Lecture 07: Continuity. Compactness. Equivalent Norms.

2.7 Continuity

Definition 1. *Let X and Y be normed linear spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $f : M \subset X \to Y$.*

- *f is continuous at $x_0 \in M$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(x_0)\| < \varepsilon$ for all x so that $\|x - x_0\| < \delta$.*
- *f is continuous on M if f is continuous at all $x_0 \in M$.*
- *f is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in M$ so that $\|x - y\| < \delta$. (Note δ does not depend either on x or y).*

Proposition 10. *Let X and Y be normed linear spaces over $\mathbb{K}$ and $f : M \subset X \to Y$. Then f is continuous at $x \in M$ if and only if for every sequence $\{x_n\}$ in M,*

$$\lim_{n\to\infty} x_n = x \quad \textit{implies} \quad \lim_{n\to\infty} f(x_n) = f(x).$$

Proof. Exercise. □

Proposition 11. *Let $f : X \to Y, g : Y \to Z$, where X, Y, Z are normed linear spaces. If f is continuous at $a \in X$ and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.*

Proof. Exercise. □

2.8 Compactness

Definition 1. *Let S be a set in a normed linear space X.*

- *S is called relatively compact iff each sequence $\{u_n\}$ in S has a convergent subsequence $u_{n_k} \to u \in X$ as $k \to \infty$.*
- *S is called compact iff each sequence $\{u_n\}$ in S has a convergent subsequence $u_{n_k} \to u \in S$ as $k \to \infty$.*
- *S is called bounded iff there is a number $r \geq 0$ such that $\|u\| \leq r$ for all $u \in S$.*

Proposition 12. *Let S be a set in a normed linear space X. Then*

1. *The set S is compact iff it is relatively compact and closed.*
2. *If S is relatively compact, then S is bounded.*
3. *If S is compact, then S is closed and bounded. The reverse might not be true.*

Proof. 1. Exercise.

2. Suppose S is relatively compact but S is not bounded. Then, there exists a sequence $\{u_n\} \subset S$ such that $\|u_n\| \geq n$ for all n. Since S is relatively compact, there exists a convergent subsequence $\{u_{n_k}\}_k$. Therefore, $\{u_{n_k}\}_k$ is bounded. On the other hand, $|u_{n_k}| \geq n_k \geq k$, a contradiction.

3. Combining (1) and (2), we have the conclusion that if S is compact, then S is closed and bounded. Below is a counter example, where the reverse might not be true.

Example 1. *In $(\ell_2, \|\cdot\|_2)$, consider*

$$\overline{B_1(0)} := \{x = (x_1, x_2, \dots) : \|x\|_2 \leq 1\}.$$

The closed ball $\overline{B_1(0)}$ is closed and bounded, but $\overline{B_1(0)}$ is not compact. Indeed, consider the following sequence in $\overline{B_1(0)}$:

$$e_k = (0, \dots, 0, 1, 0, \dots), \quad \text{where the } k\text{th position of } e_k \text{ is 1 and other positions are 0's}, \quad k = 1, 2, \dots$$

Since $\|e_k - e_j\|_2 = \sqrt{2}$ for every $k \neq j$, the sequence $\{e_k\}_k \subset \overline{B_1(0)}$ has no convergent subsequences since no subsequence can be a Cauchy sequence. Therefore, $\overline{B_1(0)}$ is not compact.

□

Theorem 1. *Let X and Y be normed linear spaces and $T : X \to Y$ be a continuous mapping. Then the image of a compact subset S of X under T is compact.*

Proof. Let $\{y_n\}$ be a sequence in $T(S)$. Then $y_n = T(x_n)$ for some $x_n \in S$. Since S is compact, there is a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x_* \in S$. Since T is continuous, $y_{n_k} = T(x_{n_k}) \to T(x_*) \in T(S)$ as $k \to \infty$. Therefore $T(S)$ is compact. □

An important consequence is the following theorem, which is a generalization of the Extreme Value Theorem for continuous functions over bounded and closed intervals on $\mathbb{R}$.

Theorem 2 (The Weeirstrass Theorem)**.** *Let $T : S \to \mathbb{R}$ be a continuous function on the compact, nonempty subset S of a normed linear space. Then T has a minimum and maximum on S.*

Proof. By Theorem 1, $T(S)$ is compact in $\mathbb{R}$. Therefore, $T(S)$ is closed and bounded. Therefore, $\alpha = \inf_{x\in S} T(x)$ is finite. By the definition of the infimum, there exists a sequence $\{x_n\}$ in S such that $\lim_{n\to\infty} T(x_n) = \alpha$. Since $T(S)$ is closed, $\lim_{n\to\infty} T(x_n) \in T(S)$, i.e., $\alpha \in T(S)$. Thus T has a minimum on S. The same argument can be used to show T has a maximum on S. □

Note:

- Image of a closed set under a continuous mapping might not be closed. For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \exp^x$ and $S = (-\infty, 0] \subset \mathbb{R}$. The function f is continuous on $\mathbb{R}$, S is a closed subset in $\mathbb{R}$ but $f(S) = (0, 1]$ is not a closed subset of $\mathbb{R}$.

- Image of a bounded set under a continuous mapping might not be bounded. For example, $f : (0,1) \to \mathbb{R}$, $f(x) = \dfrac{1}{x}$ and $S = (0,1)$ is a bounded set but $f(S) = (0, \infty)$ is not bounded.

Now, we will present some compactness critera for a set in a normed linear space.

2.8.1 Compactness in Finite-Dimensional Normed Linear Spaces

Next, we recall a theorem in real analysis.

Theorem 3 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Using Bolzano-Weierstrass Theorem, we have the following result.

Theorem 4. *In $(\mathbb{K}^n, \|\cdot\|_\infty)$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, a subset $S \subset \mathbb{K}^n$ is compact if and only if S is closed and bounded.*

Proof. Case $\mathbb{K} = \mathbb{R}$. It is sufficient to prove that if S is bounded in $\mathbb{R}^n$, then S is relatively compact. Consider a sequence in S:

$$\{u_m = (u_{m,1}, \ldots, u_{m,n})\}_m \subset S.$$

Since S is bounded, there is a constant $M > 0$ such that

$$M \geq \|u_m\|_\infty \geq |u_{m,k}|, \quad \text{for all} \quad k = 1, 2, \ldots, n, \quad \text{and} \quad m = 1, 2, \ldots.$$

The real sequence $\{u_{m,1}\}_m$ is bounded, so by the Bolzano-Weierstrass theorem, there is a subsequence $\{u_m^{(1)}\}$ of $\{u_m\}$ such that $\{u_{m,1}^{(1)}\}$ converges.
By the Bolzano-Weierstrass theorem, there is a subsequence $\{u_m^{(2)}\}$ of $\{u_m^{(1)}\}$ such that $\{u_{m,2}^{(2)}\}$ converges. Thus $\{u_{m,1}^{(2)}\}$ and $\{u_{m,2}^{(2)}\}$ converge.
Repeating this process n times, we have constructed a subsequence $\{u_m^{(n)}\}_m$ of $\{u_m\}_m$ such that $\{u_{m,k}^{(n)}\}_m$ converges for all $k = 1, 2, \ldots, n$. Using the $\varepsilon - N_\varepsilon$ definition of convergent sequences, we can easily verify that $\{u_m^{(n)}\}_n$ converges in $\mathbb{R}^n$. Therefore, S is relatively compact.

Case $\mathbb{K} = \mathbb{C}$. (Sketch of the proof): Any $u \in \mathbb{C}^n$ can be written as $u = v + iw$, where $v, w \in \mathbb{R}^n$. Use $\|u\|_\infty \geq \|v\|_\infty$ and $\|u\|_\infty \geq \|w\|_\infty$.

□

Definition 2. *Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X are called equivalent iff there are positive numbers $\alpha, \beta > 0$ such that*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1, \quad \textit{for all} \ \ x \in X.$$