coefficients** of the polynomial. The domain of any polynomial is $(-\infty, \infty)$ and if $a_n \neq 0$, the **degree** of the polynomial is n.

When a polynomial is in factored form, we can get useful information to help sketch the graph.
Note that a linear function is a polynomial of degree 1.
A polynomial of degree 2 is called a **quadratic function** and is of the form $P(x) = ax^2 + bx + c$. Its graph is always a parabola obtained by shifting the parabola $y = ax^2$ as we have already seen.

Harder Problem

Euclid 2015 4a

Find the x and y intercepts of $y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4)$. Video Solution

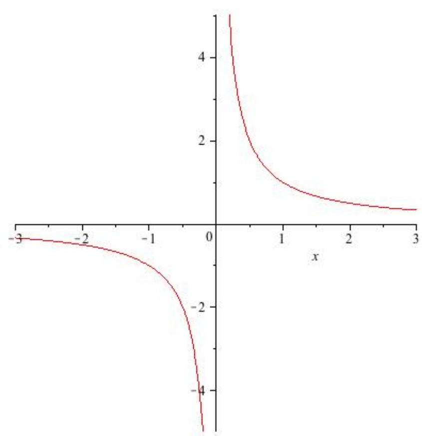
Harder Problem

Euclid 2015 4b

The graphs of the equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$ intersect at exactly two points. Determine all possible values of a. Video Solution Written Solution

Reciprocal Functions

A function of the form $f(x) = x^{-1} = 1/x$ is called the **reciprocal function**.

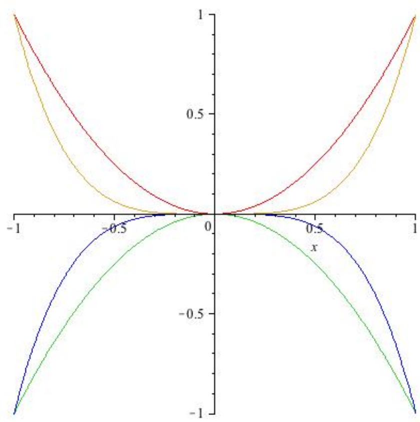


When looking at more general reciprocal functions of the form $g(x) = \dfrac{1}{h(x)}$, it can be useful to examine the zeros of $h(x)$ which lead to the asymptotes of $g(x)$. We can also examine the end behaviour of g and h for very large positive or negative values of x. We will explore this concept in more details in later sections when we discuss function limits.

Power Functions

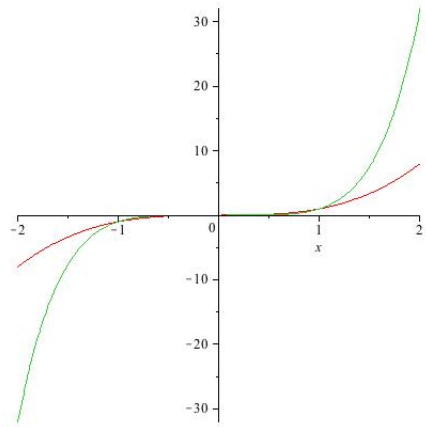
A function of the form $f(x) = x^a$, where a is a constant is called a **power function**.

If a is an even positive integer, the graph of $y = x^a$ is an even function and similar to the graph of $y = x^2$.



$y = x^2$, $y = x^4$, $y = -x^2$, $y = -x^4$

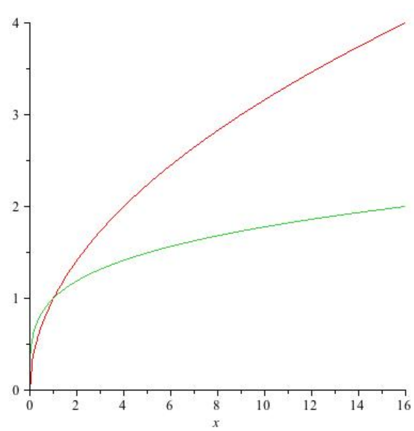
If a is an odd positive integer, the graph of $y = x^a$ is an odd function and similar to the graph of $y = x^3$.



$y = x^3$ $y = x^5$

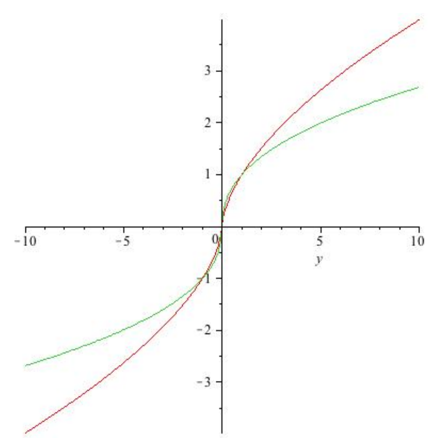
Root Functions

If n is even, the graph of $y = \sqrt[n]{x}$ is similar to the graph of $y = \sqrt{x}$, the upper half of the parabola $y^2 = x$.



$y = \sqrt{x}$, $y = \sqrt[4]{x}$

If n is odd, the graph of $y = \sqrt[n]{x}$ is similar to the graph of $y = \sqrt[3]{x}$, which is equivalent to the cubic function $x = y^3$.



$y = \sqrt[5]{x}$, $y = \sqrt[7]{x}$

Rational Functions

A rational function f is a ratio of polynomials.

$$f(x) = \frac{x^3 - 2x + 1}{2x^7 - x^6 + 4x^2 - 9}$$

Algebraic Functions

A function that can be expressed using the operations addition, subtraction, multiplication, division, raising to a power, and taking a root.

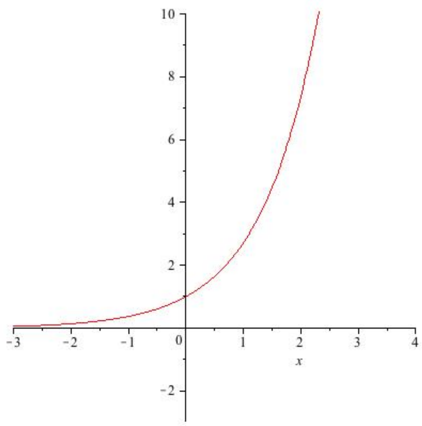
$$f(x) = \frac{\sqrt[3]{1-x^2}}{x^{2/5} - \sqrt{2}x^5}$$

Transcendental Functions

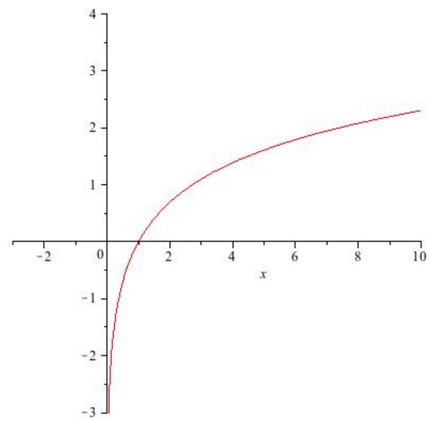
A function that cannot be expressed as a finite combination of the algebraic operations of addition, subtraction, multiplication, division, raising to a power, and taking a root. The next few function types are examples of transcendental functions.

Exponential, Logarithmic and Trigonometric Functions

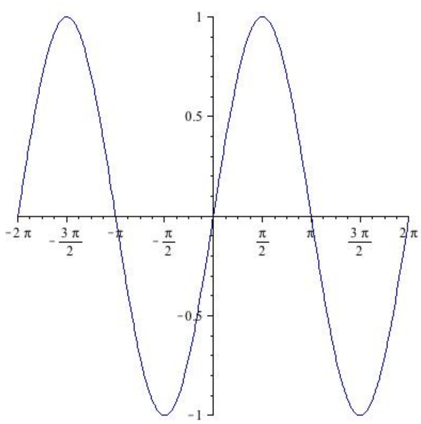
We will study these in greater depth over the next two weeks. For now, here are some basic graphs.



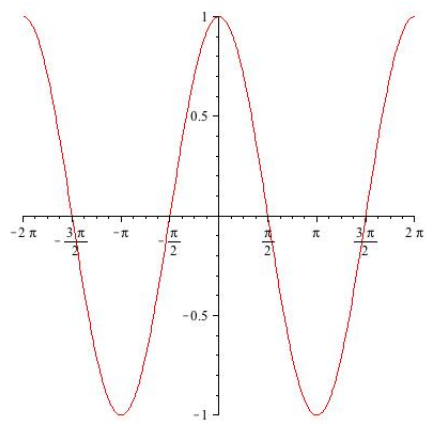
$y = e^x$



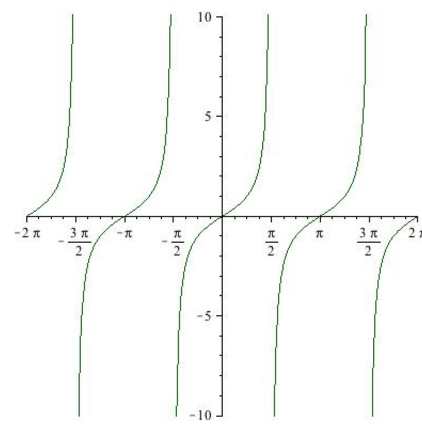
$y = \ln x$



$y = \sin x$



$y = \cos x$



$y = \tan x$

1.5 New Functions From Old

We can take some of the basic functions described above and obtain new functions by shifting, stretching, reflecting as well as applying algebraic operations and composition.

Vertical and Horizontal Shifts

Suppose we know the graph of a function $y = f(x)$ and c is a constant $c > 0$.
To obtain the graph of:

$$\begin{aligned}
y &= f(x) + c, \text{ shift the graph of } y = f(x) \text{ up } c \text{ units} \\
y &= f(x) - c, \text{ shift the graph of } y = f(x) \text{ down } c \text{ units} \\
y &= f(x + c), \text{ shift the graph of } y = f(x) \text{ to the left } c \text{ units} \\
y &= f(x - c), \text{ shift the graph of } y = f(x) \text{ to the right } c \text{ units}
\end{aligned}$$

Stretches and Reflections

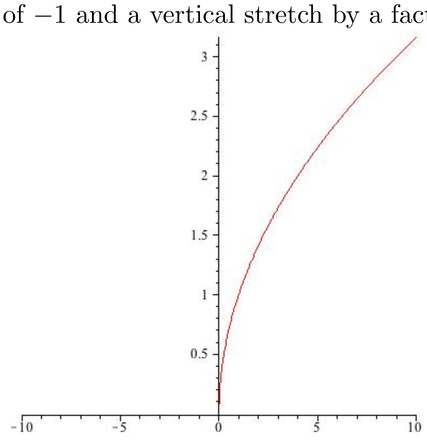
Suppose we know the graph of a function $y = f(x)$ and c is a constant $c > 1$.
To obtain the graph of:

$$\begin{aligned}
y &= cf(x), \text{ stretch the graph of } y = f(x) \text{ vertically (away from } x\text{-axis) by a factor of } c \\
y &= (1/c)f(x), \text{ shrink the graph of } y = f(x) \text{ vertically (towards } x\text{-axis) by a factor of } c \\
y &= f(cx), \text{ shrink the graph of } y = f(x) \text{ horizontally (towards } y\text{-axis) by a factor of } c \\
y &= f(x/c), \text{ stretch the graph of } y = f(x) \text{ horizontally (away from } y\text{-axis) by a factor of } c \\
y &= -f(x), \text{ reflect the graph of } y = f(x) \text{ about the } x\text{-axis} \\
y &= f(-x), \text{ reflect the graph of } y = f(x) \text{ about the } y\text{-axis}
\end{aligned}$$

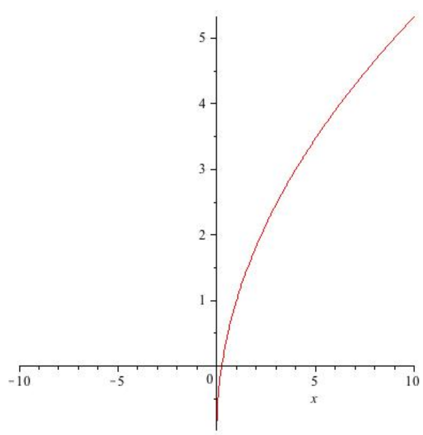
1.5.1 Find Graph from Basic Functions Example:

Sketch the graph of $y = 2\sqrt{-3x + 9} - 1$.

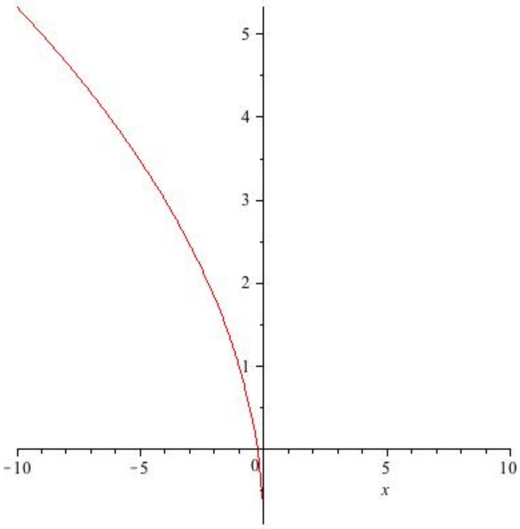
The base function is $f(x) = \sqrt{x}$. We notice a vertical shift of -1 and a vertical stretch by a factor of 2.



Let $g(x) = 2f(x) - 1 = 2\sqrt{x} - 1$.

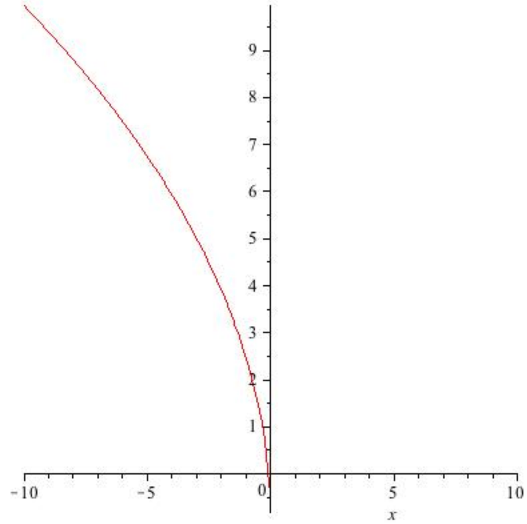


The negative sign under the square root is a reflection in the y-axis.



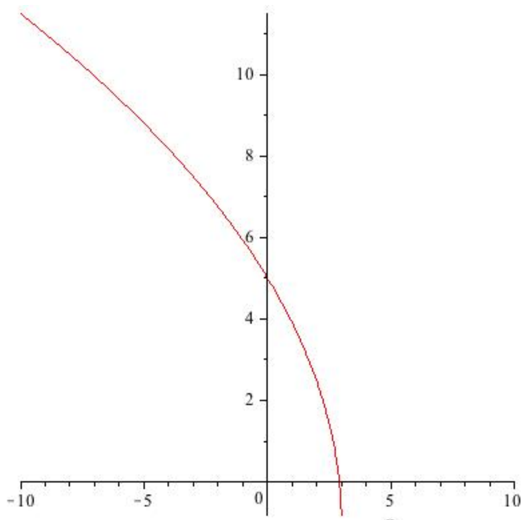
Let $h(x) = g(-x) = 2f(-x) - 1 = 2\sqrt{-x} - 1$, the reflection of g in the y-axis.

Now we deal with the factor of 3.



Let $i(x) = h(3x) = 2f(-3x) - 1 = 2\sqrt{-3x} - 1$, shrinking h towards the y-axis by a factor of 3.

Finally we look at the $+9$ to identify the horizontal shift. Note that we do not have simply a shift to the left of 9:
That is $i(x + 9) = h(3(x + 9)) = g(-3(x + 9)) = 2f(-3(x + 9)) - 1 = 2\sqrt{-3x + 27} - 1 \neq y$.
Instead we need to look at what happened to x alone by noting that $-3x + 9 = -3(x - 3)$.
Let $j(x) = i(x - 3) = h(3(x - 3)) = g(-3(x - 3)) = 2f(-3(x - 3)) - 1 = 2\sqrt{-3(x - 3)} - 1 = y$ shifts i to the right 3 units.



Now we can see exactly what happened to $f(x) = \sqrt{x}$: a vertical shift of -1, a vertical stretch by a factor of 2, a reflection in the y-axis, a shrink towards the y-axis by a factor of 3 and a shift to the right 3 units.

1.5.2 Domain and Range using Transformations

If we can obtain the graph of a function, we can use it to discover its domain and range.

Example

Sketch the graphs of the following functions and determine their domain and range.
(a) $y = 3\sqrt{x-2} - 1$ (b) $y(x-2)^2 + \frac{4}{3}$ Video Solution

1.5.3 Heaviside Example

Sketch the graph of $h(t-3)$, where h is the heaviside function. Geogebra Solution

Harder Problem

Sketch the graph of $h(t-3) - h(t-5)$. Video Solution

1.5.4 Combinations and Composition of Functions

Two functions f with domain A and g with domain B can be combined to form new functions as follows:

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) \quad \forall x \in A \cap B \\
(f-g)(x) &= f(x) - g(x) \quad \forall x \in A \cap B \\
(fg)(x) &= f(x)g(x) \quad \forall x \in A \cap B \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \quad \forall x \in A \cap B \mid g(x) \neq 0 \\
(f \circ g)(x) &= f(g(x)) \quad \forall x \in B \mid g(x) \in A
\end{aligned}$$

Combine Functions Example

If $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{x}{x-3}$, find each function and its domain.

(a) $f+g$ (b) $f-g$ (c) fg (d) f/g (e) $f \circ g$ (f) $g \circ f$ (g) $f \circ f$ (h) $g \circ g$

Solution

(b)

$$(f-g)(x) = f(x) - g(x) = \frac{1}{x+2} - \frac{x}{x-3}$$

Domain of $f-g$ is $x \neq -2$ and $x \neq 3$.
In interval notation, $x \in (-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

(g)

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x+2}\right) = \frac{1}{\dfrac{1}{x+2} + 2} = \frac{1}{\dfrac{1+2(x+2)}{x+2}} = \frac{x+2}{1+2x+4} = \frac{x+2}{2x+5}$$

Domain of $f \circ f$: $x \neq -2$ (domain of f) and $x \neq -\frac{5}{2}$. (so that $f(x)$ is in the domain of f.)
Alternatively we could note that for x to be in the domain of f we need $x \neq -2$ and for $f(x)$ to be in the domain of f, we need $\dfrac{1}{x+2} \neq -2$ or $1 \neq -2x - 4$ or $x \neq -\frac{5}{2}$.
In interval notation, $x \in (-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, -2) \cup (-2, \infty)$

(h)

$$g(g(x)) = g\left(\frac{x}{x-3}\right) = \frac{\dfrac{x}{x-3}}{\dfrac{x}{x-3} - 3} = \frac{\dfrac{x}{x-3}}{\dfrac{x-3(x-3)}{x-3}} = \frac{x}{x-3}\left(\frac{x-3}{x-3x+9}\right) = \frac{x}{9-2x}$$

Domain of $g \circ g$: $x \neq 3$ (domain of g) and $x \neq \frac{9}{2}$ (so $g(x)$ is in the domain of g.)
Alternatively we could note that for x to be in the domain of g we need $x \neq 3$ and for $g(x)$ to be in the domain of g, we need $\dfrac{x}{x-3} \neq 3$ or $x \neq 3x - 9$ or $x \neq \frac{9}{2}$.
In interval notation, $x \in (-\infty, 3) \cup (3, \frac{9}{2}) \cup (\frac{9}{2}, \infty)$

Decompose Function Example

Write the function $f(x) = 2x^2\sqrt{4x^4+5}$ as the composition of two functions g and h. Video Solution

1.5.5 Composite Functions in Science

Composite functions are useful in many applications when a first quantity depends on a second quantity that in turn depends on a third quantity. Forming the composite function allows us to see how the first quantity behaves relative to the third quantity.

Example

In fish, brain weight B is a function of body weight W in fish according to the model $B = 0.007W^{2/3}$. A model for body weight as a function of body length L is found to be $W = 0.12L^{2.53}$. Find the composite function $B \circ W$ and state what it represents.

Solution: Since B is a function of W, we can write $B(W) = 0.007W^{2/3}$ and since W is a function of L, we can write $W(L) = 0.12L^{2.53}$. Then the composite function

$$\begin{aligned} B \circ W = B(W(L)) &= B(0.12L^{2.53}) \\ &= 0.007(0.12L^{2.53})^{2/3} \\ &= 0.00084L^{5.06/3} \end{aligned}$$

Since $B \circ W$ is a function of L, this gives us the relationship between brain weight and body length.

Stewart HW: 1.1 #7 -10, 14, 22, 25, 31, 33, 35, 37, 47, 50, 69, 73 1.2 #1, 2, 3 ,4, 8, 1.3 #2, 3, 4, 8
Geogebra: Experiment with Graphs

2 Week 2

2.1 Inverse Functions

2.1.1 One-to-One

A function is called **one-to-one** if it never takes on the same value twice. More precisely, f is one-to-one iff

$$\begin{aligned} f(x_1) = f(x_2) &\implies x_1 = x_2 \\ \text{Equivalently, } x_1 \neq x_2 &\implies f(x_1) \neq f(x_2) \end{aligned}$$

We can examine the graph of a function to see whether or not it is one-to-one.

2.1.2 Horizontal Line Test

A function is one-to-one iff no horizontal line intersects its graph more than once.

Example

Sketch the graph of each given function, then determine if it is one-to-one using the Horizontal Line Test.

(a) $f(x) = \begin{cases} 3x & x < -1 \\ -3 & -1 \leq x \leq 3 \\ \sqrt{x-3} & x > 3 \end{cases}$
Video Solution

(b) $y = 2x^2 + 4x - 1,\ x \geq -1$
Video Solution

(c) $g(x) = |6-x|$
Video Solution

One-to-one functions are important because they are the only functions that have an inverse function.

Definition

Let f be a one-to-one function with domain D and range R. Then its inverse function f^{-1} has domain R and range D and is defined by

$$f^{-1}(y) = x \iff f(x) = y \quad \forall y \in R$$

Note that if f were not one-to-one, there would exist two x values for a single y and so f^{-1} would not be defined uniquely, and as such would not be a function.
Note that when we compose a function and its inverse we get back our original input.

$$f^{-1}(f(x)) = x$$

Example: If $f(x) = 3x + x^2 - 3$ for $x \geq 0$, determine $f^{-1}(1)$. Video Solution

2.1.3 Finding the Inverse Function

When we focus on the inverse function, we often reverse the roles of x and y since we are used to x being the independent variable. This is done in step 3 below in finding the inverse of a one-to-one function:

1. Write y as a function of x, $y = f(x)$.
2. Solve this equation for x in terms of y. (This is already the inverse function, but as we mentioned we would like to express the inverse as a function of x.)
3. To express f^{-1} as a function of x, interchange x and y so the resulting equation is $y = f^{-1}(x)$.

Note: To graph f^{-1} as a function of x, we simply interchange x and y in the graph of f or reflect the graph of f about the line $y = x$.

Example

Find the inverse of $f(x) = x^3 + 2$ and sketch the graph of $f^{-1}(x)$. Video Solution

2.1.4 Domain and Range of Inverse Functions

We noted above that the domain and range are reversed in the inverse function. This is because of the step where we swap the roles of x and y when finding the inverse. Sometimes it is easier to find the range of a function indirectly by finding the domain of the inverse.

Example

Find the domain and range of the following functions and their inverse: (a) $y = \dfrac{7x}{3-2x}$ (b) $y = 3\sqrt{x-2} - 1$
Video Solution

Harder Problem

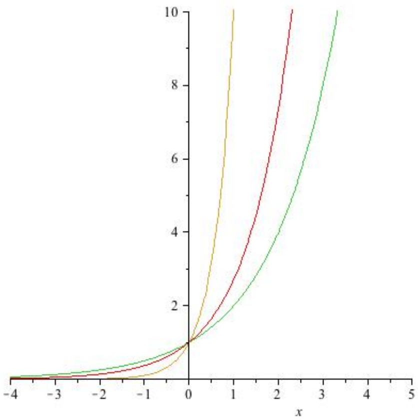
Source: Euclid 7a 2020 Video Solution Written Solution

Suppose that the function g satisfies $g(x) = 2x - 4$ for all real numbers x and that g^{-1} is the inverse function of g. Suppose that the function f satisfies $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$ for all real numbers x. What is the value of $f(\pi)$?

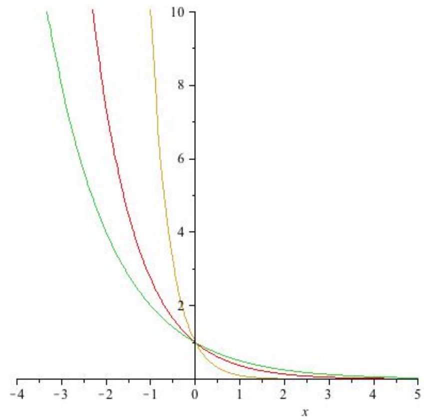
Stewart HW: 1.4 #1, 3, 13, 15 1.5 #5, 6, 8, 9, 12, 15, 17, 18
Lyryx HW: 2.3.1 - 2.3.4, 2.4.1 - 2.4.3

2.2 Exponential Functions

In general, an exponential function is of the form $f(x) = a^x$, where a is a positive constant. We examine the graphs of some exponential functions for different values of the base a. The base $e \approx 2.71828$ is an important base in calculus and we will discuss this in more detail as we move through the course.



$y = 10^x$, $y = e^x$, $y = 2^x$



$y = \left(\frac{1}{10}\right)^x$, $y = \left(\frac{1}{e}\right)^x$, $y = \left(\frac{1}{2}\right)^x$

Note the following:

- $y = a^x$ is always positive for all $a > 0$
- all graphs pass through the point $(0, 1)$ since $1 = a^0$
- $y = \left(\frac{1}{a}\right)^x = a^{-x}$ is the reflection of $y = a^x$ about the y-axis
- $y = -a^x$ is the reflection of $y = a^x$ about the x-axis

Example:

Sketch the graph of $y = 3 - 2^{x+5}$ and state the domain and range. Video Solution

2.2.1 Spread of Infection

The early stages of a disease outbreak can be modelled by exponential growth. The more infected people we have, the more people they will infect and the more the cases will rise. This means that as the number of infected people increases so does the rate of infection. This growth cannot be maintained indefinitely as those infected are not infected again and either recover or die from their illness.

> **Example:**
>
> Without intervention, the COVID-19 virus doubles every 5 days. Assuming the outbreak starts with 10 infected individuals, write a mathematical model for this scenario. Construct a table of values that gives the number of infected individuals after $t = 0, 1, 2 \ldots, 10$ days. Solution

3 Week 3

3.1 Logarithms

If $a > 0$ and $a \neq 1$, $f(x) = a^x$ is either increasing or decreasing and is one-to-one by the horizontal line test. Thus it has an inverse function f^{-1} which is called the **logarithm function with base a** and is denoted $\log_a$. If we use the definition of inverses we have

$$\log_a x = y \iff a^y = x$$

3.1.1 Cancellation Properties of Logs and Exponents

$$\begin{aligned} \log_a a^x &= x \\ a^{\log_a x} &= x \end{aligned}$$

Proof:

Let $y = a^x$. Then $\log_a y = x$. Sub in $y = a^x$ to obtain $\log_a(a^x) = x$.
Let $y = \log_a x$. Then $a^y = x$. Sub in $y = \log_a x$ to obtain $a^{\log_a x} = x$.

> **Examples:**
>
> Use the relationship between exponentials and logs and the cancellation properties to do the following:
>
> (a) Simplify $\log_2 \frac{1}{16}$ (b) Solve: $\log_e(x + 1) = 7$ (c) Solve: $e^{5x} = 4$
>
> Video Solution

Note: We give $\log_e$ a special notation ln and we call it the **natural logarithm**.

3.1.2 Algebraic Properties of Logarithms

1. $\log_a m + \log_a n = \log_a(mn)$
2. $\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$
3. $\dfrac{\log_a m}{\log_a n} = \log_n m$ (*Change of Base Formula)
4. $\log_a m^r = r \log_a m$

Proof:

1. We will show $a^{\log_a m + \log_a n} = a^{\log_a(mn)}$

$$\begin{aligned} a^{\log_a m + \log_a n} &= a^{\log_a m} a^{\log_a n} \\ &= mn \\ &= a^{\log_a(mn)} \end{aligned}$$

Therefore, since the exponential is a one-to-one function, $\log_a m + \log_a n = \log_a(mn)$.

2. Next we will show $a^{\log_a m - \log_a n = a^{\log_a(mn)}}$

$$\begin{aligned} a^{\log_a m - \log_a n} &= \frac{a^{\log_a m}}{a^{\log_a n}} \\ &= \frac{m}{n} \\ &= a^{\log_a(\frac{m}{n})} \end{aligned}$$

Since the exponential a^x is one-to-one, then $\log_a m - \log_a n = \log_a(\frac{m}{n})$.

3. To prove the Change of Base formula, we first multiply both of the equation by $\log_a n$ and instead prove

$$\log_a m = \log_a n \log_n m$$

Again we will show $a^{\log_a n \log_n m} = a^{\log_a m}$.

$$\begin{aligned} a^{\log_a n \log_n m} &= \left(a^{\log_a n}\right)^{\log_n m} \\ &= n^{\log_n m} \\ &= m \\ &= a^{\log_a m} \end{aligned}$$

4. Finally we will show $a^{\log_a m^r} = a^{r \log_a m}$.

$$\begin{aligned} a^{r log_a m} &= \left(a^{\log_a m}\right)^r \\ &= m^r \\ &= a^{\log_a m^r} \end{aligned}$$

Note: Base 10 is used often so we often write $\log_{10}$ as simply log.

3.1.3 Solving Equations Involving Logarithms and Exponentials

In solving equations involving logarithms, it is helpful to note that the function $y = \log_b m$ is one-to-one which means that if $\log_b m = \log_b n$, then $m = n$. The exponential function is also one-to-one meaning if $b^x = b^y$, then $x = y$ for any base b.

Examples

Solve for x in each of the following equations:

(a) $25^{x+2} = 5^{3x-4}$

(b) $5 + 3 \cdot 4^{x-1} = 12$

Video Solution

Harder Problem

Solve for x given $\ln(x - 2e) + \ln(x - 3e) = 2$. Video Solution

Harder Problem

Euclid 2019 3b

If $16^{15/x} = 32^{4/3}$, what is the value of x? Video Solution

Harder Problem

Euclid 2019 3c

Suppose that $\dfrac{2^{2022}+2^a}{2^{2019}} = 72$. Determine the value of a. Video Solution

3.2 Modelling with Logarithms

Logarithms tell us something about the magnitude of an amount. During the pandemic, a logarithmic scale was often used to model the number of people infected in a population. The logarithmic scale shows us clearly if the rate of change is constant, increasing or decreasing.

3.2.1 Spread of Infection Example Part 2:

Use the table of values you constructed in the Example in Section 2.1.1 and create a 3rd column with $\log_2(y)$. Plot this data with time in days as the horizontal axis and the values of $\log_2 y$ as the vertical axis. This is an example of a logarithmic scale. Another way to do this is by Solution

3.3 Trigonometry

Angles and Arcs: Video Lesson

The measure of an angle is an amount of rotation, where one complete revolution is divided into 360 equal parts called **degrees**. One revolution is equivalent to 360 degrees or 360°. We do not use degrees in calculus. Instead we use **radians** because they have no units so we can manipulate their value with other expressions through multiplication, division etc, without worrying about units of measurement.

$$\text{number of radians} = \frac{a}{r} = \frac{\text{arc length of sector}}{\text{radius}}$$

We often use θ to denote the number of radians. Since the circumference (or arc length) of a circle is $2\pi r$, the number of radians in an entire circle is $\theta = \dfrac{2\pi r}{r} = 2\pi$. Since 2π radians $= 360°$, we have π radians $= 180°$. We can use this to convert between degrees and radians.

$$\_\_\_\ \text{rad} \times \frac{180°}{\pi\ \text{rad}} = \_\_\_°$$

An angle is in **standard position** if it is drawn in the xy-plane with its vertex at the origin and its initial arm on the positive x-axis. Angles are **coterminal** if their terminal arms coincide. We measure an angle in the positive direction by travelling counterclockwise from the positive x-axis to its terminal arm.

3.3.1 Trig Ratios: Video Lesson

Let $P(x, y)$ be a point on a circle of radius r and let θ be the angle made by P with the x-axis. Then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$
$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

3.3.2 Special Triangles: Video Lesson

Example

Find the trig ratios of $\dfrac{2\pi}{3}$ and $\dfrac{7\pi}{6}$.

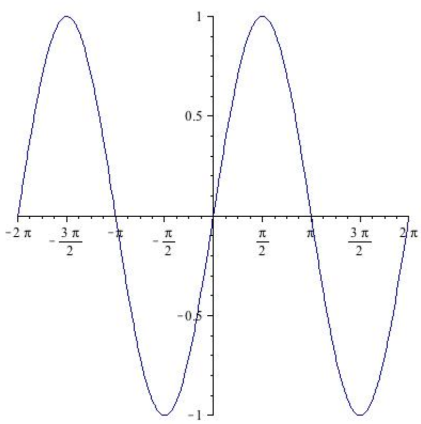
3.3.3 Trig Identities: Video Lesson

The following identities are valid for any angle θ. You should familiarize yourself with this list.

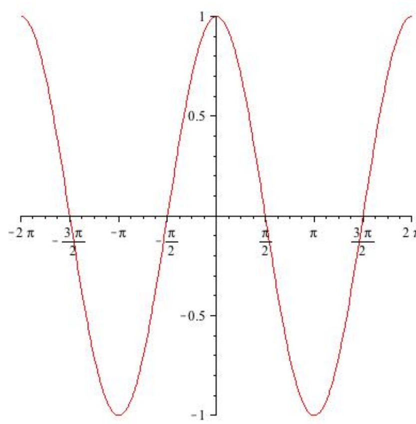
$$\begin{aligned}
\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta &= \frac{1}{\tan\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \\
\sin^2\theta + \cos^2\theta &= 1 \\
\tan^2\theta + 1 &= \sec^2\theta \\
\cot^2\theta + 1 &= \csc^2\theta \\
\sin(x \pm y) &= \sin x\cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \\
\sin 2x &= 2\sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x \\
\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) &= \cos\theta \qquad \tan(-\theta) = -\tan\theta
\end{aligned}$$

3.3.4 Graphs of Trig Functions: Video Lesson

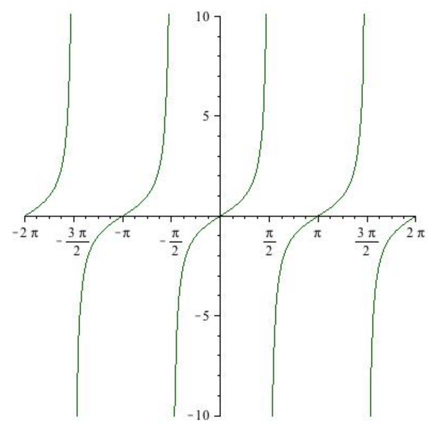
By examining the graphs of trig functions, we see that $y = \sin x$ repeats the same values every 2π radians. We say that sin has period 2π. Note that cos has period 2π and tan has period π. We will note here that these three trig functions are certainly not one-to-one.



$y = \sin x$



$y = \cos x$



$y = \tan x$

Example:

Sketch the graph of $3\sin\left(2x - \frac{\pi}{4}\right)$. Video Solution

Example

Suppose we have data that can be modelled by a trig function of the form $y = A\cos(B(t + C)) + D$, where A, B, C, D are constants. The maximum point occurs at $(7, 23)$ and the minimum occurs at $(2, 7)$. Determine the cosine function that fits the data. Video Solution

3.4 Solving Trig Equations

When solving equations involving trig ratios, we need to remember that within 0 and 2π, there are two angles that produce the same trig ratio. The period of the trig ratio tells us how often each of these angles repeats when the domain is unrestricted.

Examples:

Solve the following equations for the angle θ.

(a) $\sin 2\theta - \cos\theta = 0$.

(b) $2\cos^2\theta - 7\cos\theta + 3 = 0$, $\theta \in [0, 2\pi]$

(a) Video Solution

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ or $\sin\theta = \dfrac{1}{2}$

From $\cos\theta = 0$, we find $\theta = \dfrac{\pi}{2} + 2k\pi$ or $\theta = \dfrac{3\pi}{2} + 2k\pi$.

From $\sin\theta = \dfrac{1}{2}$, we find $\theta = \dfrac{\pi}{6} + 2k\pi$ or $\theta = \dfrac{5\pi}{6} + 2k\pi$.
(where $k \in \mathbb{Z}$)

(b) Video Solution

Harder Problem

Euclid 2020 7b Written Solution Video Solution

Determine all pairs of angles (x, y) with $0 \le x < \pi$ that satisfy the following system of equations:

$$\begin{aligned} \log_2(\sin x\cos y) &= -\frac{3}{2} \\ \log_2\left(\frac{\sin x}{\cos y}\right) &= \frac{1}{2} \end{aligned}$$

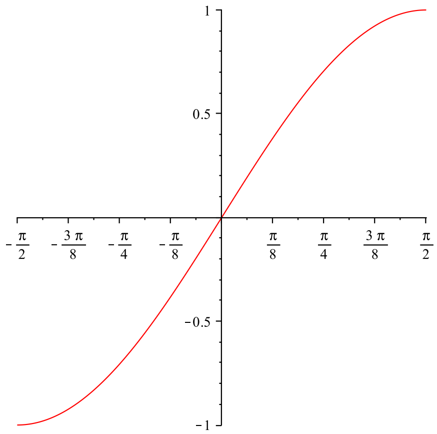
3.5 Inverse Trig Functions

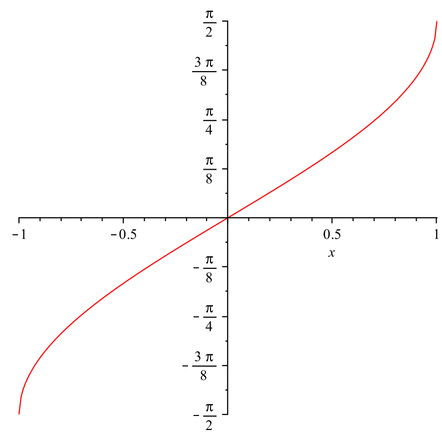
3.5.1 Introduction to Inverse Trig Functions: Video Lesson

As we mentioned above, trig functions are not one-to-one since they do not pass the horizontal line test. However, if we restrict their domains so that the functions only attain each function value exactly once, we will have a one-to-one function that will thus have an inverse.

Let's look first at the sine function $y = \sin x$. If we restrict the domain so that $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then there are no two values of x within this interval that give the same value of $\sin x$. Thus, on this interval, $y = \sin x$ is one-to-one.

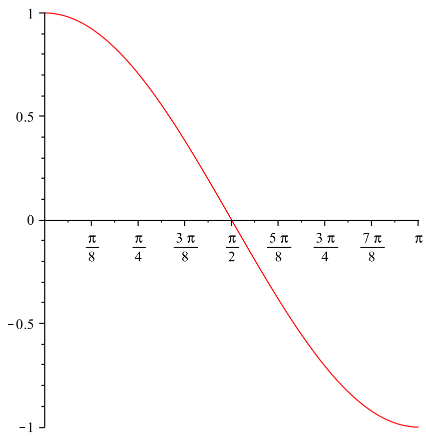
Swapping the roles of x and y so we can graph the inverse trig function, we have $y = \sin^{-1} x$, $\quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is the domain of $y = \sin x$ and the range of $y = \sin^{-1} x$. Also $[-1, 1]$ is the range of $y = \sin x$ and the domain of $y = \sin^{-1} x$.



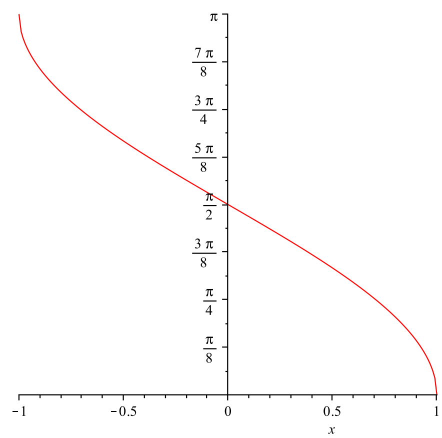


3.5.2 Graphs of Trig and Inverse Trig: Video Lesson

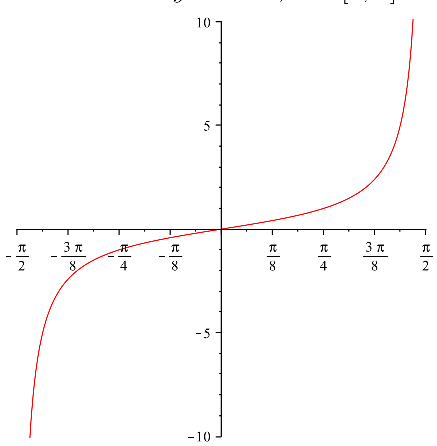
There are lots of different ways we could restrict the domain so that our trig functions will have an inverse. We will follow the convention of the text and define the six inverse trig functions with restricted domain as below.



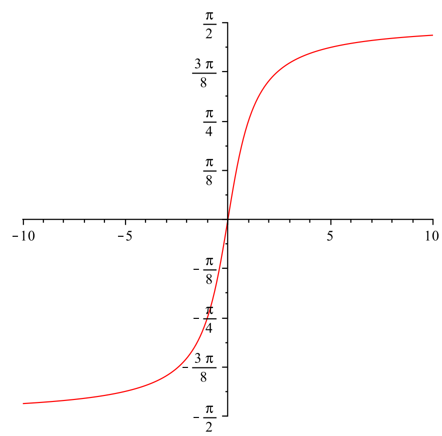
$y = \cos x,\ x \in [0, \pi]$



$y = \cos^{-1} x,\ y \in [0, \pi]$



$y = \tan x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$y = \tan^{-1} x,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = \sec x,\ x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

$y = \sec^{-1} x,\ y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

$y = \csc x,\ x \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

$y = \csc^{-1} x,\ y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$

$y = \cot x,\ x \in (0, \pi)$ $\qquad$ $y = \cot^{-1} x,\ y \in (0, \pi)$

Example: Simplify the following expressions:

(a) $\sec(\cot^{-1} \frac{7}{5})$ $\qquad$ (b) $\cos(\arctan x)$ $\qquad$ (c) $\csc^{-1}\left(\csc \frac{3\pi}{4}\right)$

Video Solutions

Stewart HW: 1.5 #21, 25, 35(b), 37, 39, 53, 55, 57, 63, 64, 66, 68, 71, Appendix D #23, 29, 45, 46, 53, 65, 78, 79
Lyryx HW: 1.3.1 - 1.3.4, 1.3.10, 1.3.11, 2.5.1 - 2.5.6, 2.6.1-2.6.3, 2.8.10, 2.8.13

4 Week 4

4.1 Limits of Sequence

With some sequences, it is possible to write a formula for the nth term as a function of n. For example, the sequence of positive integers $\{1, 2, 3, \ldots, n \ldots\}$ has nth term equal to n. Since $a_n = n$, we can write the sequence of positive integers as $\{n\}_{n=1}^{\infty}$ or simply $\{n\}$. In general, a sequence whose nth term is a_n can be referred to as $\{a_n\}$.

We will be interested in the long term behaviour of a sequence, or the value of the nth term for very large values of n. We write this limit as $\lim\limits_{n\to\infty} a_n$.

If the sequence $\{a_n\}$ has a limit L, written $\lim\limits_{n\to\infty} a_n = L$, we say that $\{a_n\}$ **converges** to L or simply say the sequence is **convergent**. If this limit does not exist, we say the sequence is divergent. Some of these sequences **diverge to infinity**. That is, as n becomes large, the terms a_n become larger without bound. In this case, we sometimes write $\lim\limits_{n\to\infty} a_n = \infty$.

4.1.1 Simple Sequence Limits

- The constant sequence $\{a\}$ has limit $L = a$.
- If $c > 0$, then $\left\{\dfrac{1}{n^c}\right\}$ has limit $L = 0$.
- If $c > 0$, then $\{\sqrt[n]{c}\}$ has limit $L = 1$.
- If $|c| < 1$, then $\{c^n\}$ has limit $L = 0$.

4.1.2 Sequence Limit Properties

If $\lim\limits_{n\to\infty} a_n = P$ and $\lim\limits_{n\to\infty} b_n = Q$, then

1. $\lim\limits_{n\to\infty} a_n + b_n = P + Q$ (Addition formula)
2. $\lim\limits_{n\to\infty} a_n - b_n = P - Q$ (Subtraction formula)
3. $\lim\limits_{n\to\infty} a_n b_n = PQ$ (Multiplication formula)
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{P}{Q}$, as long as $b_n \neq 0$ and $Q \neq 0$. (Division formula)

Example: Evaluate $\lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n + \sqrt[n]{5} - \frac{1}{n^7}}{3}$. Video Solution

Example: Evaluate $\lim_{n\to\infty} \frac{7n^3 - 3n + 5}{2n^3 + 4n - 1}$. Video Solution

4.2 Instantaneous Velocity

How does a speedometer calculate how fast your car is moving at any given point in time? Certainly, we can calculate an average velocity as the change in position over the time elapsed. If we make the time elapsed interval very small, we get closer to the instantaneous velocity.

Example:

Suppose that a ball is dropped from the top of the CN tower in Toronto. The distance in metres fallen after t seconds is given by $s(t) = 4.9t^2$ and the **average velocity** of the ball can be calculated as

$$\text{average velocity} = \frac{\text{change in position}}{\text{time period}}$$

(a) Find the average velocity of the ball over the following time intervals in the table below.

Time Interval	Average Velocity (m/s)
$5 \le t \le 6$	
$5 \le t \le 5.1$	
$5 \le t \le 5.05$	
$5 \le t \le 5.01$	
$5 \le t \le 5.001$	

(b) The **instantaneous velocity** at $t = a$ is defined to be the limiting value of the average velocities over shorter and shorter time periods that that at $t = a$. What does the instantaneous velocity at $t = 5$ appear to be using the table above?
Written Solution

The tangent line to a curve is a line that touches the curve having the same direction as the curve at the point of contact. Finding the slope of the tangent line will tell us the slope of the curve. We know how to find the slope of a line given two points so we will construct a line segment called the secant line that touches the curve at two points. As we move the second point closer to the first point, the secant line becomes closer to the tangent line.

Try experimenting with dragging two points closer together in this demonstration (Created with Geogebra by author mckaysm).

Both the tangent and instantaneous velocity problem involve the notion of a limit. We will study limits in further detail in the next few sections.

4.3 Limit of a Function at a Point

4.3.1 Definition

Suppose $f(x)$ is defined when x is near a. Then we say "the limit of $f(x)$ as x approaches a equals L", and we write

$$\lim_{x\to a} f(x) = L$$

if the values of $f(x)$ get closer to L as x gets closer to a on either side of a, but $x \neq a$.

Sometimes a limit oscillates forever as x approaches a or the limit may take on different values when x approaches a from different directions. In each of these cases, since a single limiting value L cannot be attained, we say the limit does not exist.

4.3.2 One Sided Limits

We say the limit of $f(x)$ as x approaches a **from the left** is equal to L and write

$$\lim_{x \to a^-} f(x) = L$$

if the values of $f(x)$ get closer to L as x gets closer to a where $x < a$.
We say the limit of $f(x)$ as x approaches a **from the right** is equal to L and write

$$\lim_{x \to a^+} f(x) = L$$

if the values of $f(x)$ get closer to L as x gets closer to a where $x > a$.
Note that

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Example:

Evaluate $\displaystyle\lim_{x \to 1} \frac{H(x-1) - \lfloor x \rfloor}{1-x}$ where $H(x)$ is the Heaviside step function.
Written Solution

4.3.3 Removable and Jump Discontinuity

A function f has a **jump discontinuity** at $x = a$ when both one-sided limits at a exist, but have different values. That is, f has a jump discontinuity at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exists but

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

A function f has a **removable discontinuity** at $x = a$ when the limit exists but it is different from the function value. That is, f has a removable discontinuity at $x = a$ when

$$\lim_{x \to a} f(x) \neq f(a)$$

Example: Suppose $g(x) = \begin{cases} x, & \text{if } x < 1 \\ 3, & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$ Evaluate each of the following if it exists:

(a) $\displaystyle\lim_{x \to 1^-} g(x)$ (b) $\displaystyle\lim_{x \to 1} g(x)$ (c) $g(1)$

(d) $\displaystyle\lim_{x \to 2^-} g(x)$ (e) $\displaystyle\lim_{x \to 2^+} g(x)$ (f) $\displaystyle\lim_{x \to 2} g(x)$

Video Solution 1 (with mistake) Video Solution 2 (with discontinuity discussion)

4.3.4 Infinite Limits

Let f be a function defined on both sides of a. Then we say the limit of $f(x)$ as x approaches a approaches infinity and we write

$$\lim_{x\to a} f(x) = \infty$$

if the values of $f(x)$ increase without bound as x approaches a from either side of a, but not equal to a.
We say the limit of $f(x)$ as x approaches a approaches negative infinity and we write

$$\lim_{x\to a} f(x) = -\infty$$

if the values of $f(x)$ decrease without bound as x approaches a.
Note that even though we we write "$= \infty$" or "$= -\infty$", this does not imply that infinity is a number or even that the limit exists. We symbolically write $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$ to indicate the particular way in which the limit does not exist.

Definition

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x\to a} f(x) = \infty & \lim_{x\to a^-} f(x) = \infty & \lim_{x\to a^+} f(x) = \infty \\ \lim_{x\to a} f(x) = -\infty & \lim_{x\to a^-} f(x) = -\infty & \lim_{x\to a^+} f(x) = -\infty \end{array}$$

Example:

Find the vertical asymptotes of the function $y = \dfrac{x^2+1}{3x-2x^2}$ and check your answer by graphing the function using Geogebra. Video Solution

4.3.5 Infinite Discontinuity

A function has an **infinite discontinuity** at $x = a$ when both one-sided limits are infinite. That is, f has an infinite discontinuity at $x = a$ when

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ and } \lim_{x\to a^+} f(x) = \pm\infty$$

Here we used the notation $\pm\infty$ to mean plus or minus infinity to save from writing out each of the two cases separately.

4.4 Calculating Limits Using Limit Laws

In order to evaluate limits, we will use the following list of limit laws. Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$.

6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, where n is a positive integer.

7. $\lim_{x\to a} c = c$

8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$, where n is a positive integer.

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer (and $a > 0$ if n is even).

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer (and $\lim_{x\to a} f(x) > 0$ if n is even).

4.4.1 Direct Substitution

If f is a polynomial or rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

This fact arises from the fact that a polynomial or rational function is simply a combination of operations in the above limit laws.

Example: Evaluate $\lim_{x\to 2} \dfrac{1-x}{\sqrt{x-1}+1}$. Video Solution

4.4.2 Simplifying Before Substituting

If we would like to evaluate $\lim_{x\to a} f(x)$, but f is a polynomial or rational function where a is NOT in the domain of f, we can use factoring, rationalizing denominator, or other algebra tricks to get a simpler function. Note that this new function will be equal to f everywhere except possibly at a itself.

Example: Determine $\lim_{x\to 1} f(x)$ given the function $f(x) = \begin{cases} \dfrac{|x^2+2x-3|}{x-1}, & \text{if } x < 1 \\ \dfrac{4(1-\sqrt{2-x})}{x-1}, & \text{if } 1 < x \le 2 \end{cases}$ Video Solution

4.4.3 Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a, (except possibly at a), and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

Example: If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim_{x\to 4} f(x)$. Video Solution

Harder Problem

Use the squeeze theorem to determine $\lim_{t\to 2^-} f(t)$ given that $-\dfrac{1}{t^3} \le f(t) \le \dfrac{\sqrt{t+2}-\sqrt{2t}}{t^2-2t}$ on the interval $(0, 2)$. Video Solution

Stewart HW: 2.1 #2, 5, 2.2 # 5, 11, 2.3 #15, 21, 27, 45
Lyryx HW: 3.5.1(adefgjo) 3.3.1, 3.4.1, 3.5.2, 3.5.4, 3.5.6(abc)

5 Week 5

5.1 Continuity

Definition

A function is **continuous** at $x = a$ if all three of the following conditions are met:

1. $f(a)$ is defined 2. $\lim_{x\to a} f(x)$ exists 3. $\lim_{x\to a} f(x) = f(a)$

If one or more of these is false, f is **discontinuous** at $x = a$.
Geometrically, a function is continuous if its graph can be drawn without lifting your pen from the page.
A function f is **continuous from the right** at $x = a$ if

$$\lim_{x\to a^+} f(x) = f(a)$$

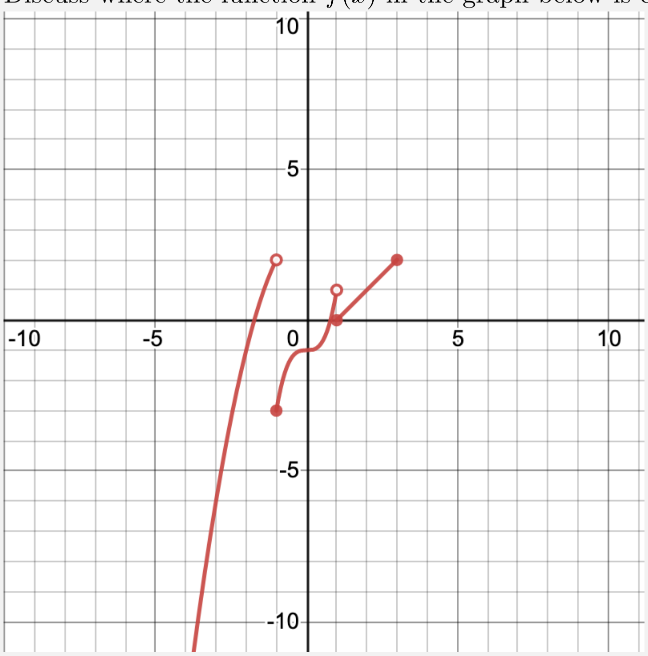
and f is **continuous from the left** at $x = a$ if

$$\lim_{x\to a^-} f(x) = f(a)$$

A function f is **continuous on an interval** $[a, b]$ if it is continuous at every number in the interval and is continuous from the right at the start of the interval, a, and continuous from the left at the end of the interval, b.

5.1.1 Example:

Discuss where the function $f(x)$ in the graph below is continuous.



Video Solution

Theorem

The following types of functions are continuous at every number in their domain:

polynomials, trig functions, exponential functions
rational functions, root functions, log functions

This tells us that when searching for discontinuities, we should look at places where the function is undefined. A type of function that is not always continuous is a piecewise function. When searching for discontinuities, we should check at "meeting points" of each piece of the function.

This theorem also tells us that we can use direct substitution when evaluating limits for any of the above functions as long as we are substituting a value in the domain of the function. (See how condition 3 of continuity is precisely the definition of direct substitution).

Theorem

If f and g are continuous at a, then so are the functions $f+g$, $f-g$, cf, fg and $\frac{f}{g}$ if $g(a) \neq 0$.

5.1.2 Example:

Show that the function $g(x) = \dfrac{\sqrt{x^2-9}}{x^2-2}$ is continuous on its domain and state its domain. Video Solution

Theorem

If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then $f \circ g$ is continuous at a.

5.1.3 Example

Evaluate $\lim\limits_{x\to 2} \arctan\left(\dfrac{x^2-4}{3x^2-6x}\right)$. Video Solution

5.1.4 Example

The gravitational force exerted by the planet Earth on a unit mass at a distance r from the centre of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3}, & r < R \\ \\ \dfrac{GM}{r^2}, & r \geq R \end{cases},$$

where M is the mass of the Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r? Video Solution

5.1.5 Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

We can use IVT to locate roots of a function. Recall that a root of a function f is a value of x that makes $f(x) = 0$.

Example: Use the Intermediate Value Theorem to show that the equation $2\sqrt{x^2+1} = x^2 - 1$ has a solution in $(\sqrt{3}, \sqrt{8})$. Video Solution

5.1.6 Bisection Algorithm

The bisection method for finding a root r such that $f(r) \approx 0$ is as follows:

1. Choose an interval $[a,b]$ such that $f(a)$ and $f(b)$ have opposite signs
2. Find the midpoint of $[a,b]$, $M = \dfrac{a+b}{2}$.
3. Determine whether the root lies in $[a,M]$ or $[M,b]$.

Using this new interval, repeat the steps until the interval is sufficiently small.

An approximation for r will then be the midpoint of the final interval with **margin of error** equal to half the width of the final interval.

In other words, if we find that r lies in the interval $[x_1, x_2]$, an approximation for r is $\frac{x_1+x_2}{2}$ with margin of error (or precision) $\frac{1}{2}(x_2 - x_1)$.

Example: Approximate the solution to $x^3 + 2x^2 - 3x = 1$ with a precision of 0.125 using the Bisection method.
Solution: The solution to the equation is the same as the solution to the equation $x^3 + 2x^2 - 3x - 1 = 0$. If we define the function $f(x) = x^3 + 2x^2 - 3x - 1$, then we are looking for r such that $f(r) = 0$.
Let's try to find an interval that a root r could lie in.
Note that $f(0) = -1 < 0$ and $f(2) = 8 + 8 - 6 - 1 = 9 > 0$.
Since f is a continuous function (polynomial), then by the Intermediate Value Theorem, there exists a value r between 0 and 2 so that $f(r) = 0$. We will take the midpoint m_1 of this first interval $[0, 1]$ and test the value of f here.

$$m_1 = \frac{0+2}{2} = 1$$

Now $f(1) = 1 + 2 - 3 = -1 < 0$.
Since $f(2) > 0$ and $f(1) < 0$, we will choose $[1, 2]$ for our next interval. Taking the midpoint, we obtain

$$m_2 = \frac{1+2}{2} = \frac{3}{2}$$

Now $f(\frac{3}{2}) = 2.375 > 0$.
Since $f(\frac{3}{2}) > 0$ and $f(1) < 0$, we will choose $[1, \frac{3}{2}]$ for our next interval. Taking the midpoint, we obtain

$$m_3 = \frac{1+\frac{3}{2}}{2} = \frac{\frac{5}{2}}{2} = \frac{5}{4}$$

Now $f(\frac{5}{4}) \approx 0.328 > 0$ and $f(1) < 0$ so we will choose $[1, \frac{5}{4}]$ as our next interval.
Note that the width of this interval is $\frac{5}{4} - 1 = 0.25$ and half this interval is 0.125 so we are within the desired margin of error.
Our approximation for r is the midpoint of this interval or $\dfrac{\frac{5}{4}+1}{2} = \dfrac{\frac{9}{4}}{2} = \frac{9}{8} = 1.125 \pm 0.125$.

5.2 Limits at Infinity; Horizontal Asymptotes

We have already looked at vertical asymptotes of a function. That is $\lim_{x\to a} f(x) = \pm\infty$ tells us there is a vertical asymptote at $x = a$. Now we are going to let x become arbitrarily large or small (approach $\pm\infty$) and see what happens to the function value. That is, $\lim_{x\to\pm\infty} f(x)$.

Definition

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L$$

Example: Evaluate $\displaystyle\lim_{x\to-\infty} \frac{2\arctan(x)}{\pi}$.

Solution: Since $\arctan(x)$ is continuous everywhere and that $\tan(x) \to -\infty$ as $x \to -\frac{\pi}{2}$. Therefore

$$\lim_{x\to-\infty} \frac{2\arctan(x)}{\pi} = \frac{2(\frac{-\pi}{2})}{\pi} = -1$$

Theorem

If $r > 0$ is a rational number then $\lim_{x\to\infty} \frac{1}{x^r} = 0$, and $\lim_{x\to-\infty} \frac{1}{x^r} = 0$ as long as x^r is defined for all x.

Example: Evaluate $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Note that as x becomes large, both the numerator and denominator become large, so it is not obvious what happens to their ratio. To evaluate the limit at infinity of any rational function, we first divide the numerator and denominator by the highest power of x that occurs in the denominator. (We can assume $x \neq 0$ since we are interested in values of x very far away from 0).
Video Solution

Example

Evaluate $\lim_{x\to\infty} (\sqrt{x^2+1} - x)$. Hint: Multiply numerator and denominator by conjugate. Video Solution

5.2.1 Infinite Limits at Infinity

If one of the following occurs,

$$\lim_{x\to\infty} f(x) = \infty \text{ or } \lim_{x\to-\infty} f(x) = \infty$$
$$\lim_{x\to\infty} f(x) = -\infty \text{ or } \lim_{x\to-\infty} f(x) = -\infty$$

we have an infinite limit at infinity. We do not have a vertical or horizontal asymptote, but we still get useful information about the graph of $f(x)$.

Harder Problem

Find vertical and horizontal asymptotes of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$. Video Solution

5.2.2 Limits of Logs and Exponentials

The following limits are useful to remember and can be verified by examining the graphs of the exponential and logarithmic functions.

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\infty} \ln x = \infty$$
$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to0^+} \ln x = -\infty$$

Examples

Determine the following limits.

1. (a) $\lim_{x\to0^-} \frac{5}{\log_4(1-3^{1/x})}$ Video Solution
   (b) $\lim_{x\to\infty} \frac{3\ln x + 2\ln^2 x}{3\ln^2 x - 2\ln x}$ Video Solution

Stewart HW: 2.5# 18, 21, 25, 29, 36, 43, 55, 2.6 #15, 17, 23, 24, 28, 31
Lyryx HW: 3.7.1-3.7.4

5.3 Derivatives and Rates of Change

Recall that the tangent one to a curve is a line that touches the curve having the same slope as the curve at the point of contact and a secant line is a line between two points of a curve. As the two points on the secant line approach one another, we get closer to the tangent line to the curve at a single point. Recall this Geogebra demonstration that explores Secant and Tangent lines. We are interested in finding the slope of the tangent line at a point $P(a, f(a))$ of a curve with equation $y = f(x)$.

Definition

The **tangent line** to a curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

as long as this limit exists.
Another way of writing this definition is if we let $h = x - a$ and so $x = a + h$. Then as $x \to a$, $h \to 0$, and we have

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This is the precise definition of the **derivative** of f at $x = a$, denoted $f'(a)$.

> **Example**
>
> Find the derivative of $f(x) = x^2 - 8x + 9$ at a using the precise definition of a derivative. Video Solution

Recall that the point-slope form of the equation of a line through the point $P(x_0, y_0)$ with slope m is $y - y_0 = m(x - x_0)$. Here we can write the equation of the tangent one to the curve $y = f(x)$ at the point $(a, f(a))$ with slope $f'(a)$ is

$$y - f(a) = f'(a)(x - a)$$

> **Example**
>
> Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$. Video Solution

5.4 The Derivative as a Function

We have looked at the derivative of a function f at a point a, now we will let a vary by replacing it by x. Then f' is a new function called the **derivative** of f defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

How does the graph of this new function f' compare to the graph of the original function f?
This activity (created in GeoGebra by Author Tim Brzezinski) allows you to move the tangent line along the curve graphing its slope at each point. This produces the graph of f' by moving the tangent line along the curve of f.

5.4.1 A Note about Notation

If $y = f(x)$ is a differentiable function, we can write its derivative as $f'(x)$ or y' or $\frac{dy}{dx}$ or $\frac{d}{dx}(f(x))$.

If we are finding the derivative of f at a point $x = a$, we can write $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.

Harder Problem

Use the limit definition of the derivative to find $g'(3)$, where $g(x) = \dfrac{1}{\sqrt{x+1}}$. Video Solution

5.4.2 Differentiability

Definition

A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) if it is differentiable at every number in that interval.

Differentiable and Continuous

If f is differentiable at a then f is also continuous at a. However, if f is continuous at a, it may or may not be differentiable at a.

There are three ways for a function to fail to be differentiable at $x = a$: a corner/cusp, a discontinuity or a vertical tangent at $x = a$.

Example

Graph the following functions and determine where they are (i) continuous (ii) differentiable. If there are places where the function fails to be differentiable, determine whether it is due to a corner/cusp, a discontinuity or a vertical tangent.

(a) $h(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 1 \end{cases}$ (b) $g(x) = (x-1)^{2/3}$ (c) $f(x) = |x-2|$ (d) $j(x) = \ln x$

Video Solution

Harder Problem

Consider the function $f(x) = \begin{cases} 8\sqrt{2+x} - 10, & x < 2 \\ 2|x-5|, & x \geq 2 \end{cases}$. Determine if f is differentiable at $x = 2$. Video Solution

6 Week 6

6.1 Derivatives of Polynomials and Exponentials

We will use the limit definition of the derivative to come up with some simple differentiation rules that can be applied to functions that we would like to differentiate quickly. In solving examples with derivatives, you may use the rules for differentiation without proving them.

6.1.1 Simple Differentiation Rules

Constant Function

If $f(x) = c$, then $f'(x) = 0$.

Power Functions

If $f(x) = x^n$, where n is any real number, then $f'(x) = nx^{n-1}$.

Constant Multiple Rule

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$$

Sum/Difference Rule

If f and g are both differentiable, then

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

6.1.2 Proofs of Simple Differentiation Rules

Here are the proofs of the simple rules for differentiation so you can see that all these rules come from the limit definition of the derivative. You will not be responsible for memorizing or presenting proofs in this course.

Proof of Constant Rule

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

Proof of Power Rule

We will prove the Power Rule for n a positive integer only. We will prove this in general next week.
Here we use the Binomial Theorem which gives the expansion of

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n$$

where n is a positive integer.
Now suppose $f(x) = x^n$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n}{h} \text{ (the } x^n \text{ terms cancel out)} \\
&= \lim_{h\to 0} h\frac{nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}}{h} \text{ (factoring out } h\text{)} \\
&= \lim_{h\to 0} [nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}] \text{ (cancel the } h \neq 0\text{)} \\
&= nx^{n-1}
\end{aligned}$$

Proof of Constant Multiple Rule:

Let $g(x) = cf(x)$.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h\to 0} c\frac{f(x+h) - f(x)}{h} = c\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= cf'(x)
\end{aligned}$$

Proof of Sum/Difference Rule:

We will prove the difference rule. The proof for the sum rule is similar.
Let $F(x) = f(x) - g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - g(x+h) - (f(x) - g(x))}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x) - (g(x+h) - g(x))}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) - g'(x) \end{aligned}$$

Example

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Solution

Using the 4 rules above, we have $y' = 4x^3 - 6(2x) + 0$
Tangent line is horizontal when $y' = 0$ so $4x^3 - 12x = 0$ or $4x(x^2 - 3) = 0$ or $4x(x - \sqrt{3})(x + \sqrt{3}) = 0$.
This is true when $x = 0$ or $x = \pm\sqrt{3}$.
When $x = 0$, $y = 4$. When $x = \pm\sqrt{3}$, $y = -5$.
Points are $(0, 4), (\sqrt{3}, -5), (-\sqrt{3}, -5)$.

6.2 Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function so f' may have a derivative of its own, denoted $(f')' = f''$. This new function is called the second derivative of f. We could also express the second derivative of $y = f(x)$ as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. This process can be continued. That is, the third derivative of f is the derivative of the second derivative and so on. In general, the nth derivative of f is denoted $f^{(n)}$ for $n \geq 4$ and we write $f^{(n)}(x) = \frac{d^ny}{dx^n}$.

Example

In the previous example, $y = x^4 - 6x^2 + 4$ and $y' = 4x^3 - 12x$. We can also find higher derivatives: $y'' = 12x^2 - 12$, $y''' = 24x$ and $y^{(4)} = 24$. Note that $y^{(n)} = 0$ for $n \geq 5$.

6.3 The Product and Quotient Rules

Even though the derivative of a sum or difference is equal to the sum or difference of the derivatives, the same is NOT true for derivatives of products and quotients.

6.3.1 Product Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

6.3.2 Quotient Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

6.3.3 Proofs of Product and Quotient Rules

Proof of Product Rule:

Let $F(x) = f(x)g(x)$. Then,

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \end{aligned}$$

Next we will add and subtract $f(x)g(x+h)$ so that we can regroup and factor in the next step.

$$\begin{aligned} &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)+f(x)g(x+h)-f(x)g(x+h)}{h} \\ &= \lim_{h\to 0}\frac{g(x+h)[f(x+h)-f(x)]+f(x)[g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{g(x+h)[f(x+h)-f(x)]}{h} + \lim_{h\to 0}\frac{f(x)[g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0} g(x+h)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= g(x)f'(x)+f(x)g'(x) \end{aligned}$$

Proof of Quotient Rule

Let $F(x) = \frac{f(x)}{g(x)}$. We want to find $F'(x)$. Instead of applying the definition of the derivative to the quotient directly, we will cross multiply to obtain $f(x) = F(x)g(x)$. Now we can use the Product Rule instead.

$$f'(x) = F(x)g'(x) + F'(x)g(x)$$

Solving for $F'(x)$, we get

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$$

Substituting in $F(x) = \frac{f(x)}{g(x)}$, we have

$$F'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$

Finally we multiply by $\frac{g(x)}{g(x)}$ to clear the fraction in the numerator:

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Examples

Find $\frac{dy}{dx}$ if $y = \frac{4x^2+3}{2x-1}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x-1)(8x)-(4x^2+3)(2)}{(2x-1)^2}\\ &= \frac{16x^2-8x-8x^2-6}{(2x-1)^2}\\ &= \frac{2(4x^2-4x-3)}{(2x-1)^2}\\ &= \frac{2(2x+1)(2x-3)}{(2x-1)^2}\end{aligned}$$

Example: Show that $\frac{d}{dx}\left(\frac{x^e+5x^2}{x\sqrt{x}}\right) = \frac{(2e-3)x^{e-2}+5}{2\sqrt{x}}$.

First we will write the denominator as a single power of x: $x\sqrt{x} = x(x^{\frac{1}{2}}) = x^{\frac{3}{2}}$

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^e+5x^2}{x^{\frac{3}{2}}}\right) &= \frac{x^{\frac{3}{2}}(ex^{e-1}+10x)-(x^e+5x^2)\frac{3}{2}x^{\frac{1}{2}}}{x^3}\\ &= \frac{ex^{e+\frac{1}{2}}+10x^{\frac{5}{2}}-\frac{3}{2}x^{e+\frac{1}{2}}-\frac{15}{2}x^{\frac{5}{2}}}{x^3}\\ &= \frac{(e-\frac{3}{2})x^{e+\frac{1}{2}}+(10-\frac{15}{2})x^{\frac{5}{2}}}{x^3}\\ &= \frac{(2e-3)x^{e+\frac{1}{2}}+5x^{\frac{5}{2}}}{2x^3}\\ &= \frac{x^{\frac{5}{2}}((2e-3)x^{e-2}+5)}{2x^3}\\ &= \frac{(2e-3)x^{e-2}+5}{2x^{\frac{1}{2}}}\end{aligned}$$

6.3.4 Derivatives of Exponential Functions

We will attempt to differentiate $f(x) = a^x$ using the precise definition of a limit.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}\\ &= \lim_{h\to 0}\frac{a^xa^h-a^x}{h}\\ &= \lim_{h\to 0}\frac{a^x(a^h-1)}{h}\\ &= \lim_{h\to 0}a^x\lim_{h\to 0}\frac{a^h-1}{h}\\ &= a^x\lim_{h\to 0}\frac{a^h-1}{h}\end{aligned}$$

If we try different bases, we note that for a between 2 and 3, this limit is close to 1. The value of a that makes this limit exactly 1 is an irrational number around 2.71828.... We denote it by e and conclude that it is the base for which the exponential function has derivative equal to itself:

$$\frac{d}{dx}e^x = e^x$$

We will come back to differentiating the general exponential function $y = a^x$ for any base after we learn the Chain Rule.

Example

If $f(x) = (3e^x + 2)(3x^2 + 2x)$, find $f'(1)$. Video Solution

6.4 Derivatives of Trigonometric Functions

If $f(x) = \sin x$, what is $f'(x)$? We will use the definition of a derivative.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \end{aligned}$$

Note:

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0 \text{ and } \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Putting this information back into the equation above, we have

$$\begin{aligned} f'(x) &= \sin x(0) + \cos x(1) \\ &= \cos x \end{aligned}$$

The two special limits we noted above can be used to help us evaluate related limits.

Harder Problem

Find $\lim_{h\to 0} \frac{1 - \cos(5h)}{\sin(4h)}$. Video Solution

One can find the derivatives of the rest of the trig functions using the limit definition of the derivative. We will list them all here.

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x & \frac{d}{dx}\csc x &= -\csc x \cot x \\ \frac{d}{dx}\cos x &= -\sin x & \frac{d}{dx}\sec x &= \sec x \tan x \\ \frac{d}{dx}\tan x &= \sec^2 x & \frac{d}{dx}\cot x &= -\csc^2 x \end{aligned}$$

Example

If $y = \frac{3\cos x}{3x - 2\sin x}$, find $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}}$. Video Solution

6.5 Chain Rule

Suppose we want to differentiate $F(x) = \sqrt{x^2+1}$.
Note that $F(x)$ is equivalent to a composite function $f \circ g(x)$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$.
The derivative of $F(x) = f(g(x))$ is $F'(x) = f'(g(x))g'(x)$.
If we let $y = f(u)$ and $u = g(x)$, then we can write this as $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

Examples

If $y = \sin(\csc x - \tan(\sqrt{\pi + 2x}))$, find y'. **Solution:**

$$\begin{aligned}
y' &= \cos(\csc x - \tan(\sqrt{\pi+2x}))\frac{d}{dx}(\csc x - \tan(\sqrt{\pi+2x})) \\
&= \cos(\csc x - \tan(\sqrt{\pi+2x}))\left(-\csc x\cot x - \sec^2(\sqrt{\pi+2x})\frac{d}{dx}(\sqrt{\pi+2x})\right) \\
&= \cos(\csc x - \tan(\sqrt{\pi+2x}))\left(-\csc x\cot x - \sec^2(\sqrt{\pi+2x})\frac{1}{2}(\pi+2x)^{-1/2}\frac{d}{dx}(\pi+2x)\right) \\
&= \cos(\csc x - \tan(\sqrt{\pi+2x}))\left(-\csc x\cot x - \sec^2(\sqrt{\pi+2x})\frac{1}{2}(\pi+2x)^{-1/2}(2)\right)
\end{aligned}$$

6.5.1 Derivative of a^x

Recall that $\dfrac{d}{dx}e^x = e^x$. What if we have a different base $a > 0$?
We can write $a = e^{\ln a}$ so then $a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$.
We differentiate using the Chain Rule as follows:

$$\begin{aligned}
\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a)x} &= e^{(\ln a)x}\frac{d}{dx}(\ln a)x \\
&= e^{(\ln a)x}(\ln a) = a^x \ln a
\end{aligned}$$

Example

If $f(x) = 2^{\sin(\pi x)}$, find $f'(x)$ and $f''(x)$.

Solution

$$\begin{aligned}
f'(x) &= 2^{\sin(\pi x)}\ln 2\frac{d}{dx}(\sin(\pi x)) \\
&= 2^{\sin(\pi x)}\ln 2\cos(\pi x)\frac{d}{dx}(\pi x) \\
&= 2^{\sin(\pi x)}\ln 2\cos(\pi x)(\pi)
\end{aligned}$$

Stewart HW: 2.7#5, 13, 17, 31, 33, 35, 59 2.8 #5,7,11,25,29 3.1 # 7, 9, 15, 23, 26-32, 3.2#1, 2, 15, 17, 19, 21, 27, 29, 54, 3.3 #1-15(odd), 23, 39, 41, 43, 3.4 #1, 7, 11, 21, 25, 37, 47
Lyryx HW: 4.1.4, 4.1.6, 4.2.1(b,c), 4.2.4, 4.2.5, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.4.1, 4.4.2, 4.5.all, 4.6.all

6.6 Implicit Differentiation

So far we have looked at differentiating functions, where one variable can be expressed in terms of another as in $y = f(x)$ or $x = g(y)$. We would like to be able to differentiate non-functions as well as functions that are defined implicitly by a relationship between x and y. It is possible to do this without having to solve for y as an explicit function of x or x as an explicit function of y. We can accomplish this using the method of implicit differentiation. If we would like to find $\frac{dy}{dx}$, we simply differentiate both sides of the equation with respect to x and then solve the equation for $\frac{dy}{dx}$. We must remember to use the chain rule when we see a function of y since this is a function of y which is in turn a function of x (composite function).

Examples

If $x^3 + y^3 = 6xy$, find $\frac{d^2y}{dx^2}$ (or y'').

Solution

First we differentiate both sides with respect to x:

$$3x^2 + 3y^2y' = 6y + 6xy' \tag{1}$$

$$\begin{aligned} y'(3y^2 - 6x) &= 6y - 3x^2 \\ y' &= \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

To find the second derivative (which we will write as y'' here), we will go back to equation (1) above and differentiate implicitly once again:

$$\begin{aligned} 6x + 6yy'y' + 3y^2y'' &= 6y' + 6y' + 6xy'' \\ y''(3y^2 - 6x) &= 12y' - 6x - 6y(y')^2 \\ y'' = \frac{12y' - 6x - 6y(y')^2}{3y^2 - 6x} &= \frac{4y' - 2x - 2y(y')^2}{y^2 - 2x} \end{aligned}$$

We will sub our expression for the first derivative in to this equation and clear the denominator:

$$\begin{aligned} y'' &= \frac{4\left(\frac{2y - x^2}{y^2 - 2x}\right) - 2x - 2y\left(\frac{2y - x^2}{y^2 - 2x}\right)^2}{y^2 - 2x} \times \frac{(y^2 - 2x)^2}{(y^2 - 2x)^2} \\ &= \frac{4(2y - x^2)(y^2 - 2x)) - 2x(y^4 - 4xy^2 + 4x^2) - 2y(4y^2 - 4x^2y + x^4)}{(y^2 - 2x)^3} \\ &= \frac{8y^3 - 16xy - 4x^2y^2 + 8x^3 - 2xy^4 + 8x^2y^2 - 8x^3 - 8y^3 + 8x^2y^2 - 2x^4y}{(y^2 - 2x)^3} \\ &= \frac{-16xy + 12x^2y^2 - 2xy(y^3 + x^3)}{(y^2 - 2x)^3} \\ &= \frac{-16xy + 12x^2y^2 - 2xy(6xy)}{(y^2 - 2x)^3} \\ &= \frac{-16xy}{(y^2 - 2x)^3} = \frac{16xy}{(2x - y^2)^3} \end{aligned}$$

6.6.1 Derivatives of Inverse Trig Functions

We can use implicit differentiation to find the derivatives of the inverse trig functions. Recall the inverse sine function is $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by $\sin^{-1} x = y \iff \sin y = x$. We will differentiate both sides of $\sin y = x$ implicitly with respect to x:

$$\begin{aligned}\cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}\end{aligned}$$

Now since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we have $\cos y \geq 0$ and using the trig identity $\cos^2 y = 1 - \sin^2 y$, we can conclude $\cos y = \sqrt{1-\sin^2 y}$. Substituting this into the equation for $\frac{dy}{dx}$ above we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-\sin^2 y}} \\ &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$

Recall that the inverse cosine function is $\cos^{-1}$ with domain $[-1,1]$ and range $[0,\pi]$ defined by $\cos^{-1} x = y \iff \cos y = x$. Differentiate both sides of $\cos y = x$ implicitly with respect to x:

$$\begin{aligned}-\sin y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{-\sin y}\end{aligned}$$

Now since $0 \leq y \leq \pi$, we have $\sin y \geq 0$ and using the trig identity $\sin^2 y = 1 - \cos^2 y$, we can conclude $\sin y = \sqrt{1-\cos^2 y}$. Substituting this into the equation for $\frac{dy}{dx}$ above we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{-\sqrt{1-\cos^2 y}} \\ &= \frac{1}{-\sqrt{1-x^2}}\end{aligned}$$

$$\boxed{\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}}$$

Recall that the inverse tangent function is $\tan^{-1}$ with domain $[-\infty,\infty]$ and range $(-\frac{\pi}{2},\frac{\pi}{2})$ defined by $\tan^{-1} x = y \iff \tan y = x$. Differentiate both sides of $\tan y = x$ implicitly with respect to x:

$$\begin{aligned}\sec^2 y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y}\end{aligned}$$

Using the trig identity $\sec^2 y = 1 + \tan^2 y$, we substitute this into the equation for $\frac{dy}{dx}$ to obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}\end{aligned}$$

$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}}$$

The remaining inverse trig functions can be differentiated in the same way and the results are summarized below.

Function	Domain	Range	Derivative
$y = \csc^{-1} x \iff x = \csc y$	$(-\infty, -1] \cup [1, \infty)$	$\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$	$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$
$y = \sec^{-1} x \iff x = \sec y$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$
$y = \cot^{-1} x \iff x = \cot y$	$(-\infty, \infty)$	$(0, \pi)$	$\frac{dy}{dx} = \frac{-1}{1+x^2}$

Examples

Differentiate the function $f(x) = \cos(\sin^{-1}(x^2 + 3x))$. Video Solution

Harder Problem

Find $\frac{dy}{dx}$ if $2y = \cos^{-1}(e^{x^2} \tan x - \sin(x^3))$. Hint: Use the inverse to rewrite the given relation and then use implicit differentiation. Express your final answer in terms of x only. Video Solution

6.6.2 Derivatives of Logarithmic Functions

We can use implicit differentiation to find the derivative of logarithmic functions.
Suppose $y = \log_a x$. Rewrite this as an exponential $a^y = x$ and differentiate both sides implicitly with respect to x.

$$\begin{aligned} \frac{d}{dx} a^y &= \frac{d}{dx} x \\ a^y \ln a \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{a^y \ln a} \\ &= \frac{1}{x \ln a} \end{aligned}$$

$$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x \ln a}}$$

Note that when $a = e$, we have $\frac{d}{dx} \ln x = \frac{1}{x}$.
In fact, we have $\frac{d}{dx} \ln |x| = \frac{1}{x}$, for $x \neq 0$. This is because

$$\begin{aligned} \frac{d}{dx} \ln |x| &= \begin{cases} \frac{d}{dx}(\ln x) & x > 0 \\ \frac{d}{dx}(\ln(-x)) & x > 0 \end{cases} \\ &= \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{-x}(-1) & x < 0 \end{cases} \\ &= \frac{1}{x}, x \neq 0 \end{aligned}$$

Example

Simplify before differentiating: $f(x) = \ln\left(\frac{3 - 2x}{\sqrt{3x^2 + 5}}\right)$. Video Solution

Make sure you can also distinguish between constants and variables when differentiating exponential functions. If a and b are constants,

$$\frac{d}{dx}(a^b) = 0 \qquad \frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x) \qquad \frac{d}{dx}(f(x))^b = b[f(x)]^{b-1}f'(x)$$

Example

Find the derivatives of the following functions: (a) $f(x) = 4^{\sqrt{1-x^2}}$ (b) $g(x) = (\sqrt{1-x^2})^4$. Video Solution

Interesting Limit

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Proof:

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.
Also,

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} \\ &= \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} \\ &= \ln[\lim_{x\to 0}(1+x)^{1/x}] \\ &= 1 \end{aligned}$$

But $\ln e = 1$, so $\lim_{x\to 0}(1+x)^{1/x} = e$.

If we let $n = \frac{1}{x}$, then as $x \to 0^+$, $n \to \infty$ and this limit can be rewritten as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Harder Problem

Evaluate $\lim_{x\to\infty}\ln\left[\left(1+\frac{2}{x}\right)^{5x}\right]$ by making an appropriate substitution. Video Solution

7 Week 7

7.1 Related Rates

The following strategy may be helpful in solving problems that ask you to compute the rate of change of a quantity in terms of the rate of change of another quantity.

1. Draw a diagram if possible.

2. Assign variables to all quantities that are functions of time.

3. Express the given information and the required rate in terms of derivatives.
4. Write an equation that relates the various quantities of the problem. You may be able to eliminate one of the variables by substituting known relationship between the variables. (ie. a formula for equation of a triangle, volume of cylinder, Pythagorean Theorem etc.)
5. Use the Chain Rule to differentiate both sides of the equation with respect to t.
6. Substitute the given information into the resulting equation and solve for the unknown rate.

Examples

A water tank has the shape of an inverted circular cone with base radius 2 m and height 4m. If water is being pumped into the tank at a rate of $5m^3$/min and is begin pumped into the tank at a rate of $5m^3$/min, find the rate at which the water level is rising when the water is 3m deep.

Solution

Let $V(t)$ be the volume of water in the tank at time t.
Then $\frac{dV}{dt}$ =Rate of water in - Rate of water out = $5m^3$/min$-3m^3$/min= $2m^3$/min. Also $V = \frac{1}{3}\pi r^2 h$. We are asked to find $\frac{dh}{dt}$ when $h = 3$.
We can eliminate the variable r from our equation for V using similar triangles and the given radius and height of the tank. $\frac{r}{h} = \frac{2}{4}$ so $r = \frac{h}{2}$.
Substituting this into our equation for V we have $V = \frac{1}{3}\pi(\frac{h}{2})^2 h = \frac{1}{12}\pi h^3$.
Now we differentiate both sides with respect to t and then substitute in $\frac{dV}{dt} = 2$ and $h = 3$:

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{12} 3h^2 \frac{dh}{dt} \\ 2 &= \frac{\pi}{12} 3(3)^2 \frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{8}{9\pi} \end{aligned}$$

7.2 Rates of Change

If $s(t) = f(t)$ is the position function of a particle, then $v = \frac{ds}{dt}$ represents the instantaneous velocity and $a = \frac{d^2 s}{dt^2} = v'(t)$ is the particle's acceleration at time t.

Example

The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$, where t is measured in seconds and s in meters.

(a) Find the velocity after 4 s.

(b) When is the particle at rest?

(c) When is the particle moving forward?

(d) Find the total distance travelled by the particle during the first five seconds.

(e) When is the particle speeding up?

Graphs in GeoGebra
Video Solution

7.3 Exponential Growth and Decay

The Number e

As we have learned, the most convenient base for the purpose of calculus is the value $e \approx 2.71828$ since this is the base for which the exponential function is equal to its rate of change. There are applications of an amount growing or decaying exponentially such as population growth, radioactive decay and compound interest.

Suppose we have the function $y = e^{kt}$.
Since $\frac{d}{dt}e^t = e^t$, then by the Chain Rule $\frac{d}{dt}(e^{kt}) = ke^{kt}$, where k is a constant.
Now if $y = e^{kt}$, then $\frac{dy}{dt} = ky$.
If we think of t as time and y as an amount of some quantity present at time t, we can interpret this equation as the rate of change of an amount is proportional to the amount present. In general we will be solving problems whose growth/decay can be described by the equation $y = y_0e^{kt}$, where y_0 is the initial amount present (at time $t = 0$). The constant k will vary in each example and we will often have to use some given information to uncover its value.

7.3.1 Radioactive Decay Problem

If $m(t)$ is the mass of a radioactive substance remaining after time t, it has been found experimentally to satisfy $\frac{dm}{dt} = km$, where $k < 0$. The rate of decay can be determined by the half-life of the substance, which is the amount of time it takes for any given amount to decay to half its size. As we have already seen, the solution to this differential equation is $m(t) = m_0e^{kt}$, where m_0 is the initial amount.

Example

The half-life of ^{14}C is 5730 years. Carbon 14 is an unstable element in the atmosphere that is ingested by plants and animals. When an organism dies, ^{14}C starts to decay to ^{12}C. Suppose a piece of parchment is found and has 90% the ^{14}C content compared with paper today. What is the age of the artifact? Video Solution

7.3.2 Newton's Law of Cooling/Warming

The rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. If we let $T(t)$ be the temperature of the object at time t and let T_s be the temperature of the surroundings (a constant). Then we have

$$\frac{dT}{dt} = k(T(t) - T_s)$$

Note that $k < 0$. If $T_0 < T_s$, then the object is warming up in its particular surroundings and if $T_0 > T_s$, the object is cooling down.

If we let $y(t) = T(t) - T_s$, note that $y'(t) = T'(t) - 0$ (since T_s is a constant) so we have that $y'(t) = T'(t)$. Then the equation

$$\frac{dT}{dt} = k(T(t) - T_s)$$
$$y'(t) = ky(t)$$

As we have seen before, the solution to $y'(t) = ky(t)$ is $y(t) = y_0e^{kt}$.
In terms of $T(t)$, we can write this as $T(t) - T_s = (T(0) - T_s)e^{kt}$ or $T(t) = T_s + (T_0 - T_s)e^{kt}$.
We can use this as our model in solving warming and cooling problems.

7.3.3 Cooling Example

A bottle of pop at room temperature of 72°F is placed in a fridge where the temperature is 44°F. After half an hour, the pop has cooled to 61°F. What is the temperature of the pop after another half hour and how long will it take for it to cool to 50°F? Video Solution

7.3.4 Logistic Model for Population Growth Video Lesson

The equation $\frac{dP}{dt} = kP$ is a simple model for population growth. In reality, due to limited resources and environmental conditions, the population levels off eventually rather than continuing to increase exponentially forever. If we let M denote the **carrying capacity** of a population (determined experimentally), then a more realistic model is the Logistic Model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Notes:

- If $P << M$, $\frac{P}{M} \to 0$, so $\frac{dP}{dt} = kP$.
- As $P \to M$, $\frac{dP}{dt} \to 0$.
- If $P > M$, $\frac{dP}{dt} < 0$ and $P \to M^+$.

Stewart HW: 3.5 #5, 7, 9, 19, 27, 29, 49, 51, 57, 3.6 # 5, 13, 15, 19, 3.7 #1(a-f), 5, 7, 23, 3.8 #1, 5, 9, 3.9 #3, 5, 17, 21, 25
Lyryx HW: 4.7.1, 4.8.6, 4.8.7, 4.9.5, 4.9.6(a,c)

7.4 Linear Approximations and Differentials

Recall that $f'(a)$ is the slope of the tangent line to the curve $y = f(x)$ at the point $x = a$, and so $(a, f(a))$ is a point on the curve and on the tangent line.
Therefore, $f(a)$ can be calculated from the equation of the curve $f(x)$ or the equation of the tangent line at a.
We will call the equation of the tangent line $L(x)$ and note that the equation of $L(x)$ is
$L(x) = f(a) + f'(a)(x - a)$. Clearly $f(a)$ satisfies this equation.
Now, suppose we would like to find an approximation for $f(x)$ when x is near a. If we have a complicated function for our curve f, it will be easier to approximate $f(x)$ from the equation of the tangent line $L(x)$.
We are approximating

$$f(x) \approx L(x) = f'(a)(x - a) + f(a)$$

Approximating $f(x)$ by $L(x)$ is called the **linear approximation** of f at a and $L(x)$ is called the **linearization** of f at a. This activity (created with GeoGebra, by Tim Brzezinski) shows the linearization of a function.

Example

Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.
Video Solution

7.4.1 Differentials

We will examine the Leipniz notation $\frac{dy}{dx}$. We can think of dy and dx as variables making up the fraction $\frac{dy}{dx}$.
To express this, we rewrite $f'(x) = \frac{dy}{dx}$ as

$$dy = f'(x)dx$$

We call dy the differential of y. It is a dependent variable, depending on the values of x and the differential dx.

7.4.2 Geometric Meaning of dy and dx

Let $P = (x, f(x))$ and $L(x)$ be the tangent line at P.
Suppose we introduce a small change in x, $\Delta x = dx$.
The corresponding change in y is $\Delta y = f(x + dx) - f(x)$.
If $f(x + dx)$ is difficult to evaluate, we can approximate Δy by $dy = f'(x)dx$.
Note that Δy is the amount the curve rises/falls when x changes by $\Delta x = dx$ and dy is the amount the tangent line rises or falls when x changes by $dx = \Delta x$.
When dx is very small, the approximation $\Delta y \approx dy$ is fairly accurate.
In terms of differentials, we could rewrite our linear approximation for $f(x)$ near a as $f(x) \approx L(x) = f(a) + dy$ or $f(a + dx) \approx L(a + dx) = f(a) + dy$.

Example

Compare the values of dy and Δy for $y = \dfrac{16}{x}$, taking $\Delta x = 0.1 = dx$, for the following values of x:

x	4	2	1	0.5
Δy				
dy				

Video Solution

We can also think of the differential $dx = \Delta x$ as an error in measurement and Δy as the associated error in a computed formula $y = f(x + dx)$. We can approximate the associated error Δy by the differential dy.
The **relative error** in $y = f(x + dx)$ can be estimated by $\dfrac{\Delta y}{y} \approx \dfrac{dy}{y}$, where dx is a measurement error, Δy is the error in the calculated value of y, and $dy = f'(x)dx$.
The **percentage error** is the relative error $\times$ 100%.

Example

A window has the shape of a square topped by a semicircle. The width of the window is measured to be 80 cm, accurate to within 0.2cm. Use differentials to estimate the maximum error and the percent error in the calculated surface area of the window. Video Solution

8 Week 8

8.1 Newton's Method

There are many mathematical equations that are impossible to solve using algebraic techniques.
For example,

$$\cos x = x$$

If we let $f(x) = \cos x - x$, then we are really interested in finding the roots of this equation or the x-intercepts. The way your calculator or computer approximates solutions to complicated mathematical equations involves the use of **Newton's Method**. The idea is to use the x-intercept of the tangent to the curve as an approximation for the x-intercept of the curve.
Suppose we want to find a root r of $f(x)$. That is, a value for which $f(r) = 0$. We start by guessing a value x_1. Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, which we will call x_2.
Note that

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, if we set $y = 0$ we have

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

As long as $f'(x_1) \neq 0$, we can solve for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This is our second approximation for r.
We can improve upon this approximation if we repeat the process. This time we will consider the tangent line to the curve at the point $(x_2, f(x_2))$. This gives us a third approximation $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2))}$ for r.
In general, if the nth approximation for r is x_n and if $f'(x_n) \neq 0$, the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

Use Newton's Method to find an approximation accurate to 2 decimals for the root of $f(x) = \cos 2x - 3x + 4$. (Hint: Sketch the graph of $y = \cos 2x$ and $y = 3x - 4$ to obtain an initial guess for the root.) Video Solution

8.2 Indeterminate Forms and L'Hospital's Rule

Suppose we would like to evaluate $\lim\limits_{x \to 1} \dfrac{\ln x}{x-1}$. We cannot use the quotient law since the denominator is approaching 0.
In general, if we have a limit of the form $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$, where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $\dfrac{0}{0}$.
Some limits of this form, we can handle by simplifying first:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1}$$

However, we cannot use this approach on $\lim\limits_{x \to 1} \dfrac{\ln x}{x-1}$, although this limit does in fact exist.

Another limit which is not obvious is $\lim\limits_{x \to \infty} \dfrac{\ln x}{x-1}$. Here, both the numerator and denominator approach infinity as $x \to \infty$. In general, if we have a limit of the form $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$, where both $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $\dfrac{\infty}{\infty}$.
We have also seen this type of limit before, which we have handled by dividing the numerator and denominator by the highest power of x in the denominator. This will not work in evaluating $\lim\limits_{x \to \infty} \dfrac{\ln x}{x-1}$

8.2.1 L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

(ie. We have an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$).
Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Note:

1. We must verify that we have an indeterminate form before using L'Hospital's Rule.
2. L'Hospital's Rule also applies to one-sided limits and infinite limits.

Examples

Evaluate the following limits:

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$.

(b) $\lim_{x \to \infty} \frac{e^x}{x^2}$.

(c) $\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x}$.

(d) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2+1}}$

Video Solution

We can also use L'Hospital's Rule to verify the special limits $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

8.2.2 Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \pm\infty$, then $\lim_{x \to a} f(x)g(x)$ is called an **indeterminate form of type** $0 \cdot \infty$. We can deal with these limits by writing the product fg as a quotient

$$fg = \frac{f}{1/g} \text{ or } fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so we can then use L'Hospital's Rule.

Example

Determine $\lim_{x \to 0^+} \sqrt[3]{x} \ln x$. Video Solution

8.2.3 Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} [f(x) - g(x)]$ is called an **indeterminate form of type** $\infty - \infty$. We can deal with these limits by converting the difference into a quotient (find a common denominator, rationalize or factor) so we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

8.2.4 Example

Evaluate the following limit $\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$. Video Solution

Harder Problem

Evaluate the limit $\lim_{x \to 0} \frac{\sin^{-1} x - x}{2(x - \tan x)}$. Video Solution

Stewart HW: 4.8 # 7, 13, 3.10 #1, 5 4.4 # 9, 15, 19, 27, 43, 49, 63
Lyryx HW: 5.4.1-5.4.7, 5.4.14, 5.5.1-5.5.20

9 Week 9

9.1 Maximum and Minimum Values

One use of derivatives is to answer questions such as:

- What shape of a can minimizes the cost of manufacturing?
- What is the maximum acceleration of a space shuttle?
- What radius of a contracted windpipe expels air most rapidly during a cough?
- What angle should blood vessels branch to minimize the energy expended by the heart pumping blood?

Definitions:

A function f has a **local maximum** at $x = M$ if there is an interval I containing M on which $f(M) \geq f(x)$ for all x in I. The **local maximum value** is $f(M)$. A function f has an **absolute/global maximum** at $x = M$ if $f(M) \geq f(x)$ for all x in the domain of f.

A function f has a **local minimum** at $x = m$ if there is an interval I containing m on which $f(m) \leq f(x)$ for all x in I. The **local minimum value** if $f(m)$. A function f has an **absolute/global minimum** at $x = m$ if $f(m) \leq f(x)$ for all x in the domain of f.

We call values that are either a maximum or minimum, **extreme** values or **extrema**. Local extrema are also called **relative** extrema since they are the largest or smallest when compared to nearby points. Absolute extrema are the very highest and very lowest points of the entire graph, not just over a small region.

9.1.1 Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute/global maximum value $f(M)$ and an absolute/global minimum value $f(m)$ at some numbers M and m in $[a, b]$.

Definition:

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example:

Find the critical numbers of $f(x) = \dfrac{\ln x}{x^2}$. Video Solution

9.1.2 Fermat's Theorem

If f has a local maximum or minimum at c then c is a critical number of f.

Fermat's Theorem and the Extreme Value Theorem tells us that global extrema must occur at critical values or at endpoints of a closed interval.

9.1.3 Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.

3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example:

Find the absolute maximum and absolute minimum values of $f(x) = (\sin x)^2 - \cos(2x)$ on $\left[0, \frac{3\pi}{4}\right]$. Video Solution

9.2 Derivatives and the Shape of a Graph

9.2.1 What does f' say about f?

Recall that the derivative tells us the slope of the curve. For the parts of the graph where the slope is positive, the curve is moving up and to the right. When the slope is negative, the curve is moving down and to the right. Try dragging the tangent line along the curve in this activity (Created with GeoGebra by Author Ken Schwartz) to see the slope of the curve at various points.

9.2.2 Increasing/Decreasing Test

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Recall: If f has a local maximum/minimum at c then c is a critical number. We could not say: If c is a critical number, then it is a local maximum/minimum. We also need the function to change from increasing to decreasing at c. This is described in the following test.

9.2.3 First Derivative Test

Suppose c is a critical number of a continuous function f:

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f does not change sign at c, then f has no local maximum or minimum at c.

Example

Find the local extrema and intervals of increase and decrease for the function $f(x) = 5x^{2/3} - 2x^{5/3}$. Video Solution

9.2.4 What does f'' say about f?

Take another look at the GeoGebra activity and note the intervals where the tangent line lies above or below the curve.
When the curve lies above its tangent lines, the curve is said to be **concave up**.
When the curve lies below its tangent lines, the curve is said to be **concave down**.

Definition:

If the graph of f lies above all of its tangent lines on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangent lines on I, it is called **concave downward** on I.
In the first diagram above, the slopes of the tangent lines increase as we move from left to right. This means f' is increasing and so $f'' > 0$.
In the second diagram, the slopes of the tangents lines decrease as we move from left to right. This means f' is decreasing and so $f'' < 0$.

9.2.5 Concavity Test

- If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.
- If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

Definition:

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there, and the curve changes from concave up to concave down or concave down to concave up at P.
Note: There is a point of inflection at any point where the second derivative changes sign. Thus, we should look for possible inflection points where $f''(x) = 0$ or $f''(x)$ does not exist.

Example

For the function $f(x) = 5x^{2/3} - 2x^{5/3}$, find the intervals of concavity and the inflection points. Video Solution

9.2.6 The Second Derivative Test

We can also use the second derivative to look for local maximum and minimum values. Suppose f'' is continuous near c:

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then this test FAILS.

Harder Problem

Sketch a possible graph of a function f that satisfies the following:

- $f'(1) = f'(-1) = 0$
- $f'(x) < 0$ if $|x| < 1$
- $f'(x) > 0$ if $1 < |x| < 2$
- $f'(x) = -1$ if $|x| > 2$
- $f''(x) < 0$ if $-2 < x < 0$, $f''(x) > 0$ if $0 < x < 2$ and $f''(x) = 0$ at $x = 0$.

Video Solution

9.3 Guidelines for Sketching a Curve

The following steps will be useful information to help you sketch the graph of $f(x)$.

1. **Domain:** Determine the set of values x for which $f(x)$ is defined.
2. **Intercepts:** To obtain the y-intercept(s), set $x = 0$ and solve for y.
   To obtain the x-intercept(s), set $f(x) = 0$ and solve for x.
3. **Symmetry:**
   (a) If $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and so the curve is symmetric about the y-axis.
   (b) If $f(-x) = -f(x)$ for all x in the domain of f, then f is an odd function and so the curve is symmetric about the origin.

(c) If $f(x+p)=f(x)$ for all x in the domain of f, then f is a periodic function and the smallest value p is called the period.

4. **Asymptotes:**

   (a) If $\lim_{x\to\pm\infty} f(x)=L$ then $y=L$ is a horizontal asymptote of the curve. If $\lim_{x\to\pm\infty} = \pm\infty$, we do not have a horizontal asymptote but this information will still help us sketch the graph.

   (b) Look for places $x=a$ where the curves is undefined (refer to domain found in step 1). If $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x)=\pm\infty$, then $x=a$ is a vertical asymptote of the curve. Note that if the domain of f is an open interval (a,b), (where $f(a)$ and $f(b)$ are undefined), then we should compute the one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$.

5. **Local Maxima and Minima** Find critical values c of f by finding where $f'(c)=0$ or $f'(c)$ does not exist. Use the first derivative test to see if f' change from positive to negative (local max at c) or negative to positive (local min at c). Alternatively, use the second derivative test by substituting any critical values c into the second derivative. If $f''(c)<0$, local max at c. If $f''(c)>0$, local min at c.

6. **Intervals of Increase and Decrease** Compute $f'(x)$ and find the intervals on which $f'(x)>0$ (f is increasing) and $f'(x)<0$ (f is decreasing).

7. **Concavity and Points of Inflection** Compute $f''(x)$ and use the Concavity Test to determine when $f''(x)>0$ (f is concave up) and when $f''(x)<0$ (f is concave down). To find points of inflection, look at points P where $f''(P)=0$ or $f''(P)$ does not exist. If f'' changes sign at P, then P is a point of inflection.

8. **Sketch the Curve** Using the information found in steps 1-7, sketch your curve, labelling intercepts, asymptotes, local and absolute extreme and points of inflection.

Example:

Use the guidelines above to sketch the curve of the function $f(x)=5x^{2/3}-2x^{5/3}$. You may use the information found in earlier examples. Check your final graph with this GeoGebra graph. Video Solution

Example:

Use the guidelines above to sketch the graph of $g(x)=\dfrac{x^2+x+1}{x^2}$. Video Solution

9.4 Optimization Problems

We can use calculus to solve word problems associated with maximizing or minimizing certain quantities. These are known as optimization problems. The following steps will be useful to solve this type of word problem.

1. **Understand the problem** What information is given and what are you begin asked to maximize/minimize?

2. **Introduce notation** Assign a variable to the quantity Q that is to be maximized or minimized and symbols to any other unknown quantities.

3. **Draw a diagram** Include given and required quantities on the diagram.

4. **Write an equation** Express Q in terms of other variable(s).

5. **Substitute given information** If Q is expressed as a function of more than variable, use any given information to eliminate other variables so that $Q=f(x)$. State the domain of this function.

6. **Optimize** Find the absolute maximum or minimum value of $Q = f(x)$. Make sure you have verified that this value is in fact a maximum or minimum. You may use the Closed Interval Method if the domain of f is a closed interval. Otherwise, you will need to use the First Derivative Test for absolute extreme values (below).

7. **Answer the Question** Make sure that your answer makes sense in the context of the problem and that you have completely answered the question.

9.4.1 First Derivative Test for Absolute Extreme Values

Suppose c is a critical number of a continuous function f defined on an interval.

- If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
- If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

In other words, if c is the only local extrema on an interval, then it is also an absolute extrema.
Note that we could alternatively use the **second derivative test** to verify that we have a local extrema. Again, if it is the only one on the given interval, it is also an absolute extrema.

Example:

If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is $P = \dfrac{E^2R}{(R+r)^2}$. If E and r are fixed but R varies, what is the maximum value of the power? Video Solution

Example:

What is the maximum possible area of a rectangle whose base lies on the x-axis, with its two upper vertices on the graph of $y = 4 - x^2$?

Solution: Following the steps outlined above:

1. We are asked to maximize the area of a rectangle.
2. Let A be the area of rectangle. Let (x, y) be the coordinates of the upper vertex in the first quadrant. Then the other upper vertex is $(-x, y)$ due to the symmetry of the parabola about the y-axis. The length of the base is $x - (-x) = 2x$ and the height is y. Note that $0 \leq x \leq 2$ for the point (x, y) to be in the first quadrant.
3. $A = 2xy$, but $y = 4 - x^2$, so we can write A in terms of x only: $A = 2x(4 - x^2) = 8x - 2x^3$.
4. Draw a diagram with base of rectangle on x-axis and upper vertices on parabola $y = 4 - x^2$.
5. We want to maximize A, so we find $A'(x)$ to find the absolute maximum of A.
   $A'(x) = 8 - 6x^2 = 0$ when $x^2 = \frac{4}{3}$ or $x = \pm\frac{2}{\sqrt{3}}$.
   Since $x \in [0, 2]$ in the first quadrant, $x = \frac{2}{\sqrt{3}}$.
   We will use the second derivative test to show this is a maximum.
   $A''(x) = -12x$ and $A''\left(\frac{2}{\sqrt{3}}\right) < 0$.
   Therefore, $x = \frac{2}{\sqrt{3}}$ is a local maximum. Since it is the only local maximum on $[0.2]$, it is also the absolute maximum.
6. To answer the question, the maximum area is

$$A\left(\frac{2}{\sqrt{3}}\right) = 8\left(\frac{2}{\sqrt{3}}\right) - 2\left(\frac{2}{\sqrt{3}}\right)^3 = \frac{16}{\sqrt{3}} - \frac{16}{3\sqrt{3}} = \frac{(48-16)\sqrt{3}}{9} = \frac{32\sqrt{3}}{9}$$

Stewart HW: 4.1 # 5, 9, 13, 27, 31, 35, 49, 53, and 57, 4.3 #1, 19, 27, 41, and 47 4.5 # 3, 11, 21, 25, 29, 35, 39, 45, 49, 51 4.7 #3, 7, 13, 17, 25, 35, 37, 45, 55
Lyryx HW: 5.2.1-5.2.9, 5.2.16-5.2.22, 5.6.1-5.6.7, 5.6.17-5.6.23, 5.6.35-5.6.41, 5.6.56-5.6.57, 5.6.65, 5.6.60, 5.6.77, 5.7.1-5.7.14, 5.7.23-5.7.25

10 Week 10

10.1 Antiderivatives

Definition:

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

If two functions have the same derivative, then they must differ by a constant. Thus, if F is any antiderivative of f, then the **general antiderivative** of f is $F(x) + C$, where C is an arbitrary constant.

Examples

Find the general antiderivative of $f(t) = \frac{1}{1+t^2} - \sqrt[3]{t}^2$.
Video Solution

10.1.1 Table of Common Antiderivatives

Function $f(x) = F'(x)$	General Antiderivative $F(x)$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln a} + C, a > 0$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
x^n	$\frac{x^{n+1}}{n+1} + C (n \neq -1)$

Sometimes, we need to antidifferentiate more than once to uncover the original function. Since each time we take an antiderivative, there is an arbitrary constant introduced, we need multiple data points on the original function or some of its derivatives in order to solve for these constants.

Example

Find the function f if $f'''(x) = \frac{\pi^3}{3}\sin\left(\frac{\pi}{3}x\right)$, $f''(0) = 4 - \pi^2$, $f'(0) = 0$, $f(0) = \frac{21}{2}$.
Video Solution

Antidifferentiation is often used in describing the motion of an object moving along a straight path. Since the velocity function $v(t)$ is the derivative of the displacement function $s(t)$ and acceleration $a(t)$ is the derivative of velocity, we can start with acceleration and a couple of pieces of information about displacement and velocity at particular times to uncover the displacement function.

Example:

A stone is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) How long does it take the stone to hit the ground and with what velocity?

(b) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

Video Solution

10.2 Sigma Notation

Suppose we want to add together the first 100 integers. We can write this using three dots to convey the idea of continuing the pattern.

$$S = 1 + 2 + 3 + \ldots + 98 + 99 + 100$$

Another way to express the sum S is

$$S = \sum_{i=1}^{100} i$$

This is called *sigma notation*.
In general when we want to add together the numbers a_i for i taking on the integer values m through n inclusive,

we express this as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_n$$

We need to recognize the description of a_i in our sum. The subscript i just means we want a formula in terms of i. (Another way of saying a_i would be $f(i)$.)
Suppose we want to represent $3^2 + 4^2 + 5^2 + \ldots + 9^2$ using summation notation.
We recognize that we are adding together squares of numbers between 3 and 9.
So $m = 3$, $n = 9$ and $a_i = i^2$, and we have $3^2 + 4^2 + 5^2 + \ldots + 9^2 = \sum_{i=3}^{9} i^2$.
Check by substituting values for i starting at 3 and going all the way up to 9.

Example:

Write the sum $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \ldots + \frac{23}{27}$ using sigma notation. Video Solution

Properties of Summations

1. $\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$
2. $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$
3. $\sum_{i=1}^{n} c = cn$. In particular, $\sum_{i=1}^{n} 1 = n$

The following three formulas will be useful in evaluating sums.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

We will demonstrate how to obtain the first formula.
We will define the sum S as follows:

$$S = 1 + 2 + 3 + \ldots + (n-1) + n$$

Writing this another way, we have

$$S = n + (n-1) + \ldots + 3 + 2 + 1$$

Adding the two equations together, we have

$$2S = (n+1) + (n+1) + \ldots + (n+1) + (n+1) + (n+1)$$
$$2S = n(n+1)$$
$$S = \frac{n(n+1)}{2}$$

Example:

Find the value of the sum $\sum_{i=3}^{6} i(i+2)$. Video Solution

Harder Problem

Evaluate the limit at infinity $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$. Video Solution

10.3 Areas and Distances

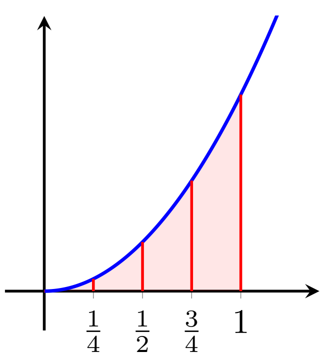
10.3.1 The Area Problem

In this section, we will solve the problem of how to find the area of the region S that lies below the continuous function $y = f(x)$ between $x = a$ and $x = b$.

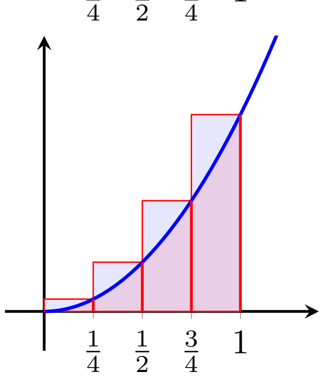
It is easy to find the area of objects with straight edges but it is more difficult when a shape has curved edges. We will divide the area into strips of equal width and then approximate the area of each strip by a rectangle and finally take the limit of these areas as we increase the number of rectangles.

Estimate Area Using Rectangles

Estimate the area under the parabola $y = x^2$ from 0 to 1 using 4 rectangles of equal width.



We will divide S into 4 strips S_1, S_2, S_3, S_4 of width $\frac{1}{4}$. We can approximate each strip by a rectangle whose base is the same as the base of the strip and whose height is the same as the height of the right edge of the strip. That is, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals: $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]$.



Each rectangle has width $\frac{1}{4}$ and the heights are: $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, 1^2. If we let R_4 denote the sum of the areas of the rectangles, we have

$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32} \approx 0.46875$$

Note that the area of S, A_S is less than R_4.

Instead of using the larger rectangles above, we could use the smaller rectangles whose heights are the values of f at the left endpoints of the subintervals. The sum of these approximating rectangles is

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 = \frac{7}{32} \approx 0.21875$$

Note that $L_4 < A_S < R_4$ or $0.21875 < A_S < 0.46875$.
We can obtain better estimates for A_S by increasing the number of strips.

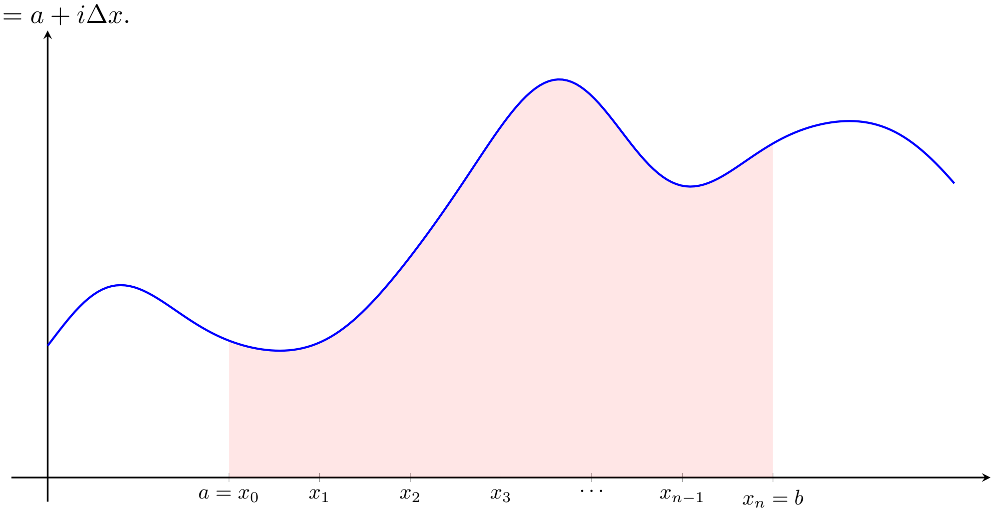
n	L_n	R_n
10	0.30875	0.35875
50	0.3234	0.3434
100	0.32835	0.33835
1000	0.3328335	0.3338335

From these values it looks as if R_n and L_n are approaching $\frac{1}{3}$ as n increases. We will verify this observation once we find an expression for the area under a curve in general.

10.3.2 Riemann Sums

In general, if we want to find the area S under a curve $y = f(x)$, we start dividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width.

The width of the interval $[a, b]$ is $b - a$ and the width of each strip is $\Delta x = \dfrac{b-a}{n}$.
These strips divide the interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, where $x_0 = a$, $x_n = b$ and $x_i = a + i\Delta x$.



If we approximate the areas S_i by rectangles using

(a) right endpoints, then the area of each rectangle i is $\Delta x f(x_i)$ and

(b) using left endpoints, then the area of each rectangle i is $\Delta x f(x_{i-1})$.

An approximation for the area of S is simply the sum of the areas of rectangles, so

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$
$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

If we want the exact area A of the region S that lies under the graph of the continuous function f, we simply take the limit of the sum of the areas of the approximating rectangles. That is,

$$\begin{aligned} A &= \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x \end{aligned}$$

In fact, we can choose any sample point x_i^* that lies in the ith subinterval $[x_{i-1}, x_i]$. Common choices for x_i^* are left endpoints, right endpoints and midpoints. Left endpoints are given by $x_{i-1} = a + (i-1)\Delta x$, right endpoints are given by $x_i = a + i\Delta x$ and the midpoint of the ith subinterval $\overline{x_i} = \dfrac{x_{i-1} + x_i}{2}$. The exact area can be found by the following formula

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a **Riemann Sum**.

Example:

Express the area of the region bounded $f(x) = x^2$ on the interval $[0, 1]$ as the limit of a Riemann sum using right endpoints. Show that the limit of this Riemann sum approaches $\frac{1}{3}$.

Solution: We divide the interval $[0, 1]$ into n subintervals of equal width $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. The ith right endpoint is given by $x_i = a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$. Thus, we have

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} \\ &= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \\ &= \sum_{i=1}^{n} \frac{i^2}{n^3} \\ &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= \frac{2n^3+3n^2+n}{6n^3} \end{aligned}$$

Finally, we take the limit of R_n as $n \to \infty$ to obtain $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

Example:

Find the exact area of the region bounded by $f(x) = x^2 + 3x - 4$ and the x-axis on the interval $[2, 4]$ using a Riemann sum and left endpoints. Video Solution

10.3.3 The Distance Problem

Suppose an object moves with velocity $v = f(t)$, where $a \le t \le b$ and $f(t) \ge 0$. If we take velocity readings at equally spaced times $t_0 = a, t_1, t_2, \ldots, t_n = b$, we will assume the velocity to be constant on each subinterval. Then the total distance travelled during the time interval $[a, b]$ (using left endpoints) is approximately

$$f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t = \sum_{i=1}^{n} f(t_{i-1})\Delta t \text{ where } \Delta t = \frac{b-a}{n}.$$

The more frequently we take velocity measurements, the more accurate our estimates become, so the exact distance d travelled is the limit of such an expression.

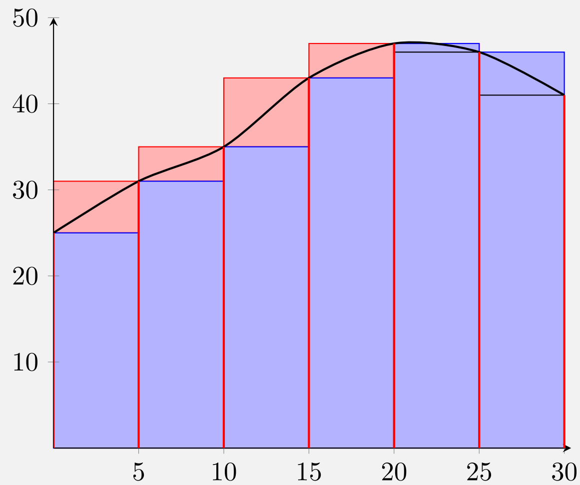
$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\Delta t$$

Example

Suppose the speedometer readings taken every 5 seconds give the following table:

Time (sec)	0	5	10	15	20	25	30
Velocity (m/s)	25	31	35	43	47	46	41

Then an approximation for the distance travelled in 30 seconds is
$5(25)+5(31)+5(35)+5(43)+5(47)+5(46)=1135$ m using left endpoints and
$5(31)+5(35)+5(43)+5(47)+5(46)+5(41)=1215$ m using right endpoints.



Stewart HW: #4.9 # 3, 5, 9, 15, 17, 25, 27, 31, 33, 36, 45, 66 Appendix E # 1-19, 5.1# 1, 5, 13, 17, 19
Lyryx HW: 6.1.1, 6.1.2

11 Week 11

11.1 The Definite Integral

The following sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a **Riemann Sum** and the limit of this sum

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

is called the **definite integral** of f from a to b.
$\int$ is called an **integral sign**, which means the limit of sums.
$f(x)$ is called the **integrand**
a and b are called the **lower and upper limits of integration**, respectively.
The process of calculating a definite integral is called **integration**.
Note that $\int_a^b f(x)dx$ is a number not a function. It represents the area below the curve $y=f(x)$ between $x=a$ and $x=b$.

Example

Express $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$ as a definite integral on the interval $[0, \pi]$. Video Solution

11.1.1 Properties of Definite Integrals Video Lesson

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b cdx = c(b-a)$
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
6. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

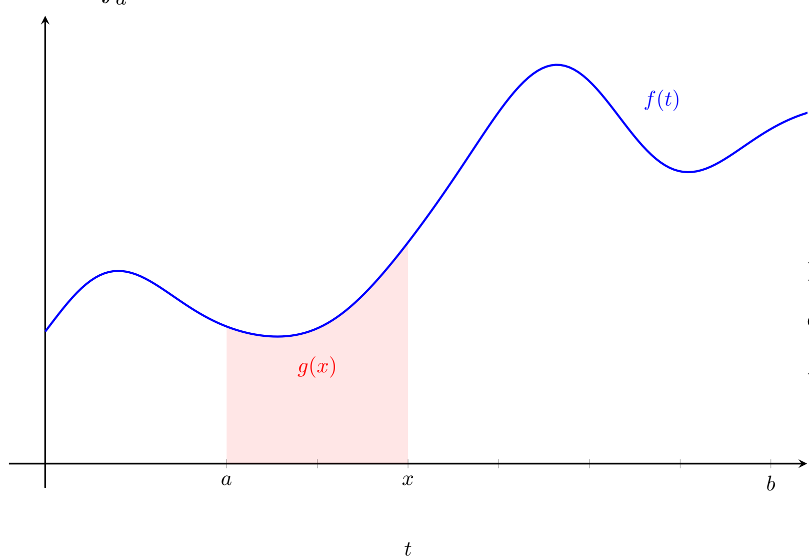
11.1.2 Comparison Properties of Integrals

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

11.2 Fundamental Theorem of Calculus

11.2.1 FTOC Part I Video Lesson

Let $g(x) = \int_a^x f(t)dt$, where f is a continuous function on $[a, b]$ and x varies between a and b.



FTOC I says that $g(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$, differentiable on (a, b) and $g'(x) = f(x)$. Alternatively, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

We can think of g as an "area so far" function - the area under f from a to x.

The proof of the Fundamental Theorem of Calculus Part I is included here for interest. It uses several theorems and properties we have seen in the course but you are not responsible for memorizing or reproducing any part of this proof on assessments.

Proof:

Let $g(x) = \int_a^x f(t)dt$. We want to show $g'(x) = f(x)$.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

We will assume x and $x+h$ are in the interval (a,b). Looking at the numerator, we have

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \end{aligned}$$

using Property 6 of Integrals.
For $h \neq 0$, $\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$.
Assume $h > 0$.
Since f is continuous on $[x, x+h]$, the Extreme Value Theroem says there exist numbers $u, v \in [x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m is the absolute minimum and M is the absolute maximum of f on $[x, x+h]$. That is, on the interval $[x, x+h]$, we have $m \leq f(t) \leq M$.
By Comparison Property 3 of Integrals,

$$\begin{aligned} (x+h-x)m \leq \int_x^{x+h} f(t)dt \leq M(x+h-x) \\ mh \leq \int_x^{x+h} f(t)dt \leq Mh \\ f(u)h \leq \int_x^{x+h} f(t)dt \leq f(v)h \end{aligned}$$

Since $h > 0$, we can divide this inequality by h:

$$\begin{aligned} f(u) \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq f(v) \\ f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v) \end{aligned}$$

We get a similar inequality for $h < 0$, being careful to change signs when we divide inequality by $h < 0$.
Finally, we want to take the limit as $h \to 0$. We note that as $h \to 0$, $u \to x$ and $v \to x$, since u and v lie between x and $x+h$. We can use the Squeeze Theorem.
We have $\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x)$ and $\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$ because f is continuous at x.
Thus, by the Squeeze Theorem, $\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x)$ and so $g'(x) = f(x)$ as required.

11.2.2 FTOC Part II Video Lesson

If f is continuous on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f. We often use the notation $F(x)\big|_a^b$ to denote $F(b) - F(a)$.

Proof:

Let $g(x) = \int_a^x f(t)dt$.
Then $g'(x) = f(x)$ by FTOC I. So g is an antiderivative of f.
If F is any other antiderivative of f on $[a, b]$, then F and g differ by a constant, or $F(x) = g(x) + C$, for $a \leq x \leq b$.

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) - g(a) \\ &= \int_a^b f(t)dt - \int_a^a f(t)dt \\ &= \int_a^b f(t)dt - 0 \end{aligned}$$

Examples

Evaluate the following

1. $\dfrac{d}{dx} \displaystyle\int_1^{x^2} \sin t dt$.
2. $\dfrac{d}{dx} \displaystyle\int_0^1 x^2 dx$.

Video Solution

Harder Problem

Prove that $\displaystyle\int_{\pi/2}^x t^2 \sin t dt = -x^2 \cos x + 2 \int_{\pi/2}^x t \cos t dt$ by first showing that the functions on the left and right sides of the equal sign have the same derivative, and so differ by a constant; then evaluate the constant. Video Solution

Stewart HW: 5.2 # 1, 3, 5, 33, 35, 39, 47, 51 5.3 # 7, 9, 17, 19, 21, 23, 27, 29, 37, 39
Lyryx HW: 8.1.1-8.1.5, 6.2.1-6.2.6

11.3 Indefinite Integrals Video Lesson

Since there is an obvious connection between antiderivatives and definite integrals, we use the notation

$$\int f(x)dx = F(x)$$

if F is an antiderivative of f or $F'(x) = f(x)$. This is called an **indefinite integral** which is a function or family of functions as opposed to a definite integral which is a number.
Note that the connection between definite and indefinite integrals is that

$$\int_a^b f(x)dx = \int f(x)dx \Big|_a^b$$

When evaluating an indefinite integral, we must always include "$+C$" in our answer to demonstrate the fact there are infinitely many antiderivatives of a function that differ by a constant.

Examples

Evaluate the following integrals:

1. $\displaystyle\int \frac{3}{x^2+1}dx$

2. $\displaystyle\int_1^3 2^x dx.$

Video Solution

11.3.1 Table of Common Indefinite Integrals Video Lesson

$$\int cf(x)dx = c\int f(x)dx \quad \int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx \quad \int kdx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}dx = \ln|x| + C \qquad \int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C \qquad \int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C \qquad \int \frac{1}{x^2+1}dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

11.3.2 Net Change Theorem Video Lesson

If $y = F(x)$, then $F'(x)$ represents the rate of change of y with respect to x and $F(b) - F(a)$ represents the change in y when x changes from a to b. Thus, the integral of a rate of change is equal to the net change in y or $\displaystyle\int_a^b F'(x)dx = F(b) - F(a)$.

11.3.3 Distance vs Displacement as an Integral Video Lesson

Suppose an object moves along a straight line with position function $s(t)$. Then its velocity is $v(t) = s'(t)$ so

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

is the net **displacement** of the particle during the time period from t_1 to t_2.
If we want to calculate the **distance** travelled, we have to consider the intervals when $v(t) \geq 0$ (particle moves forwards/to the right) and when $v(t) \leq 0$ (particle moves backwards/to the left).

$$\text{Total Distance Travelled} = \int_{t_1}^{t_2} |v(t)|dt$$

Example

Find the displacement and the distance travelled by the particle whose velocity is $v(t) = t^2 - t - 6$ during time $1 \leq t \leq 4$.

12 Week 12

12.1 Substitution Rule Video Lesson

How do we evaluate integrals such as

$$\int 2x\sqrt{1+x^2}dx$$

$2x\sqrt{1+x^2}$ is the derivative of what function?
It looks like the Chain Rule may have been used to get this derivative.
We will introduce a new variable u to be what seems to be the inner function $u = 1 + x^2$.
Recall: If $u = f(x)$, then $du = f'(x)dx$. Here $du = 2xdx$.
If we think of the dx in the integral as a differential, we have the differential $2xdx$ occurring in the given integral.

$$\int 2x\sqrt{1+x^2}dx = \int \sqrt{1+x^2}2xdx = \int \sqrt{u}du$$

Now we can integrate:

$$\int \sqrt{u}du = \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

We can check our answer by differentiating it using the Chain Rule.
In general, whenever we have an integral of the form

$$\int f(g(x))g'(x)dx$$

we use the substitution $u = g(x)$, so $du = g'(x)dx$, and solve the simpler integral

$$\int f(u)du$$

Note: Let u be a function in the integrand whose derivative also occurs. Usually, we will let u be the inner function or the complicated part of a function.

Be careful when evaluating definite integrals to remember to change the limits of integration since we are doing a substitution of variables.

Examples

Evaluate the following integrals:

1. $\displaystyle\int \frac{4x}{\sqrt{1-2x^2}}dx$
2. $\displaystyle\int \frac{x}{x^2+5}dx$

Video Solution

Look out for integrals of the form $\displaystyle\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ or $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$. You may need to **complete the square** first to put the denominator of your integral in this form. Look out for quadratics or roots of quadratics in the denominator.

12.1.1 Symmetry: Video Lesson

Suppose f is continuous on $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

More Integrals to Try: Exercise

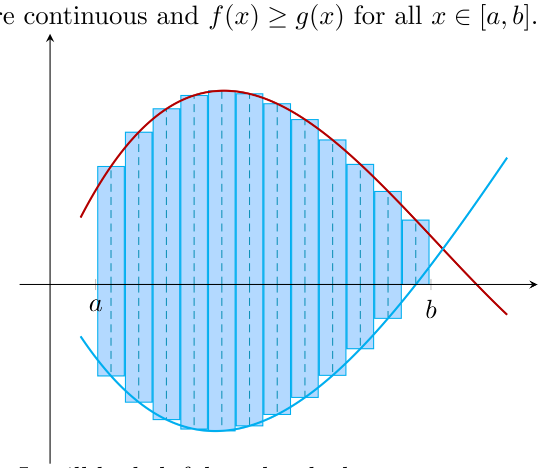
1.	$\int \frac{7e^{6x}}{4+5e^{6x}}dx$	2.	$\int \frac{8^{\ln(x^2)}}{x}dx$
3.	$\int e^{2x}\sqrt{1+4e^x}dx$	4.	$\int_1^2 (1+e^{-x})^2 dx$
5.	$\int \frac{\ln(\sqrt{1+x^2})^x}{1+x^2}dx$	6.	$\int \frac{2^{\ln(1/x)}}{x}dx$
7.	$\int \sqrt{10^{5x}}dx$	8.	$\int_0^{1/2} \frac{x^5}{\sqrt{1-2x^3}}dx$

Answers

1.	$\frac{7}{30}\ln(4+5e^{6x})+C$	2.	$\frac{1}{6\ln 2}8^{\ln(x^2)}+C$
3.	$\frac{(1+4e^x)^{3/2}}{120}(12e^x-2)+C$	4.	$1+2e^{-1}-\frac{3}{2}e^{-2}-\frac{1}{2}e^{-4}$
5.	$\frac{1}{4}\ln^2(1+x^2)+C$	6.	$\frac{-1}{\ln 2}2^{\ln(1/x)}+C$
7.	$\frac{2}{5\ln 10}\sqrt{10^{5x}}+C$	8.	$\frac{1}{9}-\frac{\sqrt{3}}{16}$

12.2 Areas Between Curves

Consider the region S that lies between two curves $y=f(x)$ and $y=g(x)$ from $x=a$ to $x=b$, where f and g are continuous and $f(x)\geq g(x)$ for all $x\in[a,b]$.



We can divide S into n strips of equal width and then approximate the area of the ith strip by a rectangle of base $\Delta x = \frac{b-a}{n}$ and height $f(x_i^*)-g(x_i^*)$, where x_i^* is any sample point in the ith subinterval. Then if we let $n\to\infty$, we get a value for the area A of this region S

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i^*)-g(x_i^*)]\Delta x = \int_a^b [f(x)-g(x)]dx$$

It will be helpful to sketch the two curves so we can see which curve lies on top. If we get these backwards, it will result in a negative area calculation. It may be helpful to express the integral as $A=\int_a^b (y_T-y_B)dx$, where y_T is the top curve and y_B is the bottom curve.

Sometimes we need to put more thought into the bounds for x because the boundaries for the region between the two curves will be defined by where the two curves intersect. Here, we will have to first find the points of intersection of the two curves then see which curve is y_T and which curve is y_B between the points of intersection. The curves may cross over each other at points of intersection and the roles of y_T and y_B will be switched. We can examine the graph to see which lies on top in each interval OR we can test a value for x in each curve between the intersection points and see which gives the larger y value. We will illustrate this in the next example.

Example

Find the area of the region bounded by the curves $f(x)=20-x^2$ and $g(x)=(x-6)^2$. Video Solution

Some regions are better treated by regarding x as a function of y. Then the area between the curves $x=f(y)$

and $x = g(y)$ from $y = c$ to $y = d$ if $f(y) \geq g(y)$ for all $y \in [c, d]$ is given by

$$\int_c^d [f(y) - g(y)]dy$$

In order to produce a positive value for the area we will need to subtract the curve on the left from the curve on the right, so we could write the area formula as

$$\int_c^d (x_R - x_L)dy$$

Again, we may need to look for points of intersection first to see where curves overlap to get bounds for y and determine which has the larger x value on each interval.

Example

Find the area of the region bounded by $x = y^2 - 2y$ and $y = x - 4$. Video Solution

Stewart HW: 5.4#1, 3, 5, 9, 11, 12, 15, 21, 27, 31, 37, 53, 63, 5.5#1, 3, 5, 9, 11, 13, 23, 25, 39, 45, 47, 53, 55, 57, 59, 63, 69, 73 6.1 # 1, 5, 7, 9, 11, 17, 20
Lyryx HW: 6.3.1-6.3.9, 7.1.1-7.1.15 8.2.1-8.2.5, 8.2.9

That's a wrap for MATH 127. In MATH 128, you will continue to explore more advanced techniques of integration. You will also study volumes of solids of revolution and find solutions to differential equations. You will learn about how functions can be expressed as a polynomial with an infinite number of terms. This will help us to understand how calculators and computers are able to approximate function values and definite integrals of more complicated functions.