4.10 Sturm-Liouville Problem

Consider the Sturm-Liouville problem:

$$\begin{cases} Lu + \lambda r u = 0 \quad \text{on} \quad [a,b],\ Lu = (pu')' + qu \\ R_1 u = 0,\ R_2 u = 0,\ R_1 u = \alpha_1 u(a) + \alpha_2 u'(a), R_2 u = \beta_1 u(b) + \beta_2 u'(b)\,, \end{cases}$$

$|\alpha_1| + |\alpha_2| > 0$ and $|\beta_1| + |\beta_2| > 0$, $p, p', q, r \in C[a,b]$, $p(x),\ r(x) > 0$ on $[a,b]$, $D(L) = \{f \in C^2[a,b] :\ R_1 f = 0,\ R_2 f = 0\}$, $L : D(L) \to C[a,b]$.

Theorem 4.16: *If $\lambda = 0$ is not an eigenvalue of L then*

$$L^{-1}v(x) = \int_a^b g(x,y)v(y)dy \quad \textit{where} \quad g \in C([a,b]^2) \quad \textit{and} \quad g(y,x) = g(x,y)\,.$$

Proof: $L[u] = pu'' + p'u' + qu = 0$ has nonzero solutions

$$u_1(x),\ u_2(x) \quad \text{on} \quad [a,b] \quad \text{so that} \quad R_1 u_1 = 0,\ R_2 u_2 = 0$$

Consider the Wronskian $w(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$.

$$w'(x) = -\frac{p'(x)}{p(x)} w(x) \Rightarrow \ln\left|\frac{w(x)}{w(a)}\right| = -\ln\left|\frac{p(x)}{p(a)}\right| \Rightarrow w(x)p(x) = w(a)p(a)$$

$w(a) \neq 0$, since if $w(a) = 0$ then $R_1 u_2 = 0$ and then u_2 is a nonzero solution to $Lu_2 = 0,\ R_1 u_2 = 0,\ R_2 u_2 = 0$, a contradiction. Thus $w(x) \neq 0 \quad \forall x \in [a,b]$ so that u_1 and u_2 are independent solutions. Given $v \in C[a,b]$ we solve $\begin{cases} L[u] = v \\ R_1 u = R_2 u = 0 \end{cases}$ by variation of parameters:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + z(x),\ z(x) = u_1(x)v_1(x) + u_2(x)v_2(x)$$

$$\begin{cases} u_1 v_1' + u_2 v_2' &= 0 \\ u_1' v_1' + u_2' v_2' &= \frac{v}{p} \end{cases} \Rightarrow \begin{cases} v_1' &= -\dfrac{u_2 v}{pw} = \dfrac{-u_2 v}{p(a)w(a)} \\ v_2' &= \dfrac{u_1 v}{pw} = \dfrac{u_1 v}{p(a)w(a)} \end{cases}$$

Choose c_1, c_2 to satisfy the BC's:

$$R_1 u = 0 \Rightarrow c_2 = 0$$

$$R_2 u = 0 \Rightarrow c_1 = \int_a^b \frac{u_2(y)v(y)dy}{p(a)w(a)}$$

$$\begin{aligned}
\Rightarrow u(x) &= u_1(x)\int_a^b \frac{u_2(y)v(y)}{p(a)w(a)}dy + \int_a^x \frac{[-u_1(x)u_2(y)+u_2(x)u_1(y)]}{p(a)w(a)}v(y)dy \\
&= \int_x^b \frac{u_1(x)u_2(y)}{p(a)w(a)}\,v(y)dy + \int_a^x \frac{u_2(x)u_1(y)}{p(a)w(a)}v(y)dy \\
&= \int_a^b g(x,y)v(y)dy,\ g(x,y) = \frac{1}{p(a)w(a)}\cdot\begin{cases} u_1(x)u_2(y), & a \le x \le y \le b \\ u_2(x)u_1(y), & a \le y \le x \le b \end{cases}
\end{aligned}$$

Example 4.11: $Lu = u'', u(0) = u(1) = 0 \Rightarrow p = 1$

$$u_1(x) = x,\ u_2(x) = x-1 \Rightarrow w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 1$$

$$\Rightarrow g(x,y) = \begin{cases} x(y-1), & 0 \le x \le y \le 1 \\ (x-1)y, & 0 \le y \le x \le 1 \end{cases}$$

We now return to the general Sturm-Liouville problem. Suppose that $\lambda = 0$ is not an eigenvalue of L. $Lu + \lambda r u = 0 \Rightarrow u = -\lambda L^{-1}(ru)$.

Let $Tu(x) = -L^{-1}(ru) = -\int_a^b g(x,y)r(y)u(y)dy. \Rightarrow u = \lambda T(u)$.

Let $\mu = \frac{1}{\lambda}$ (for $\lambda \neq 0$) $\rightarrow T(u) = \mu u$.

Let $\langle u, v\rangle_r = \int_a^b u(x)v(x)r(x)dx$, $\|u\|_r = \sqrt{\int_a^b u^2(x)r(x)dx}$ since $0 < r_0 \le r(x) \le r_1$, on $[a,b]$, r_0, r_1 constants,

$\|u\|_r$ is equivalent to $\|u\| = \sqrt{\int\limits_a^b u^2(x)dx}$.

$$\langle Tv, w\rangle_r = -\int\limits_a^b r(x)w(x)\int\limits_a^b g(x,y)r(y)v(y)dydx$$
$$= -\int\limits_a^b r(y)v(y)\int\limits_a^b g(y,x)r(x)w(x)dxdy = \langle v, Tw\rangle_r$$

Thus T is self-adjoint and compact on $(L_2[a,b], \langle\,,\rangle_r)$. By the Spectral Theorem T has eigenfunctions ϕ_i with eigenvalues μ_i, $\mu_i \neq 0 : T\phi_i = \mu_i\phi_i$ or $-L^{-1}(r\phi_i) = \mu_i\phi_i$

$$\phi_i \in L_2[a,b] \Rightarrow \phi_i \in C[a,b] \Rightarrow \phi_i \in D(L) = \{f \in C^2[a,b] : R_1 f = R_2 f = 0\}$$
$$= T(C[a,b])$$

Claim: $\{\phi_i\}$ is an orthonormal basis for $L_2[a,b]$ (equivalently $\mu = 0$ is not an eigenvalue for T).

$$Tf = \sum_{i=1}^{\infty}\langle Tf, \phi_i\rangle\phi_i\,.$$

Let $S = D(L)$. S is dense in $L_2[a,b]$ (with respect to the L_2 norm). Given $h \in L_2[a,b]$ there exists $\{h_n\} \subset S$ so that $h_n \overset{L_2}{\to} h$.

$$h_n = \sum_{i=1}^{\infty}\langle h_n, \phi_i\rangle\phi_i \quad \text{and} \quad \sum_{i=1}^{\infty}\langle h, \phi_i\rangle\phi_i \quad \text{converges to} \quad \overline{h}$$

$$\|h_n - \overline{h}\| = \|\sum_{i=1}^{\infty}\langle h_n - h, \phi_i\rangle\phi_i\| \leq \|h_n - h\|$$

Thus $h_n \to \overline{h}$ so $\overline{h} = h \Rightarrow h = \sum\limits_{i=1}^{\infty}\langle h, \phi_i\rangle\phi_i$.
Therefore $\{\phi_i\}$ is an orthonormal basis for $L_2[a,b]$.
If $\lambda = 0$ is an eigenvalue for L with regard to $R_1 u = R_2 u = 0$ take λ^* not an eigenvalue. Replace $q(x)$ by $\hat{q}(x) = q(x) + \lambda^* r(x)$, $\hat{L}u = (pu')' + \hat{q}u$. $\hat{L}$ has eigenvalues $\hat{\lambda}_n = \lambda_n - \lambda^*$ which are never zero.

Theorem 4.17: *The Sturm-Liouville problem has a set of eigenfunctions $\{\phi_n\}$ which form an orthonormal basis for $L_2[a,b]$.*

More is known: $\lambda_1 < \lambda_2 < \cdots$, $\lim\limits_{n\to\infty}\lambda_n = \infty$, each eigenvalue has multiplicity one.