2 Normed Linear Spaces

2.1 Normed Linear Spaces: Definitions and Examples

Definition 1. *Let X be a real (or complex) vector space. A real-valued function $\|\cdot\| : X \to \mathbb{R}$ is a norm on X if*

1. $\|x\| \geq 0$ *for all* $x \in X$ *(positivity)*
2. $\|x\| = 0$ *if and only if* $x = 0$ *(strict positivity)*
3. $\|\alpha x\| = |\alpha|\|x\|$ *for any scalar* α *and for all* $x \in X$ *(homogeneity)*
4. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$ *(triangle inequality)*

The pair $(X, \|\cdot\|)$ is called a normed linear space.

Example 1. 1. *The following functions are norms on $\mathbb{R}^n$:*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad p \geq 1,$$

and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Proof. For $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, with $p \geq 1$, the first three requirements can be verified from the definition of $\|x\|_p$. The triangle inequality can be verified using the Minkowski's inequality for finite sums. □

2. *The following functions are norms on $X = C[a,b] = \{f : [a,b] \to \mathbb{R}$ continuous on $[a,b]\}$:*

$$\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} \qquad (1 \leq p < \infty)$$

and

$$\|f\|_\infty = \max_{a \leq t \leq b} |f(t)|.$$

Proof. Let $f \in C[a,b]$. Since f is continuous on $[a,b]$, f is integrable on $[a,b]$ and achieves its maximum and minimum in $[a,b]$. Therefore, $\|f\|_p$ and $\|f\|_\infty$ are well-defined.

For $\|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p}$ with $1 \leq p < \infty$,

- Positivity: By the definition of $\|f\|_p$, we have $\|f\|_p \geq 0$.

- Homogeneity: By the definition of $\|f\|_p$, we have

$$\|\alpha f\|_p = \left(\int_a^b |\alpha f(t)|^p \, dt\right)^{1/p} = |\alpha| \left(\int_a^b |f(t)|^p \, dt\right)^{1/p} = |\alpha| \|f\|_p.$$

- Triangle inequality: Using the Minkowski's inequality for integrable functions, we have

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p, \quad \text{for all} \quad f, g \in CC[a,b].$$

- Strict positivity: Prove by contradiction. Suppose there exists $f \in C[a,b]$ with $\|f\|_p = 0$ but $f \not\equiv 0$. That is, there exists $x_0 \in [a,b]$ such that $f(x_0) \neq 0$. Since f is continuous, there exists a subinterval of width δ, $x_0 \ni I \subset [a,b]$ such that $\frac{|f(x_0)|}{2} \geq |f(x) - f(x_0)|$ for every $x \in I$. Since $|f(x) - f(x_0)| \geq |f(x_0)| - |f(x)|$, we have

$$\frac{|f(x_0)|}{2} \geq |f(x) - f(x_0)| \geq |f(x_0)| - |f(x)|, \quad |f(x)| \geq \frac{|f(x_0)|}{2},$$

for all $x \in I$. So,

$$0 = \|f\|_p = \left(\int_a^b |f(t)|^p \, dt\right) \geq \left(\int_I |f(t)|^p \, dt\right)^{1/p} \geq \delta^{1/p} \frac{|f(x_0)|}{2} > 0,$$

a contradiction. Therefore the assumption is wrong. That means if $\|f\|_p = 0$ for some $f \in C[a,b]$, then $f \equiv 0$.

For $\|f\|_\infty = \max_{a \leq t \leq b} |f(t)|$, DIY.
Question: what is the best norm for $C[a,b]$? □

3. *For $1 \leq p < \infty$, the vector space*

$$L_p[a,b] = \{f : [a,b] \to \mathbb{R} \text{ measurable s.t. } \int_a^b |f(x)|^p dx < \infty\}/\sim$$

(where $f \sim g$ iff $f = g$ a.e.) is a normed space, with the norm defined as

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}.$$

Proof. The positive and homogenous properties are obvious. If $f \in L_p[a,b]$ and $\|f\|_p = 0$, then $f = 0$ a.e., which proves the strict positive property. The triangle inequality comes from the Minkowski's inequality for integrable functions. □

More examples:

4. *For* $C^1[a,b]$,

$$\|f\|_{1,\infty} = \max_{a\le t\le b} \{|f(t)|, |f'(t)|\}$$

and

$$\|f\|_{1,2} = \left(\max_{a\le t\le b} |f(t)|^2 + \max_{a\le t\le b} |f'(t)|^2\right)^{1/2}$$

are norms.

5. *For* $1 \le p < \infty$, *the vector space* $\ell_p = \{x = \{x_i\}, x_i \in \mathbb{R} \mid \sum\limits_i |x_i|^p < \infty\}$ *is a normed space, with the norm defined as* $\|x\|_p = \left(\sum\limits_i |x_i|^p\right)^{\frac{1}{p}}$.

6. *The vector space* $\ell_\infty = \{x = \{x_i\}, x_i \in \mathbb{R} \mid \sup_i |x_i| < \infty\}$ *is a normed space, with the norm defined as* $\|x\|_\infty = \sup_i |x_i|$.

Definition 2. *Let* $(X, \|\cdot\|)$ *be a normed linear space. A sequence* $\{x_n\} \subset X$ *is said to converge or to be convergent if there is an* $x \in X$ *such that*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

x *is called the limit of* $\{x_n\}$ *and we write* $\lim\limits_{n\to\infty} x_n = x$.

Proposition 3 (Uniqueness of Limits)**.** *Let* $(X, \|\cdot\|)$ *be a normed linear space. A sequence in* X *converges to at most one point in* X.

Proof. Consider a sequence $\{x_n\}$ in X. If $\{x_n\}$ diverges, the proof is done. Suppose $\{x_n\}$ converges to two elements $x, y \in X$. Then

$$\|x - y\| = \|(x - x_n) + (x_n - y)\| \le \|x - x_n\| + \|x_n - y\| \to 0 \quad \text{as } n \to \infty.$$

Hence $\|x - y\| = 0$, i.e., $x = y$. □