Lecture 04: Riesz-Fischer Theorem

Lemma 4. *Let $(X, \|\cdot\|)$ be a normed linear space and $\{x_n\}$ be a Cauchy sequence in X. Then there exists a subsequence $\{x_{n_k}\}_k \subset \{x_n\}$ such that*

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}, \quad \textit{for all } k = 1, 2, \dots$$

Proof. Since $\{x_n\}$ is a Cauchy sequence,

- ◇ For $\varepsilon = \frac{1}{2}$, there exists $n_1 > 0$ such that $\|x_n - x_m\| < \frac{1}{2}$ for every $n, m \geq n_1$.
- ◇ For $\varepsilon = \frac{1}{2^2}$, there exists $n_2 > n_1$ such that $\|x_n - x_m\| < \frac{1}{2^2}$ for every $n, m \geq n_2$.
- ◇ For $\varepsilon = \frac{1}{2^3}$, there exists $n_3 > n_2$ such that $\|x_n - x_m\| < \frac{1}{2^3}$ for every $n, m \geq n_3$.

...

We have constructed a subsequence $\{x_{n_k}\}_k$ with

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}, \quad \text{for every} \quad k \geq 1.$$

□

Lemma 5. *Let $(X, \|\cdot\|)$ be a normed linear space and $\{x_n\}$ be a Cauchy sequence in X. If there is a subsequence $\{x_{n_k}\}_k \subset \{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x \in X$, then $\{x_n\}$ also converges to that limit.*

Proof. Pick $\varepsilon > 0$. Since $\lim_{k\to\infty} x_{n_k} = x$, there exists N_1 such that

$$\|x_{n_k} - x\| < \frac{\varepsilon}{2}, \quad \text{for all} \quad k \geq N_1.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists $N_2 > N_1$ such that

$$\|x_n - x_m\| < \frac{\varepsilon}{2}, \quad \text{for all} \quad n, m \geq N_2.$$

Note that $n_{N_2} \geq N_2 > N_1$. For all $n \geq N_2$, we have

$$\|x_n - x\| \leq \|x_{n_{N_2}} - x\| + \|x_{n_{N_2}} - x_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $\lim_{n\to\infty} x_n = x$. □

Recall Some Important Results from Measure Theory

Theorem 4 (Lebesgue Monotone Convergence Theorem)**.** *Assume $\Omega \subset \mathbb{R}^d$ is measurable. If $\{f_n : \Omega \to [0, \infty]\}_n$ is a sequence of nonnegative measurable functions satisfying*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \quad \textit{for a.e.} \quad x \in \Omega,$$

then

$$\lim_{n\to\infty} \int_\Omega f_n(x)\, dx = \int_\Omega \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

Theorem 5 (Lebesgue Dominated Convergence Theorem)**.** *Assume $\Omega \subset \mathbb{R}^d$ is measurable. Let $\{f_n : \Omega \to [-\infty, \infty]\}_n$ be a sequence of measurable functions that converge pointwise for a.e. $x \in \Omega$. If there is a measurable function g such that*

$$|f_n(x)| \leq g(x) \quad \textit{for every } n \textit{ and a.e. } x \in \Omega,$$

then

$$\lim_{n\to\infty} \int\limits_{\Omega} f_n(x)\,dx = \int\limits_{\Omega} \left(\lim_{n\to\infty} f_n(x)\right)\,dx.$$

Recall:

$$L_p[a,b] = \{f : [a,b] \to \mathbb{R} \text{ measurable s.t. } \int\limits_a^b |f(x)|^p dx < \infty\}/W,$$

where $W = \{f : [a,b] \to \mathbb{R} \mid f = 0 \quad a.e.\}$. In practice, we consider $[f] \in L_p[a,b]$ as a function $f : [a,b] \to \mathbb{R}$ with $\int\limits_a^b |f(x)|^p dx < \infty$ and functions that coincides μ-almost everywhere are the same.

Theorem 6 (Riesz-Fischer theorem)**.** *The set $(L_p[a,b], \|\cdot\|_p)$ with $1 \leq p < \infty$ is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $(L_p[a,b], \|\cdot\|_p)$.
By Lemma 4, there is a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k}$, for every $k = 1, 2, \ldots$
By Lemma 5, to prove $\{f_n\}$ converges, it suffices to show that $\{f_{n_k}\}$ converges in $(L_p[a,b], \|\cdot\|_p)$. Consider the following series

$$f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

and

$$|f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

The corresponding partial sums are

$$S_{1,m}(x) = f_{n_1}(x) + \sum_{k=1}^{m} (f_{n_{k+1}}(x) - f_{n_k}(x)) = f_{n_{m+1}}(x),$$

$$S_{2,m}(x) = |f_{n_1}(x)| + \sum_{k=1}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Since $\{S_{2,m}(x)\}$ is an increasing sequence, the limit

$$g(x) := \lim_{m\to\infty} S_{2,m}(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

always exists, where $g(x)$ could be $+\infty$ at some points.

- Step 1: Prove that $g \in L_p[a,b]$.
  The triangle inequality in $L_p[a,b]$ gives

$$\|S_{2,m}\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{m} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{m} \frac{1}{2^k} < \|f_{n_1}\|_p + 1.$$

  Therefore

$$\int_a^b (S_{2,m}(x))^p \, dx = \|S_{2,m}\|_p^p \leq (\|f_{n_1}\|_p + 1)^p \,,$$

  and

$$\lim_{m\to\infty} \int_a^b (S_{2,m}(x))^p \, dx \leq (\|f_{n_1}(x)\|_p + 1)^p < \infty.$$

  On the other hand, since $\{(S_{2,m}(x))^p\}$ is a monotone increasing sequence of nonnegative functions, the Lebesgue monotone convergence theorem implies

$$\lim_{m\to\infty} \int_a^b (S_{2,m}(x))^p \, dx = \int_a^b (\lim_{m\to\infty} (S_{2,m}(x))^p) \, dx = \int_a^b g(x)^p \, dx.$$

  Hence $\int_a^b g(x)^p \, dx < \infty$ and $g \in L_p[a,b]$. It also implies $g(x)$ is finite a.e. in $[a,b]$. In other words, $S_{2,m}(x)$ pointwise converges a.e. in $[a,b]$. Hence $S_{1,m}(x)$ pointwise converges a.e. in $[a,b]$ to a finite value $f(x)$:

$$f(x) := \lim_{m\to\infty} S_{1,m}(x) = \lim_{m\to\infty} f_{n_m}(x).$$

- Step 2: Prove that $f \in L_p[a,b]$.
  Since $|S_{1,m}(x)| \leq S_{2,m}(x) \leq g(x)$, we have $|f(x)| \leq g(x)$. Since $g \in L_p[a,b]$, we conclude that $f \in L_p[a,b]$.

- Step 3: Prove that $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$.
  We have

$$|f_{n_m}(x) - f(x)|^p \leq (2 \max\{|f(x)|, |S_{1,m-1}(x)|\})^p \leq (2g(x))^p.$$

  Since $(2g(x))^p$ is measurable, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{m\to\infty} \int_a^b |f_{n_m}(x) - f(x)|^p \, dx = \int_a^b \left(\lim_{m\to\infty} |f_{n_m}(x) - f(x)|^p \right) dx = 0.$$

  It means $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$.
  Note: we also can prove $f \in L_p[a,b]$ after proving $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$. For $\varepsilon = 1$, there exists N so that $\|f_{n_m} - f\|_p < 1$ for all $m \geq N$. Then

$$\|f\|_p \leq \|f_{n_N} - f\|_p + \|f_{n_N}\|_p < 1 + \|f_{n_N}\|_p < \infty.$$

In conclusion, we have proved that $(L_p[a,b], \|\cdot\|_p)$ is a Banach space. □