Lecture 06: Applications of the Banach Fixed Point Theorem to ODEs and Integral Equations

2.6 Applications to Ordinary Differential Equations

Given $(x_0, u_0) \in \mathbb{R}^2$, let $F(x, w)$ be a continuous function on a rectangle

$$S = \{(x, w) \in \mathbb{R}^2 : \; |x - x_0| \leq a \text{ and } |w - u_0| \leq b\},$$

and thus bounded on S,

$$|F(x, w)| \leq c, \quad \text{for all } (x, w) \in S.$$

For $0 < h \leq a$, consider the following initial value problem:

$$\begin{cases} u' &= F(x, u), \quad x_0 - h \leq x \leq x_0 + h \\ u(x_0) &= u_0. \end{cases} \tag{4}$$

We are looking for a differentiable function $u : [x_0 - h, x_0 + h] \to \mathbb{R}$ that satisfies Equation (4) and

$$(x, u(x)) \in S \quad \text{for all } x \in [x_0 - h, x_0 + h].$$

Questions: When does the IVP (4) have a solution? Is the solution unique? What is the value of h with respect to a, b, c?

Denote $X := C[x_0 - h, x_0 + h]$ and $M := \{u \in X : \|u - u_0\|_\infty \leq b\}$.
Consider the following integral equation (Picard integral equation)

$$u(x) = u_0 + \int_{x_0}^{x} F(y, u(y))dy, \quad x_0 - h \leq x \leq x_0 + h, \quad u \in M, \tag{5}$$

along with the iteration method

$$u_0(x) = u_0, \quad u_{n+1}(x) = u_0 + \int_{x_0}^{x} F(y, u_n(y))dy, \quad x_0 - h \leq x \leq x_0 + h, \; n = 0, 1, \dots \tag{6}$$

Proposition 8. *Suppose $F(x, u)$ is a continuous function on S. A function $u \in C^1[x_0 - h, x_0 + h]$ is a solution to the IVP (4) on S iff u is a solution to the integral equation (5).*

Proof. ($\Rightarrow$) Suppose $u \in C^1[x_0, x_0 + h]$ is a solution to the IVP. Integrating the ODE w.r.t x shows that the function u is also the solution of (5).
($\Leftarrow$) Suppose u is a solution to the integral equation (5). Then $u(x_0) = u_0$. Also, by the Fundamental theorem of calculus, $u \in C^1[x_0 - h, x_0 + h]$ and $u' = F(x, u(x))$. So u is a solution to the IVP (4). □

Theorem 1 (The Picard-Lindelöf Theorem)**.** *Assume the following:*

1. *The function $F : S \to \mathbb{R}$ is continuous.*

2. *$F(x,u)$ satisfies a Lipschitz condition with respect to u on S, that is, there exists $L \geq 0$ such that*

$$|F(x,u_1) - F(x,u_2)| \leq L|u_1 - u_2|,$$

for all $(x,u_1),(x,u_2) \in S$.

3. *We choose the real number h in such a way that*

$$0 < h \leq a, \quad hc \leq b, \quad hL < 1.$$

Then the following hold true:

(i) *The sequence $\{u_n\}$ constructed by (6) converges to some $u_* \in X$.*

(ii) *The IVP (4) has a unique solution, which is u_* in part (i) .*

(ii) *For $n = 0, 1, \ldots$, we have the following error estimates*

$$\|u_n - u_*\|_\infty \leq k^n(1-k)^{-1}\|u_1 - u_0\|_\infty,$$

$$\|u_{n+1} - u_*\|_\infty \leq k(1-k)^{-1}\|u_{n+1} - u_n\|_\infty,$$

where $k := hL$.

Proof. We know from previous lectures that $(X, \|\cdot\|_\infty)$ is a Banach space and the closed ball $M = \{u \in X : \|u - u_0\|_\infty \leq b\}$ is closed and nonempty.
For each $u \in M$, consider the following operator A

$$Au(x) := u_0 + \int_{x_0}^{x} F(y, u(y))dy, \quad \text{for } x \in [x_0 - h, x_0 + h].$$

Since F and u are continuous functions, by the Fundamental Theorem of Calculus, $Au : [x_0, x_0 + h] \to \mathbb{R}$ is also continuous. Therefore, we get the operator

$$A : M \to X.$$

We will prove that

1. $A : M \to M$.

2. The operator A is k-contractive, where $k = hL$.

Proof of (1): $A : M \to M$. Indeed, let $u \in M$. Then for every $x \in [x_0 - h, x_0 + h]$, we have

$$\left| \int_{x_0}^{x} F(y, u(y))dy \right| \leq |x - x_0| \max_{(y,u) \in S} |F(y,u)| \leq hc \leq b.$$

Therefore

$$\|Au - u_0\|_\infty = \max_{x\in[x_0-h,x_0+h]} \left| \int_{x_0}^{x} F(y,u(y))dy \right| \le b,$$

i.e., $Au \in M$.

Proof of (2): The operator A is k-contractive, where $k = hL$. Indeed, for $u, v \in M$ and for any $x \in [x_0, x_0 + h]$, we have

$$\left| \int_{x_0}^{x} (F(y,u(y)) - F(y,v(y)))\, dy \right| \le \int_{x_0}^{x} |F(y,u(y)) - F(y,v(y))|\, dy \le \int_{x_0}^{x_0+h} |F(y,u(y)) - F(y,v(y))|\, dy$$

$$\le L \int_{x_0}^{x_0+h} |u(y) - v(y)|\, dy \le L\|u - v\|_\infty \int_{x_0}^{x_0+h} dy = hL\|u - v\|_\infty.$$

The same argument holds for $x \in [x_0 - h, x_0]$. Therefore,

$$\|Au - Av\|_\infty = \max_{x\in[x_0-h,x_0+h]} \left| \int_{x_0}^{x} (F(y,u(y)) - F(y,v(y)))\, dy \right| \le hL\|u - v\|_\infty.$$

Since $0 < hL < 1$, by the Banach fixed-point theorem, the integral equation $u = Au$ has a unique solution $u_* \in M$ and the iterative method constructs a sequence $u_n \to u_*$. By Proposition 8, u_* is the unique solution to the IVP. □

Proposition 9. *If* $\dfrac{\partial F}{\partial u}$ *is continuous on* S *then* F *satisfies a Lipschitz condition with respect to* u *on* S.

Proof. Let $u_1, u_2 \in \mathbb{R}$ such that $|u - u_0| \le r$. By the Mean Value Theorem,

$$F(x,u_1) - F(x,u_2) = \frac{\partial F}{\partial u}(x,c)(u_1 - u_2),$$

for some c between u_1 and u_2. Since $\dfrac{\partial F}{\partial u}$ is continuous on S and S is compact, we have

$$L = \max_{(x,u)\in S} \left| \frac{\partial F}{\partial u} \right| < \infty,$$

and

$$|F(x,u_1) - F(x,u_2)| = \left| \frac{\partial F}{\partial u}(x,c) \right| |u_1 - u_2| \le L|u_1 - u_2|,$$

for all $(x,u_1), (x,u_2) \in S$. □

Example 1. *Consider the initial value problem*

$$u' = 1 + u^2, \quad u(0) = 0.$$

What is the maximum of h *that the P-L theory works?*

Proof. Consider $S = \{(x,w) \in \mathbb{R}^2 : |x| \leq a, |w| \leq b\}$. The function $F(x,u) = 1 + u^2$ is continuous on S with $c = \max_{(x,u)\in S} |F(x,u)| = 1 + b^2$, and F satisfies a Lipschitz condition w.r.t u on S with $L = \max_{(x,u)\in S} \left|\frac{\partial F}{\partial u}\right| = 2 \max_{(x,u)\in S} |u| = 2b$. The P-L theorem requires

$$0 < h \leq a, \quad h \leq \frac{b}{b^2+1}, \quad h < \frac{1}{2b}$$

We have $\max_b \min\left\{\frac{b}{b^2+1}, \frac{1}{2b}\right\} = \frac{1}{2}$ when $b = 1$. Therefore the P-L theorem gives a solution on $[-h, h]$ for any $0 < h < \frac{1}{2}$.

On the other hand, we can find the closed form of the IVP: $u = \tan x$ on a larger interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. There is a room to improve the P-L theorem! (see Remarks 1.1. below). □

Remark 1. *(From "Supplementary Remarks to IVP" by E. Vrscay – attached here in the next 4 pages).*

1. *The restriction on h can often be softened so that the existence of a unique solution to the IVP can be established over a larger interval.*

2. *However, h might not be arbitrarily large. There are IVPs that their solutions blow up at finite time. For example $u' = u^2,\ u(0) = u_0 > 0$. The solution of this IVP is $u(x) = \frac{u_0}{1-u_0x}$ for $0 \leq x < \frac{1}{u_0}$, which blows up at $x = \frac{1}{u_0}$.*

3. *The iterative method in the P-L Theorem provide estimates u_n to the solution u_* of the IVP.*

4. *Consider $u' = u^{1/3},\ u(0) = u_0 = 0,\ x \in [0,T]$. Then*

$$\frac{du}{u^{1/3}} = dx, \quad \int_0^x \frac{du}{u^{1/3}} = \int_0^x dy, \quad u(x)^{2/3} - u_0^{2/3} = \frac{2}{3}x.$$

So $u(x) = \left(u_0^{2/3} + \frac{2}{3}x\right)^{3/2} = \left(\frac{2}{3}x\right)^{3/2}$ is a solution. There is another solution $u(x) \equiv 0$. The reason why we can not apply the Picard-Lindelöf theorem is because the function F is not Lipschitz.

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Supplementary remarks to Section 2.7, "Initial Value Problem"

Recall that the solution to the initial value problem

$$\begin{aligned} y' &= f(t,y), \quad t_0 \le t \le a, \\ y(t_0) &= y_0, \end{aligned} \tag{1}$$

also satisfies the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds, \qquad t_0 \le t \le a, \tag{2}$$

which is obtained by simple integration of the DE. We see that the solution $y(t)$ to the IVP in (1) is the fixed point of the integral operator T, defined as follows: $h = Tg$, where

$$h(t) = (Tg)(t) = y_0 + \int_{t_0}^{t} f(s, g(s))\, ds. \tag{3}$$

The integral operator T is often called the *Picard integral operator*.

We saw that under certain conditions on f, the operator T is contractive on a complete metric space (S_a, d_∞) of functions supported on $[t_0, a]$. You will also recall that some work had to be done to obtain an estimate of a, based on the properties of f:

1. First, $a \le t_0 + \dfrac{b}{M}$, where b can be prescribed and $M = \max_R |f(t,y)|$.
2. Then $a < t_0 + \dfrac{1}{L}$, where L is the Lipschitz constant for the second argument of f.

In what follows, we show that these restrictions can often be "softened" so that the existence of a unique solution to Eq. (1) can be established over a larger interval. This is done by showing that the operator T is "eventually contractive".

Now return to the following fundamental set of identities involving the Picard integral operator:

$$\begin{aligned} |(Tg)(t) - (Th)(t)| &= \left| \int_{t_0}^{t} [f(s,g(s)) - f(s,h(s))]\, ds \right| \\ &\le \int_{t_0}^{t} |f(s,g(s)) - f(s,h(s))|\, ds \\ &\le L \int_{t_0}^{t} |g(s) - h(s)|\, ds \\ &\le L d_\infty(g,h) \int_{t_0}^{t} ds \\ &= L d_\infty(g,h)(t - t_0) \end{aligned} \tag{4}$$

Note that we have not integrated out to the value a, but rather are keeping the right-hand side as a function of t. This will be useful below.

We replace g and h in the above relations with Tg and Th, respectively:

$$\begin{aligned} |(T^2g)(t) - (T^2h)(t)| &= \left| \int_{t_0}^{t} [f(s,Tg(s)) - f(s,Th(s))]\, ds \right| \\ &\le \int_{t_0}^{t} |f(s,Tg(s)) - f(s,Th(s))|\, ds \\ &\le L \int_{t_0}^{t} |Tg(s) - Th(s)|\, ds. \end{aligned} \tag{5}$$

Now insert the final result from (4):

$$\begin{aligned} |(T^2g)(t) - (T^2h)(t)| &\leq L^2 d_\infty(g,h) \int_{t_0}^{t} (s - t_0)\, ds \\ &= \frac{1}{2} L^2 d_\infty(g,h)(t - t_0)^2 \end{aligned} \tag{6}$$

We can repeat this procedure for T^2g and T^2h, etc., to arrive at the following result, which can be proved by induction:

$$|(T^n g)(t) - (T^n h)(t)| \leq \frac{1}{n!} L^n (t - t_0)^n d_\infty(g,h), \qquad t \in [t_0, a]. \tag{7}$$

Taking the supremum over $t \in [t_0, a]$ on both sides, we obtain the important result,

$$d_\infty(T^n g, T^n h) \leq \frac{1}{n!} L^n (a - t_0)^n d_\infty(g,h). \tag{8}$$

For sufficiently large n, say $n = p$,

$$\frac{1}{p!} L^p (a - t_0)^p < 1, \tag{9}$$

which implies that the operator $U = T^p$ for some $p > 1$ is a contraction. We say that T is "eventually contractive." From Banach's Contraction Mapping Theorem, it follows that U has a unique fixed point $\bar{u}(t) \in S_a$. We now state the following result, which will be left as an exercise:

The fixed point $\bar{u}$ of T^p is also a unique fixed point of T.

This implies that $\bar{u}$ is the unique solution to the IVP in Eq. (1).

Note that the above analysis can also be extended over to the "other side" of t_0, i.e., an interval $[c, t_0]$, provided that suitable conditions on f be met.

A final comment: From Eq. (9), one might be tempted to conclude that the outer endpoint a of the interval $[t_0, a]$, over which the unique solution exists, can be made as large as possible: Given any $a > 0$, we can find a $p > 0$ which guarantees that the inequality in Eq. (9) is true. This could pose a problem, since we know that some solutions "blow up" in finite time. Consider the following IVP, which is solved in the Addendum to Page 15, posted on the Course webpage.

$$\frac{dy}{dt} = y^2\,, \quad y(0) = y_0 > 0\,, \tag{10}$$

The function $f(t,y) = y^2$ is Lipschitz in the variable y, so a unique solution exists. It is given by

$$y(t) = \frac{y_0}{1 - y_0 t}, \quad 0 \leq t < \frac{1}{y_0}\,. \tag{11}$$

Nevertheless, the solution $y(t)$ "blows up" at $t = \dfrac{1}{y_0}$.

If we return to the proof of the existence-uniqueness to initial value problems using the Contraction Mapping Theorem, we see that, in fact, no such problem exists. The proof rests on the assumption that the solution is an element of a closed ball of continuous functions – the space S_a on Page 15 of the Course Notes. These functions are necessarily bounded. As such, the endpoint a may not be arbitrarily large – it depends on the function $f(t,y)$ on the RHS of the IVP. It, i.e, a, probably won't have to be as small as the value determined in the proof given in the Course Notes. But finding larger values could be a tricky procedure, involving some kind of "juggling", along with the knowledge that the Picard operator T is eventually contractive.

Picard method of successive approximation

Finally, the contractivity of the T (or T^p) operator is the basis for the *Picard method of successive substitution/approximation* or, simply, "*Picard's method*", that provides estimates to the solution of the IVP in Eq. (1). Often, these estimates are in the form of power series about the point t_0 (which is often zero). Picard's method is often discussed in undergraduate courses in ODEs. As such, it is treated in many texts devoted to ODEs and will not be discussed in great detail here. A excellent discussion of both theoretical and practical aspects of this method is to be found in the book, *Differential Equations with Applications and Historical Notes*, by G.F. Simmons (McGraw-Hill).

Briefly, we start with a function $u_0(t)$ that will be the "seed" of the iteration procedure. It is often most convenient to start with the constant function $u_0(t) = y_0$. We then construct the iteration sequence

$$u_{n+1} = Tu_n \,, \tag{12}$$

which becomes

$$u_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, u_n(s))\, ds, \quad n = 0, 1, \cdots . \tag{13}$$

From the contractivity (or eventual contractivity) of the Picard integral operator T (over an appropriate interval), it follows that the sequence of functions $\{u_n\}$ will converge uniformly to the solution $y(t)$ to the IVP in Eq. (1) (over an appropriate interval).

Let us now illustrate with a simple example. Consider the following IVP,

$$\frac{dy}{dt} = ay, \qquad y(0) = y_0, \tag{14}$$

where a and y_0 are arbitrary, nonzero real numbers. For convenience we have set $t_0 = 0$. Of course, we know that the solution to this IVP is

$$y(t) = y_0 e^{at} \,, \tag{15}$$

but we'll pretend, for the moment, that we don' know it.

The solution of this IVP must satisfy the equivalent integral equation,

$$y(t) = y_0 + \int_0^t ay(s)\, ds \,, \tag{16}$$

which is the fixed point equation $y = Ty$, where T denotes the Picard integral operator associated with the IVP in Eq. (14). Just as a check, we differentiate both sides with respect to t:

$$\begin{aligned} y'(t) &= \frac{d}{dt}\left[\int_0^t ay(s)\, ds\right] \\ &= ay(t) \,. \end{aligned} \tag{17}$$

Furthermore, if we set $t = t_0 = 0$ in Eq. (14), we obtain

$$\begin{aligned} y(0) &= y_0 + \int_0^0 ay(s)\, ds \\ &= y_0 \,. \end{aligned} \tag{18}$$

Thus, the IVP in (14) is satisfied.

Let us now perform the Picard method of successive substitution associated with this IVP. As mentioned above, it is convenient to start with the constant function,

$$u_0(t) = y_0, \tag{19}$$

as the "seed" for the iteration procedure. Then

$$\begin{aligned} u_1(t) &= y_0 + \int_0^t au_0(s)\; ds \qquad (20) \\ &= y_0 + \int_0^t ay_0\; ds \\ &= y_0 + ay_0 t \\ &= y_0[1 + at]. \end{aligned}$$

Now repeat this procedure:

$$\begin{aligned} u_2(t) &= y_0 + \int_0^t a u_1(s)\, ds \\ &= y_0 + \int_0^t a y_0 (1 + as)\, ds \\ &= y_0 + a y_0 t + y_0 \frac{1}{2}(at)^2 \\ &= y_0 [1 + at + \frac{1}{2}(at)^2]. \end{aligned} \tag{21}$$

One can conjecture, and in fact prove by induction, that

$$u_n(t) = y_0[1 + at + \cdots + \frac{1}{n!}(at)^n], \qquad n \geq 0, \tag{22}$$

which is the nth degree Taylor polynomial $P_n(t)$ to the solution $y(t) = y_0 e^{at}$. As you will recall from MATH 128, for each $t \in \mathbf{R}$, these Taylor polynomials are partial sums of the infinite Taylor series expansion of the function $y(t)$. As such, we see that the sequence of functions $\{u_n\}$ converges to the solution. A little more work will show that the convergence is uniform over closed subintervals that include the point $t_0 = 0$.

Earlier, we commented that it was convenient to start the Picard method with the constant function $u_0(t) = y_0$. But we don't have to. We can, in fact, start with any function that satisfies the initial condition $u_0(0)y_0$. For example, let us consider

$$u_0(t) = y_0 \cos t\,. \tag{23}$$

Then

$$\begin{aligned} u_1(t) &= y_0 + \int_0^t a u_0(s)\, ds \\ &= y_0 + \int_0^t a y_0 \cos s\, ds \\ &= y_0 + a y_0 \sin s]_0^t \\ &= y_0[1 + a \sin t]. \end{aligned} \tag{24}$$

Once again:

$$\begin{aligned} u_2(t) &= y_0 + \int_0^t a u_1(s)\, ds \\ &= y_0 + \int_0^t a y_0 [1 + a \sin s]\, ds \\ &= y_0 + a y_0 t - a^2 y_0 \cos s]_0^t \\ &= y_0[1 + at - a^2 \cos t + a^2]. \end{aligned} \tag{25}$$

It is perhaps not obvious that these functions are "getting closer" to the solution $y(t) = y_0 e^{at}$. But it is not too hard to show (Exercise) that the Taylor series expansions of $u_1(t)$ and $u_2(t)$ agree, respectively, to the first two and three terms of the Taylor series expansion of $y(t)$.