**Proposition 4.** *Let $(X, \|\cdot\|)$ be a normed linear space, $\{x_n\} \subset X$, and $x_n \to x \in X$. Then $\|x_n\| \to \|x\|$.*

*Proof.* Exercise. $\square$

**Definition 3.** *Let $(X, \|\cdot\|)$ be a normed linear space. A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists an $N_\varepsilon$ so that*

$$\|x_n - x_m\| < \varepsilon, \quad \textit{for all} \quad n, m > N_\varepsilon.$$

**Proposition 5.** *Every convergent sequence is a Cauchy sequence.*

*Proof.* Let $\{x_n\} \subset X$ be a convergent sequence. For every $\varepsilon > 0$, there exists $N$ so that

$$\|x_n - x\| < \frac{\varepsilon}{2}, \quad \text{for all} \ \ n > N.$$

Then for $n, m > N$, we have

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x_m - x\| < \varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. $\square$

Conversely, there exist normed linear spaces such that not every Cauchy sequence converges.

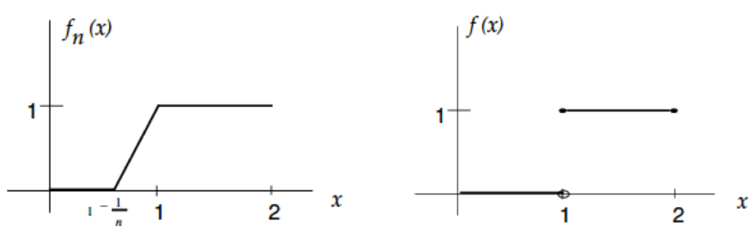
**Example 2.** *Consider the set of all rational numbers $\mathbb{Q}$. The set $\mathbb{Q}$ is a normed linear space under the standard addition $u+v$, the scalar multiplication $\alpha\, u$, and the absolute operator as a norm on $\mathbb{Q}$, $\|u\| = |u|$ $(u, v, \alpha \in \mathbb{Q})$. Consider the following sequence that approximates $\sqrt{2} = 1.4142135...$*

$$x_1 = 1, \quad x_2 = 1.4 = \frac{14}{10}, \quad x_3 = 1.41 = \frac{141}{100}, \ldots$$

*The sequence $\{x_n\}$ converges to $\sqrt{2}$ and is a Cauchy sequence in $\mathbb{Q}$. However, $\sqrt{2} \notin \mathbb{Q}$.*

**Example 3.** *Consider the following sequence of (piecewise linear) functions in $C[0,2]$:*

$$f_n(x) = \begin{cases} 0 & \textit{for} \quad 0 \leq x < 1 - \dfrac{1}{n} \\ 1 & \textit{for} \quad 1 < x \leq 2 \\ 1 + n(x-1) & \textit{for} \quad 1 - \dfrac{1}{n} \leq x \leq 1. \end{cases}$$




*Claim: The sequence $\{f_n\}$ is a Cauchy sequence in $(C[0,2], \|\cdot\|_1)$, but $\{f_n\}$ does not converge in $(C[0,2], \|\cdot\|_1)$.*

*Proof.*

- Claim 1: $\{f_n\}$ is a Cauchy sequence w.r.t. $\|\cdot\|_1$. Indeed, with $m > n$, we have

$$\begin{aligned}\|f_n - f_m\| &= \int_0^2 |f_n(x) - f_m(x)|dx \\ &= \text{area of the triangle formed by } \left(1 - \frac{1}{m}, 0\right); \left(1 - \frac{1}{n}, 0\right); (1,1) \\ &= \frac{\frac{1}{n} - \frac{1}{m}}{2} < \frac{1}{2n} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Suppose $\{f_n\}$ converges to some function $f \in C[0,2]$ w.r.t. $\|\cdot\|_1$, i.e., $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

- Claim 2: The function $f$ must be

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad 1 < x \le 2 \end{cases}$$

**Part 2.1:** Prove that $f(x) = 1$ for all $1 < x \le 2$. Suppose $f(x) \neq 1$ for all $1 < x \le 2$. Then there exists $x_1 \in (1,2]$ such that $f(x_1) \neq 1, f(x_1) - 1 \neq 0$. Since $f - 1 \in C[0,2]$, similar to the argument in Example 1 - part 2, there exists a subinterval $I$ of width $\delta$ such that $x_1 \ni I \subset (1,2]$ such that $|f(x) - 1| \ge \dfrac{|f(x_1) - 1|}{2}$ for all $x \in I$. Then

$$\|f_n - f\|_1 = \int_0^2 |f_n(x) - f(x)|dx \ge \int_I |1 - f(x)|dx \ge \delta\frac{|f(x_1) - 1|}{2}.$$

Therefore, $0 = \lim_{n\to\infty} \|f_n - f\|_1 \ge \delta\dfrac{|f(x_1) - 1|}{2} > 0$, a contradiction. That means $f(x) = 1$ for all $1 < x \le 2$.

**Part 2.2:** Prove that $f(x) = 0$ for all $0 \le x < 1$. (hint: follow the same argument as in Part 2.1).

- Claim 3: The function $f$ is not continuous at $x = 1$ since $\lim_{x\to 1^-} f(x) = 0$ and $\lim_{x\to 1^+} = 1$. Therefore, $f(x) \notin C[0,2]$, a contradiction.

In conclusion, $\{f_n\}$ is a Cauchy sequence in $(C[0,2], \|\cdot\|_1)$, but $\{f_n\}$ does not converge in $(C[0,2], \|\cdot\|_1)$.

□

## 2.2 Banach Spaces: Definitions and Examples

**Definition 1.** *A normed linear space $(X, d)$ is called a Banach space if every Cauchy sequence in $X$ converges (that is, has a limit which is an element of $X$).*
*Banach spaces are also called complete normed spaces.*

Most proofs of completeness are based on the completeness of $\mathbb{R}$.

**Theorem 1.** *$(\mathbb{R}^n, \|\cdot\|_p)$ is a Banach space for $1 \le p \le \infty$.*

*Proof.* Exercise.

**Theorem 2.** *$(C[a,b], \|\cdot\|_1)$ is not a Banach space (See Example 3 Section 2.1)*

□

**Theorem 3.** *$(C[a,b], \|\cdot\|_\infty)$ is a Banach space.*

*Proof.* Let $\{f_n\}$ be a Cauchy sequence in $C[a,b]$ w.r.t. $\|\cdot\|_\infty$. Then for every $\varepsilon > 0$, there exists $N_\varepsilon$ such that for all $n, m > N_\varepsilon$, we have

$$\varepsilon > \|f_n - f_m\|_\infty = \max_{t\in[a,b]} |f_n(t) - f_m(t)|. \tag{1}$$

- Step 1: Show that $f_n$ converges pointwise to some function $f$:

  Fixed $x \in [a,b]$. Then for every $n, m > N_\varepsilon$,

  $$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon.$$

  Therefore, $\{f_n(x)\}_{n\ge1}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{f_n(x)\}_{n\ge1}$ converges. Denote $f(x) := \lim_{n\to\infty} f_n(x)$. In other words, we have constructed a function $f : [a,b] \to \mathbb{R}$ such that $\{f_n\}$ converges pointwise to $f$.

- Step 2: Prove that $\{f_n\}$ converges uniformly to $f$, i.e., $\|f_n - f\|_\infty \to 0$. From the inequality (1), for every $\varepsilon > 0$, there exists $N_\varepsilon$ such that

  $$|f_n(t) - f_m(t)| < \varepsilon, \quad \text{for all } n, m > N_\varepsilon, \quad \text{for all } t \in [a,b].$$

  Now letting $m \to \infty$ and keeping everything else fixed, we get

  $$|f_n(t) - f(t)| \le \varepsilon, \quad \text{for all } n > N_\varepsilon, \quad \text{for all } t \in [a,b].$$

  Therefore $\{f_n\}$ converges uniformly to $f$, $f_n \xrightarrow{\|\cdot\|_\infty} f$. Since the uniform convergence of continuous functions is a continuous function, $f \in C[a,b]$. In conclusion, the Cauchy sequence $\{f_n\}$ converges in $C[a,b]$ w.r.t. the infty norm $\|\cdot\|_\infty$. That completes the proof.

□

**Theorem 4.** *The space $(L_p[a,b], \|\cdot\|_p)$ with $1 \le p < \infty$ is a Banach space.*

*Proof.* See next lecture. □