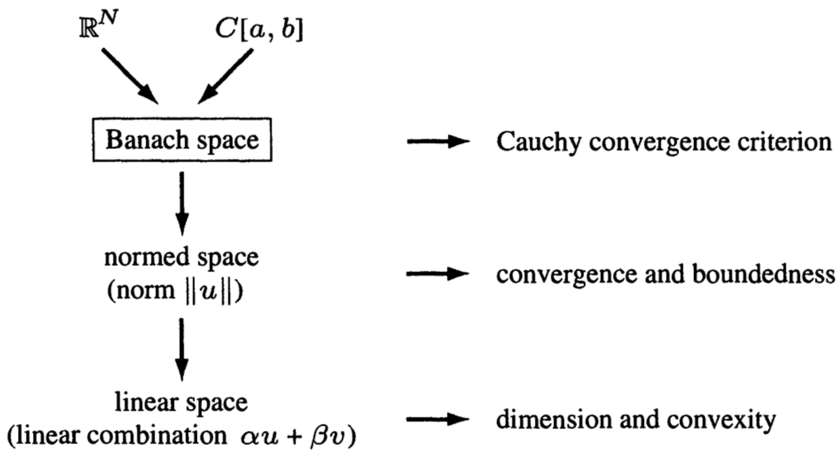

Figure 1: Source: from Zeidler's book

Recap: So far, we have been studied the completeness of the following normed linear spaces.

1. $(\mathbb{R}^n, \|\cdot\|_p)$ (with $1 \le p \le \infty$) is a Banach space.

2. $(C[a,b], \|\cdot\|_\infty)$ is a Banach space. (Proved in class)

3. $(L_p[a,b], \|\cdot\|_p)$ (with $1 \le p < \infty$) is a Banach space. (Proved in class)

   More examples of Banach spaces (Exercises):

4. $(\ell_p, \|\cdot\|_p)$ (with $1 \le p \le \infty$) is a Banach space.

5. $(L_\infty[a,b], \|\cdot\|_\infty)$ is a Banach space, where

   $L_\infty[a,b] := \{f : [a,b] \to \mathbb{R} \mid \text{There exists an } M \text{ such that } |f(x)| \le M \text{ for almost every } x \in [a,b]\ \}/W,$

$$W = \{f : [a,b] \to \mathbb{R} \mid f = 0 \quad a.e.\}$$

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} |f(x)| = \inf\{M \mid |f(x)| \le M \text{ for almost every } x \in [a,b]\}$$

   Some incomplete normed linear spaces:

6. $(C[a,b], \|\cdot\|_1)$ is not a Banach space. (Proved in class)

7. $(C[a,b], \|\cdot\|_2)$ is not a Banach space. (Exercise)

Lecture 05: Open and Closed Sets. Convexity. Banach Fixed-Point Theorem

2.3 Open and Closed Sets

Definition 1. *Let $(X, \|\cdot\|)$ be a normed linear space. Given a point $x_0 \in X$ and a real number $r > 0$, Define the following sets*

$$\begin{aligned} B(x_0, r) &= \{x \in X \mid \|x - x_0\| < r\} && \text{(open ball)} \\ \overline{B(x_0, r)} &= \{x \in X \mid \|x - x_0\| \le r\} && \text{(closed ball)} \\ S(x_0, r) &= \{x \in X \mid \|x - x_0\| = r\} && \text{(sphere)} \end{aligned}$$

In all three cases, x_0 is called the center, and r the radius.

Definition 2. *A subset M of a normed linear space X is said to be open if for every $x_0 \in M$, there exists $r > 0$ such that $B(x_0, r) \subset M$.*

A subset M of a normed linear space X is said to be closed if the situation $\{x_n\} \subset M$, $x_n \to x \in X$ implies $x \in M$.

Proposition 6. *Let $(X, \|\cdot\|)$ be a normed linear space and M be a subset of X. Then M is open if and only if $M^c := X \backslash M$ is closed.*

Proof. ($\Rightarrow$) Suppose M is open, we need to show that M^c is closed. Let $\{x_n\} \subset M^c$ and $x_n \xrightarrow{\|\cdot\|} x \in X$. Assume $x \notin M^c$, then $x \in M$. Since M is open, by definition, there exists $r > 0$ such that $B(x, r) \subset M$. On the other hand, since $x_n \to x$, there exists N_r so that $\|x_n - x\| < r$ for every $n > N_r$. Choose $n_0 = [N_r + 1]$. Then $x_{n_0} \in B(x, r) \subset M$. Therefore, $x_{n_0} \in M \cap M^c = \emptyset$, a contradiction. Hence, the assumption $x \notin M^c$ is wrong, which means $x \in M^c$.

($\Leftarrow$) Suppose M^c is closed, we need to show that M is open. We will prove by contradiction. Assume M is not open. Then there exists $x_0 \in M$ so that for every $r > 0$, $B(x_0, r) \not\subset M$, that is $B(x_0, r) \cap M^c \neq \emptyset$. Let $x_n \in B\left(x_0, \frac{1}{n}\right) \cap M^c$. Since

$$\|x_n - x_0\| < \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty,$$

the sequence $x_n \to x_0$ (due to Squeeze Limit Theorem). Since M^c is closed and $\{x_n\} \subset M^c$, by definition, $x_0 \in M^c$. We have $x_0 \in M \cap M^c = \emptyset$, a contradiction. Therefore, the assumption is wrong and M is open. □

Proposition 7. *Let $(X, \|\cdot\|)$ be a normed linear space, $x_0 \in X$, and $r \in \mathbb{R}_+$. Then $B(x_0, r)$ is open and $\overline{B(x_0, r)}$ is closed.*

Proof. (a). Let $x_1 \in B(x_0, r)$. Then $\|x_1 - x_0\| < r$. Denote $r_1 = r - \|x_1 - x_0\|$.
Claim: $B(x_1, r_1) \subset B(x_0, r)$. Indeed, for any $y \in B(x_1, r_1)$, we have

$$\|y - x_0\| \le \|y - x_1\| + \|x_1 - x_0\| < r_1 + \|x_1 - x_0\| = r,$$

which implies $y \in B(x_0, r)$. Hence $B(x_1, r_1) \subset B(x_0, r)$.

(b). Let $\{x_n\} \subset \overline{B(x_0, r)}$ and $x_n \to x \in X$. Then $\|x_n - x_0\| \leq r$. Using Proposition 4, we have $\|x_n - x_0\| \to \|x - x_0\|$. By Squeeze Limit Theorem, we have $\|x - x_0\| \leq r$. Therefore $x \in \overline{B(x_0, r)}$. □

Theorem 1. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{W}$ is a subspace of X. Then $(\mathbb{W}, \|\cdot\|)$ is a Banach space iff $\mathbb{W}$ is closed.*

Proof. ($\Rightarrow$) Suppose $(\mathbb{W}, \|\cdot\|)$ is a Banach space, $\{x_n\} \subset \mathbb{W}$, and $\lim_{n\to\infty} x_n = x \in X$. Since $\{x_n\}$ is a convergent sequence in X, $\{x_n\}$ is a Cauchy sequence in X. In addition, since $\{x_n\} \subset W$, $\{x_n\}$ is a Cauchy sequence in $\mathbb{W}$. Because $\mathbb{W}$ is a Banach space, there exists $y \in \mathbb{W}$ so that $x_n \to y \in \mathbb{W} \subset X$. By the uniqueness of the limits, $x = y$. Therefore $\mathbb{W}$ is closed.
($\Leftarrow$) Suppose $\mathbb{W}$ is closed. Let $\{x_n\} \subset \mathbb{W}$ is a Cauchy sequence. Since X is a Banach space, $x_n \to x \in X$. Since $\mathbb{W}$ is closed, $x \in \mathbb{W}$. Therefore, $\mathbb{W}$ is a Banach space. □

2.4 Convexity

Definition 1. *The set M in a linear space is called convex iff*

$$u, v \in M \quad \text{and} \quad 0 \leq \alpha \leq 1 \quad \text{imply} \quad \alpha u + (1-\alpha)v \in M.$$

The function $f : M \to \mathbb{R}$ is called convex iff M is convex and

$$f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha) f(v),$$

for all $u, v \in M$ and all $\alpha \in [0, 1]$.

Example 1. *Let X be a normed space, and let $u_0 \in X$, $r \geq 0$ be given Then the closed ball*

$$B = \{u \in X \mid \|u - u_0\| \leq r\}$$

is convex.

Proof. If $u, v \in B$ and $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} \|\alpha u + (1-\alpha)v - u_0\| &= \|\alpha(u - u_0) + (1-\alpha)(v - u_0)\| \\ &\leq \|\alpha(u - u_0)\| + \|(1-\alpha)(v - u_0)\| \\ &\leq \alpha\|u - u_0\| + (1-\alpha)\|v - u_0\| \leq \alpha r + (1-\alpha) r = r. \end{aligned}$$

□

Example 2. *Let $(X, \|\cdot\|)$ be a normed space. The function $f : X \to \mathbb{R}$, $f(u) := \|u\|$ is continuous and convex.*

Proof. Exercise. □

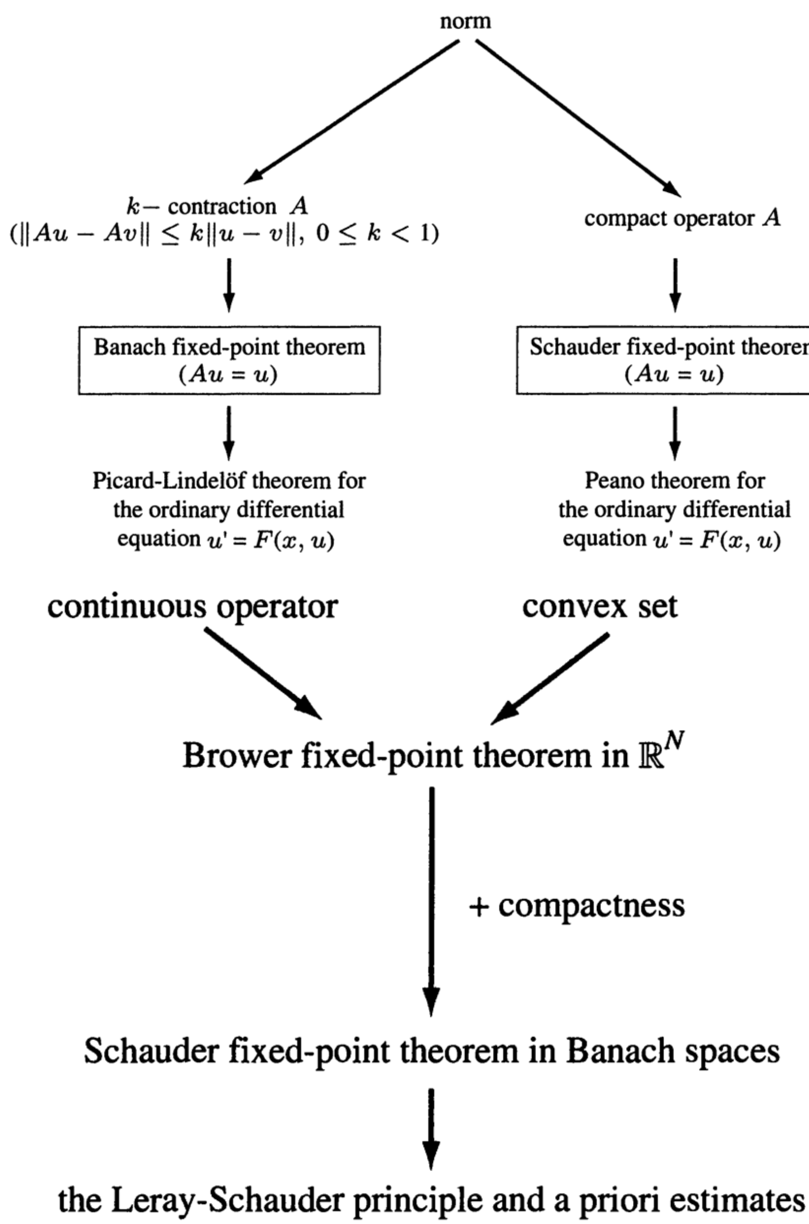

Figure 2: Source: From Zeidler's book.

2.5 The Banach Fixed-Point Theorem and the Iteration Method

Definition 1. *Let M and Y be sets. An operator $A : M \to Y$ associates to each point $u \in M$ a point $v \in Y$, denoted by $v = Au$.*

Example 1. *Let $-\infty < a < b < \infty$ and let the function*

$$F : [a, b] \times \mathbb{R} \to \mathbb{R}$$

be continuous. For each $u \in C[a,b]$, define

$$Au : [a,b] \to \mathbb{R}, \ (Au)(x) := \int_a^x F(t,u(t))dt \quad \text{for all} \ \ x \in [a,b].$$

Since u and F are continuous function, $G(t) = F(t,u(t))$ is also continuous. By the Fundamental Theorem of Calculus, Au is continuous. In conclusion, we have defined an operator from $C[a,b]$ to itself:

$$A : C[a,b] \to C[a,b], \quad (Au)(x) := \int_a^x F(t,u(t))dt \quad \text{for all} \ \ x \in [a,b].$$

Next, we will discuss about the Banach fixed-point theorem. It represents a fundamental convergence theorem for a wide class of iteration methods such as Newton's method. It is also used to prove the existence and uniqueness of solutions to certain ODEs (Picard-Lindelöf Theorem), to integral equations, and to value iteration, policy iteration, and policy evaluation of reinforcement learning.

Problem statement: Given an operator $A : M \to M$, we want to solve the operator equation

$$u = Au, \quad u \in M, \tag{2}$$

by using the iteration method:

$$u_0 \in M, \quad u_{n+1} = Au_n \quad n = 0, 1, \ldots, \tag{3}$$

Each solution of $u = Au$ is called a fixed point of the operator A.

Theorem 1 (Banach Fixed-Point Theorem)**.** *Assume that:*

(i) M is a closed, nonempty set in the Banach space X

(ii) The operator $A : M \to M$ is $k-$contractive, i.e.,

$$\|Au - Av\| \le k\|u - v\| \quad \text{for all } u, v \in M$$

and fixed $k \in [0,1)$.

Then the following hold true:

1. *Existence and uniqueness. The equation $u = Au, \ u \in M$ has exactly one solution $u_* \in M$.*

2. *Convergence of the iteration method. For each given $u_0 \in M$, the sequence $\{u_n\}$ constructed by the iteration method (3) converges to the unique solution u_* of Equation (2).*

3. *Error estimates. For all $n = 0, 1, \ldots$, we have a priori error estimate*

$$\|u_n - u_*\| \le \frac{k^n}{1-k}\|u_1 - u_0\|,$$

and for all $n = 1, 2, \ldots$, we have a posteriori error estimate

$$|u_n - u_*\| \le \frac{k}{1-k}\|u_n - u_{n-1}\|.$$

4. *Rate of convergence. For all $n = 0, 1, \dots$ we have*

$$\|u_{n+1} - u_*\| \le k\|u_n - u_*\|.$$

Proof. 1 & 2. Step 1: Show that $\{u_n\}$ is a Cauchy sequence in X. Then since X is Banach, $\{u_n\}$ to some $u_* \in X$. Since M is closed and $\{x_n\} \subset M$, $u_* \in M$.
Step 1.1: Evaluate

$$\|u_{n+1} - u_n\| = \|Au_n - Au_{n-1}\| \le k\|u_n - u_{n-1}\| \le k^2\|u_{n-1} - u_{n-2}\| \le \cdots \le k^n\|u_1 - u_0\|.$$

Step 1.2: Evaluate

$$\begin{aligned}
\|u_{n+m} - u_n\| &= \|(u_{n+m} - u_{n+m-1}) + \cdots + (u_{n+2} - u_{n+1}) + (u_{n+1} - u_n)\| \\
&\le \|u_{n+m} - u_{n+m-1}\| + \cdots + \|u_{n+1} - u_n\| \\
&\le (k^{n+m-1} + \cdots + k^n)\|u_1 - u_0\| \\
&\le k^n(k^{m-1} + \cdots + k + 1)\|u_1 - u_0\| = k^n\frac{1 - k^m}{1 - k}\|u_1 - u_0\| \\
&\le \frac{k^n}{1 - k}\|u_1 - u_0\|.
\end{aligned}$$

Since $k \in [0, 1)$, $k^n \to 0$ as $n \to \infty$. Therefore the sequence $\{u_n\}$ is Cauchy. Since X is Banach, $\{u_n\}$ to some $u_* \in X$. Also, because M is closed and $\{x_n\} \subset M$, $u_* \in M$.

Step 2: Show that $u_* = Au_*$.
Observe that

$$\|u_{n+1} - Au_*\| = \|Au_n - Au_*\| \le k\|u_n - u_* \xrightarrow{n \to \infty} 0.$$

Therefore, $Au_* = \lim_{n \to \infty} u_{n+1} = u_*$.

Step 3: Uniqueness of the solution: Show that if $u_* = Au_*$ and $v_* = Av_*$ for some $v_* \in M$ then $u_* = v_*$.
We have

$$\|u_* - v_*\| = \|Au_* - Av_*\| \le k\|u_* - v_*\|, \quad (k - 1)\|u_* - v_*\| \ge 0$$

Since $k \in [0, 1)$, this implies $\|u_* - v_*\| = 0$, and hence $u_* = v_*$.

3. From $\|u_{n+m} - u_n\| \le \dfrac{k^n}{1 - k}\|u_1 - u_0\|$, letting $m \to \infty$, we get

$$\|u_* - u_n\| \le \frac{k^n}{1 - k}\|u_1 - u_0\|, \quad \text{for all } n = 0, 1, \dots$$

Notice that

$$\|u_{n+m} - u_n\| \le \|u_{n+m} - u_{n+m-1}\| + \cdots + \|u_{n+1} - u_n\| \le (k^m + \cdots + k)\|u_n - u_{n-1}\| \le \frac{k}{1 - k}\|u_n - u_{n-1}\|$$

Letting $m \to \infty$, we get

$$|u_n - u_*\| \le \frac{k}{1 - k}\|u_n - u_{n-1}\|.$$

4. It comes from

$$\|u_{n+1} - u_*\| = \|Au_n - Au_*\| \le k\|u_n - u_*\|.$$

□

Comments: The priori error estimates can help to determine the maximal number of iterations required to attain a given precision. The posteriori error estimates base on u_n and u_{n+1} to determine the accuracy of the approximation u_{n+1}. Experience shows that a posteriori estimates are better than a priori estimates.