1 Preliminaries

1.1 Recall Some Important Notions and Results from Real Analysis

This section is based on *Real Analysis and Applications: Theory in Practice* by Davidson and Donsig (a textbook for AMATH/PMATH 331). The electronic version of the book can be downloaded from the UW Library.

Definition 1. *A sequence of real number $\{x_n\}$ is said to converge to $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an integer N_ε such that $|x_n - x| < \varepsilon$ for all $n > N_\varepsilon$.*

Note: Conversely, a sequence of real number $\{x_n\}$ is said to not converge to $x \in \mathbb{R}$ if there exists an $\varepsilon > 0$ such that for every N, there exists an $n > N$ such that $|x_n - x| \geq \varepsilon$.

Definition 2. *A sequence of real number $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists an N_ε such that $|x_n - x_m| < \varepsilon$ for all $n, m > N_\varepsilon$.*

Theorem 1 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Theorem 2 (Completeness of $\mathbb{R}$)**.** *Let $\{x_n\}$ be a sequence of real number. Then $\{x_n\}$ converges if and only if $\{x_n\}$ is a Cauchy sequence.*

Definition 3. *A set $S \subset \mathbb{R}$ is said to be bounded above by b if $x \leq b$ for all $x \in S$.*
A set $S \subset \mathbb{R}$ is said to be bounded below by a if $x \geq a$ for all $x \in S$.

Theorem 3 (Theorem and Definition)**.** *A set $S \subset \mathbb{R}$ which is bounded above has a least upper bound or supremum, written*

$$M = \sup_{x \in S} x,$$

with the properties

1. *If $x \in S$, then $x \leq M$.*
2. *If $c < M$, then there is an $x \in S$ such that $x > c$.*

Example 1. *Consider a real-valued $f : S \to \mathbb{R}$ and assume f has a supremum in S, $M = \sup_{x \in S} f(x)$. By definition of the supremum, there is a sequence $\{x_n\} \subset S$ such that $f(x_n) > M - \dfrac{1}{n}$.*

Theorem 4 (Theorem and Definition)**.** *A set $S \subset \mathbb{R}$ which is bounded below has a greatest lower bound or infimum, written*

$$m = \inf_{x \in S} x,$$

with the properties

1. *If $x \in S$, then $x \geq m$.*

2. *If $c > m$, then there is an $x \in S$ such that $x < c$.*

Note: For a set S, $\max_{x \in S} x$ and $\sup_{x \in S} x$ are not the same. There are sets where the supremum exists but the maximum does not. For example, $S = (0, 1)$.

Definition 4 (Continuous Functions, $\varepsilon - \delta$ Definition)**.** *A function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is continuous at a point $x_0 \in \Omega$ (this automatically means that $f(x_0)$ exists) iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x_0) - f(x)| < \varepsilon \text{ whenever } |x_0 - x| < \delta, \ x \in \Omega.$$

The function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is said to be continuous on Ω iff f is continuous at every point of Ω.

Definition 5 (Sequential Continuity)**.** *A function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is said to be sequentially continuous at a point $x_0 \in \Omega$ iff for every sequence $\{x_n\} \subset \Omega$ converging to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.*

Proposition 1. *A function $f : \mathbb{R} \supset \Omega \to \mathbb{R}$ is continuous at a point $x_0 \in \Omega$ iff it is sequentially continuous at x_0.*

Proposition 2. *A real-valued function that is continuous on a closed and bounded region $\Omega \subset \mathbb{R}$ is bounded, and achieves its supremum and infimum in Ω.*

Example 2. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Since $[a, b]$ is closed and bounded, f is bounded and achieves its supremum and infimum in $[a, b]$. That is there exist $x_1, x_2 \in [a, b]$ such that*

$$f(x_1) = \max_{x \in [a,b]} f(x), \quad f(x_2) = \min_{x \in [a,b]} f(x).$$

Definition 6. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $\Omega \subset \mathbb{R}$. We say that $\{f_n\}$ converges pointwise to a function $f : \Omega \to \mathbb{R}$ if*

$$\lim_{n \to \infty} f_n(x) = f(x), \quad \text{for all } x \in \Omega.$$

That is, for each $x \in \Omega$ and for every $\varepsilon > 0$, there exists $N_{\varepsilon,x}$ (depending on ε and x) such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{whenever} \quad n > N_{\varepsilon,x}.$$

Note:

- Pointwise limit of continuous functions can be discontinuous.
- Limit of integral may not be integral of limit.
- Pointwise limit of discontinuous functions can be continuous.

Definition 7. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued functions on $\Omega \subset \mathbb{R}$. We say that $\{f_n\}$ converges uniformly to a function $f : \Omega \to \mathbb{R}$ if given $\varepsilon > 0$, there exists an integer N_ε (depending on ε) so that*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in \Omega \text{ and for all } n > N_\varepsilon.$$

Note:

- Uniform convergence implies pointwise convergence.
- If $\{f_n\}$ converges pointwise to f, then f is the only potential limit for uniform convergence.
- Let $\{f_n : S \to \mathbb{R}\}$ be a sequence of continuous functions. If $\{f_n\}$ converges uniformly to a function f, then f is continuous.

Lemma 1 (Minkowski's Inequalities)**.** *Let $p \in \mathbb{R}$ and $1 \le p < \infty$.*

1. *(for finite sum) Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

2. *(for infinite sequence) Consider $\ell_p = \{x = (x_1, x_2, \ldots), x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$. Let $x, y \in \ell_p$. Then $x + y \in \ell_p$ and*

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}.$$

3. *(for integrable functions)*

$$\left(\int_a^b |f(t) + g(t)|^p \, dt\right)^{1/p} \le \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} + \left(\int_a^b |g(t)|^p \, dt\right)^{1/p}$$

1.2 Recall Some Important Notions and Results from Linear Algebra

In this section, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 8. *A vector space X over $\mathbb{K}$ is a set X together with an addition, $u + v$, and a scalar multiplication, αu, satisfies the following rules for every $u, v, w \in X$ and $\alpha, \beta \in \mathbb{K}$:*

1. $u + v \in X$
2. $(u + v) + w = u + (v + w)$
3. $u + v = v + u$
4. *There is a vector $0 \in X$, called the zero vector, such that $u + 0 = 0 + u = u$*
5. *For every $u \in X$, there exists $(-u) \in X$ such that $u + (-u) = 0$*
6. $\alpha u \in X$
7. $\alpha(\beta u) = (\alpha\beta)u$

8. $(\alpha + \beta)u = \alpha u + \beta u$

9. $\alpha(u + v) = \alpha u + \alpha v$

10. $1\, u = u$

The elements of a vector space X are called vectors.

Example 3. 1. $\mathbb{R}^n$ *is a vector space over* $\mathbb{R}$; $\mathbb{C}^n$ *is a vector space over* $\mathbb{C}$

2. $X = \{\text{all functions } f : \mathbb{R} \to \mathbb{R}\}$

3. $\ell_p = \{\{x_i\} \subset \mathbb{K} \mid \sum\limits_i |x_i|^p < \infty\}$, *where* $1 \leq p < \infty$.

4. $\ell_\infty = \{\{x_i\} \subset \mathbb{K} \mid \sup_i |x_i| < \infty, \forall n\}$

Definition 9. *Let X be a vector space over $\mathbb{K}$. If $\mathbb{S}$ is a subset of X and $\mathbb{S}$ is a vector space under the same operations as X, then $\mathbb{S}$ is called a subspace of X.*

Lemma 2. *(Subspace Test) If $\mathbb{S}$ is a nonempty set of X such that $u + v \in \mathbb{S}$ and $\alpha u \in \mathbb{S}$ for all $u, v \in \mathbb{S}$ and $c \in \mathbb{K}$ under the operation of X, then $\mathbb{S}$ is a subspace of X.*

Example 4. *Using subspace test, we can verify the following sets are vector spaces over $\mathbb{R}$.*

1. $X = P(x) = \{\text{all univariate polynomials}\}$

2. $X = P_n(x) = \{\text{all univariate polynomials of degree at most } n\}$

3. $X = C[a, b] = \{\text{all continuous functions on } [a, b]\}$

4. $X = C^1[a, b] = \{\text{all continuously differentiable functions on } [a, b]\}$

5. $X = C^\infty[a, b] = \{\text{all infinitely differentiable functions on } [a, b]\}$

6. $X = L_p[a, b] = \{\text{all Lebesgue integrable functions on } [a, b]\} = \{f : [a, b] \to \mathbb{R} \mid \int\limits_a^b |f(x)|^p dx < \infty\}$

7. $X = L_\infty[a, b] = \{\text{all bounded almost everywhere functions on } [a, b]\}$

Definition 10. *Let X be a vector space over $\mathbb{K}$. The vectors $\{u_1, \ldots, u_k\} \subset X$ are called linearly independent if the only solution to $0 = \alpha_1 u_1 + \cdots + \alpha_k u_k$ is the trivial solution $\alpha_1 = \ldots = \alpha_k = 0$.*

If the maximal number of linearly independent vectors in X is $n < \infty$, we say X is an n-dimensional vector space and $\dim X = n$. Any set of n linearly independent vectors in X is called a basis for the vector space X.

We write $\dim X = \infty$ if for each $n = 1, 2, \ldots$, there exist n linearly independent vectors in X. In this case, X is called an infinite dimensional space.

Convention: $\dim\{\vec{0}\} = 0$.

Example 5. $\dim \mathbb{R}^n = n, \quad \dim P_n(x) = n+1, \quad \dim C[a,b] = \infty.$

Lemma 3. *Let X be an n-dimensional vector space and $\{u_1, \dots, u_n\}$ be a basis for X. Then every vector $u \in X$ can be uniquely expressed as a linear combination of $\{u_1, \dots, u_n\}$.*

Definition 11 (Quotient Space)**.** *Let $\mathbb{V}$ be a vector space and $\mathbb{W}$ be a subspace of $\mathbb{V}$. Consider the relation $\sim$ on $\mathbb{V}$:*

$$\text{For } x, y \in \mathbb{V}, \quad x \sim y \Leftrightarrow x - y \in \mathbb{W}.$$

It is easy to verify that relation is an equivalent relation (symmetric, reflexivity, and transitivity). Denote

$$[x] = \{y \in \mathbb{V} \mid x \sim y\}$$

Define the following set

$$\mathbb{V}/\mathbb{W} = \{[x] \mid x \in \mathbb{V}\},$$

with the following operators:

$$[x] + [y] := [x+y], \quad \alpha[x] := [\alpha x],$$

for any $x, y \in \mathbb{V}, \alpha \in \mathbb{K}$. Those operators are well-defined and $\mathbb{V}/\mathbb{W}$ is a vector space. Moreover, if $\dim \mathbb{V} < \infty$, $\dim \mathbb{V}/\mathbb{W} = \dim \mathbb{V} - \dim \mathbb{W}$.

Example 6. *Let $\mathbb{V}$ be the set of all real-valued integrable functions on $[a,b]$ and $\mathbb{W} = \{f \in \mathbb{V} \mid f = 0 a.e.\}$. We can verify that $\mathbb{W}$ is a subspace of $\mathbb{V}$, hence $\mathbb{V}/\mathbb{W}$ is also a vector space. Indeed, $L_1[a,b] = \mathbb{V}/\mathbb{W}$.*

Definition 12. *A map $T : \mathbb{V} \to \mathbb{W}$ between two vector spaces over $\mathbb{K}$ is called a linear operator if it preseves the operations of addition of vectors and multiplication by scalars, i.e.,*

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Denote $\ker T = \{u \in \mathbb{V} \mid T(u) = 0\}$ *Kernel of T, a subspace of $\mathbb{V}$.*
$\operatorname{Im} T = \{Tu \mid u \in \mathbb{V}\}$ *Image (range) of T, a subspace of $\mathbb{W}$.*

Theorem 5. *Let $T : \mathbb{V} \to \mathbb{W}$ be a linear operator between two vector spaces $\mathbb{V}$ and $\mathbb{W}$ over $\mathbb{K}$. Then*

- *T is one-to-one iff $\ker T = \{\vec{0}\}$*
- *T is onto iff $\operatorname{Im} T = \mathbb{W}$*
- *If $\dim \mathbb{V} < \infty$, then* $\dim \ker(T) + \dim \operatorname{Im}(T) = \dim \mathbb{V}.$

2 Normed Linear Spaces

Below are contents in the next few weeks.

2.1 Normed Linear Spaces: Definitions and Examples

2.2 Banach Spaces: Definitions and Examples

2.3 Open and Closed Sets

2.4 Continuity

2.5 The Banach Fixed-Point Theorem and the Iteration Method

2.6 Applications to Ordinary Differential Equations

References:

- Sections 1.2, 1.3, 1.4, 1.5, 1.6, 1.8, 1.9 in *Applied Functional Analysis: Applications to Mathematical Physics*, by E. Zeidler
- Sections 3.1, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7 in *Applied Functional Analysis: Course Notes for AM 731*, by D. Siegel