## Lecture 08: Compact Sets. Compact Operators.

**Theorem 5.** *Two norms on a finite-dimensional linear space $X$ over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) are always equivalent.*

*Sketch of the proof.* If $\dim X = 0$, any norm on $X$ is the zero function. Therefore, all norms on $X$ are equivalent.
Let $0 < n = \dim X$ and $\|\cdot\|$ is a norm on $X$. Suppose $\{e_1, \dots, e_n\}$ is a basis for $X$. For each $x \in X$, there is a unique tuple $\alpha \in \mathbb{K}^n$ such that

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n.$$

Define $\|x\|_\infty := \|\alpha\|_\infty = \max\limits_{1 \le j \le n} |\alpha_j|$.

- Step 1: Prove that $\|\cdot\|_\infty : X \to \mathbb{R}$, $\left\| \sum\limits_{1 \le j \le n} \alpha_j e_j \right\|_\infty := \max\limits_{1 \le j \le n} |\alpha_j|$ is a norm on $X$. (Exercise).

  Set $S = \{\alpha \in \mathbb{K}^n : \|\alpha\|_\infty = 1\}$. Then $S$ is closed and bounded in $\mathbb{K}^n$ (Exercise).
  Therefore, $S$ is compact. Consider a function

$$f : S \subset \mathbb{K}^n \to \mathbb{R},\ f(\alpha) := \left\| \sum_{k=1}^n \alpha_k e_k \right\|.$$

- Step 2: Prove that $f$ is a continuous function. (Exercise). Hint: Show that

$$|f(\alpha) - f(\beta)| \le \|\alpha - \beta\|_\infty \sum_{k=1}^n \|e_k\|, \quad \text{for all } \alpha, \beta \in \mathbb{K}^n.$$

- Step 3: Prove $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.
  Applying the Weeirstrass theorem for continuous functions on the compact nonempty subset of $S$, we conclude that $f : S \to \mathbb{R}$ has a maximum and minimum on $S$. Let

$$A = \min_{\alpha \in S} f(\alpha) = \min_{\alpha \in \mathbb{R}^n} \left\{ \left\| \sum_{k=1}^n \alpha_k e_k \right\| \ \ s.t.\ \max_{1 \le j \le n} |\alpha_j| = 1 \right\}$$

  and

$$B = \max_{\alpha \in S} f(\alpha) = \max_{\alpha \in \mathbb{R}^n} \left\{ \left\| \sum_{k=1}^n \alpha_k e_k \right\| \ \ s.t.\ \max_{1 \le j \le n} |\alpha_j| = 1 \right\}$$

  Note that A and B are constants that depend on the norm $\|\cdot\|$ of $X$. Also, since $0 \notin S$, so $f(\alpha) > 0$ for all $\alpha \in S$. So $0 < A \le B$ and

$$A \le f(\beta) = \left\| \sum_{k=1}^n \beta_k e_k \right\| \le B, \quad \text{for all} \ \ \beta \in S.$$

  The above inequalities can be rewritten as

$$A \le \|y\| \le B \quad \text{for all} \quad y \in X \text{ with } \|y\|_\infty = 1.$$

For any $x \in X - \{0\}$, let $z = \dfrac{x}{\|x\|_\infty} \in X$. Then $\|z\|_\infty = 1$, so

$$A \le \|z\| \le B, \quad A \le \frac{\|x\|}{\|x\|_\infty} \le B, \quad A\|x\|_\infty \le \|x\| \le B\|x\|_\infty.$$

That inequality also holds for $x = 0$. So we have

$$A\|x\|_\infty \le \|x\| \le B\|x\|_\infty \quad \text{for all } x \in X$$

- Step 4: Prove that any two norms on $X$ are equivalent. Let $\|\cdot\|_{(2)}$ be another norm on $X$. Then there exist positive constants $A_2, B_2$ so that

  $$A_2\|x\|_\infty \le \|x\|_{(2)} \le B_2\|x\|_\infty \quad \text{for all } x \in X$$

  So for every $x \in X$, we have

  $$\frac{A}{B_2}\|x\|_{(2)} \le \|x\| \le \frac{B}{A_2}\|x\|_{(2)}$$

□

**Corollary 1.** *All norms on $\mathbb{R}^n$ are equivalent.*

**Theorem 6.** *In a finite dimensional normed linear space, any subset $M$ is compact iff $M$ is closed and bounded.*

*Proof.* Assignment 2. □

Note: In Assignment 2, we also prove a useful result: All finite dimensional normed spaces are Banach spaces.

### 2.8.2 Compactness in Infinite-Dimensional Normed Linear Spaces

Now we present without proof compactness criteria for some infinite dimensional normed spaces: $(C[a,b], \|\cdot\|_\infty)$ and $(L_1[a,b], \|\cdot\|_1)$ (see Zeidler's book page 35; See Oden and Demkowicz's book page 339-341).

**Theorem 7** (The Arzela-Ascoli Theorem)**.** *Consider the normed linear space $(C[a,b], \|\cdot\|_\infty)$ where $-\infty < a < b < \infty$. Suppose we are given a set $S$ in $C[a,b]$ such that*

1. *$S$ is bounded.*
2. *$S$ is equicontinuous, i.e., for each $\varepsilon > 0$, there is a $\delta > 0$ such that*

   $$|x - y| < \delta \quad \textit{and} \quad u \in S \quad \textit{imply} \quad |u(x) - u(y)| \le \varepsilon.$$

*Then $S$ is a relatively compact subset of $C[a,b]$.*

So, for $(C[a,b], \|\cdot\|_\infty)$, we have:

$$\text{compact sets} = \text{closed} + \text{bounded} + \text{equicontinuous sets}.$$

**Theorem 8** (Frechét - Kolmogorov Theorem). *A subset $\mathcal{F} \subset (L_p(\mathbb{R}), \|\cdot\|_p)$, $1 \le p < \infty$, is relatively compact in $L_p(\mathbb{R})$ iff the following conditions hold:*

1. *$\mathcal{F}$ is bounded, i.e., there exists an $M > 0$ such that $\|f\|_p \le M$ for every $f \in \mathcal{F}$.*

2. *For each $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$|t| < \delta \quad \textit{and} \quad f \in \mathcal{F} \quad \textit{imply} \quad \int_{\mathbb{R}} |f(t+s) - f(s)|^p ds \le \varepsilon.$$

3. *$\lim\limits_{n\to\infty} \int_{|s|>n} |f(s)|^p \, ds = 0$ for every $f \in \mathcal{F}$.*

Below is another useful compactness criteria (see Zeidler's book pages 38-39 for the proof).

**Theorem 9** (Finite $\varepsilon$-net). *Let $S$ be a nonempty set in the Banach space $X$. Then the following two statements are equivalent:*

*(i) $S$ is relatively compact.*

*(ii) $S$ has a finite $\varepsilon$-net ; that is, for each $\varepsilon > 0$, there exists a finite number of points $v_1, \ldots, v_N \in S$ such that*

$$\min_{1\le k\le N} \|u - v_k\| \le \varepsilon \quad \textit{for all} \;\; u \in S.$$

*In other words, $S \subset \bigcup\limits_{k=1}^{N} B(v_k, \varepsilon) \subset X$.*

Note: The smallest integer $N$ such that $S$ can be covered by $N$ $\varepsilon-$ balls is called the covering number $\mathcal{N}(S, \|\cdot\|, \varepsilon)$. For example, when $S$ is a subset of the unit ball in $(\mathbb{R}^n, \|\cdot\|)$,

$$\mathcal{N}(S, \|\cdot\|, \varepsilon) \le \left(1 + \frac{2}{\varepsilon}\right)^n,$$

See, for example, "A Mathematical Introduction to Compressive Sensing" by Foucart and Rauhut, page 577.

Next, we will study a useful operator, called compact operator, to generalize classical results for operator equations in finite-dimensional normed spaces to infinite-dimensional normed spaces.

### 2.8.3 Compact Operators

**Definition 3.** *Let $X$ and $Y$ be normed space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The operator $A : X \to Y$ is called a compact operator iff*

1. *$A$ is continuous, and*

2. *$A$ transforms bounded sets into relatively compact sets.*

**Example 2.**

*Let $X = Y = C[0,1]$ endowed with the $\|\cdot\|_\infty$ norm, consider the integral operator $A : C[0,1] \to C[0,1]$, where for every $u \in C[0,1]$, define*

$$Au(x) := \int_0^1 K(x,y)u(y)\,dy \quad \text{for all} \;\; x \in [0,1],$$

*where $K(x,y)$ is continuous on the square $[0,1]^2$. We shall show that $A$ is compact.*

*Since $K(x,y)$ is continuous on $[0,1]^2$, there exists a constant $M$ such that*

$$|K(x,y)| \le M \quad \text{for all} \quad (x,y) \in [0,1]^2.$$

- **Step 1:** *It is clear that $A$* **is well-defined** *(i.e., $Au \in C[0,1]$ for all $u \in C[0,1]$) since both $K(x,y)$ and $u(y)$ are continuous functions.*

- **Step 2: Show that $A$ is continuous.** *For any $u, v \in X$, we have*

$$\|Au - Av\|_\infty = \max_{x\in[0,1]} \left| \int_0^1 K(x,y)(u(y)-v(y))\,dy \right| \le \max_{x\in[0,1]} \int_0^1 |K(x,y)(u(y)-v(y))|\;dy \le M\|u-v\|_\infty.$$

*Therefore, for every $\varepsilon > 0$, pick $\delta = \dfrac{\varepsilon}{M}$, then whenever $u, v \in X$ with $\|u - v\|_\infty < \delta$, we have $\|Au - Av\|_\infty < \varepsilon$. Therefore, $A$ is continuous.*

*Suppose $S$ is a bounded set of functions of $C[0,1]$. Then there is $r > 0$ such that $\|u\|_\infty \le r$ for all $u \in S$. We will show that $A(S) \subset C[0,1]$ is relatively compact.*

- **Step 3: Show that $A(S)$ is bounded.** *For any $u \in S$, we have*

$$\|Au\|_\infty = \max_{x\in[0,1]} \left| \int_0^1 K(x,y)u(y)\,dy \right| \le Mr,$$

*therefore, $A(S)$ is bounded.*

- **Step 4: Show that $A(S)$ is equicontinuous.**
*Since $[0,1]^2$ is compact and $K$ is continuous on $[0,1]^2$, $K(x,y)$ is uniformly continuous. (Prove this!) Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|K(x_1,y) - K(x_2,y)| < \frac{\varepsilon}{r}, \quad \text{whenever} \;\; |x_1 - x_2| < \delta.$$

*Then for any $x_1, x_2 \in [0,1]$ with $|x_1 - x_2| < \delta$ and for any $u \in S$, we have*

$$|Au(x_1) - Au(x_2)| = \left| \int_0^1 (K(x_1,y) - K(x_2,y))u(y) \right| \le \frac{\varepsilon}{r} r = \varepsilon.$$

*Hence $A(S)$ is equicontinuous. So by the Arzela-Ascoli Theorem, $A(S)$ is a relatively compact set in $C[0,1]$.*
*In conclusion, $A$ is a compact operator.*

**Example 3.** *Let $X$ be an infinite dimensional Banach space, such as $(C[0,1], \|\cdot\|_\infty)$ or $(L_1[0,1], \|\cdot\|_1)$. Consider the identity operator $A : X \to X$, $A(x) = x$. $A$ is continuous, $\overline{B(0,1)}$ is bounded but $A(\overline{B(0,1)}) = \overline{B(0,1)}$ is not a relatively compact set in $X$ ( Assignment 2). Therefore, the identity is not a compact operator in this case.*

**Theorem 10** (Approximation Theorem for Compact Operators)**.** *Let $A : S \subset X \to Y$ be a compact operator, where $X$ and $Y$ are Banach spaces over $\mathbb{K}$ and $S$ is a bounded nonempty subset of $X$. Then for every $n = 1, 2, \ldots$, there exists a continuous operator $A_n : S \to Y$ such that*

$$\sup_{u \in S} \|Au - A_n u\| \leq \frac{1}{n}, \quad \dim(span\, A_n(S)) < \infty, \quad and \quad A_n(S) \subset co(A(S)).$$

Recall: For a set $B$ in a linear space $X$, $co(B)$ is the convex hull of $B$, $span\, B$ is the spanning set of $B$.

*Sketch of the Proof.*

- Since $A$ is compact, and $S$ is bounded, $A(S)$ is relatively compact. Using the finite $\varepsilon$-net theorem, for every $n = 1, 2, \ldots$, there exists a finite $\frac{1}{2n}$-net for $A(S)$. That is, there are elements $u_1, \ldots, u_N \in A(S)$ such that

$$\min_{1 \leq k \leq N} \|Au - u_k\| \leq \frac{1}{2n}, \quad \text{for all } u \in S.$$

- Define the Schauder operator on $S$:

$$A_n u := \frac{\sum_{k=1}^{N} a_k(u) u_k}{\sum_{k=1}^{N} a_k(u)}, \quad \text{for all } u \in S,$$

where

$$a_k : S \to \mathbb{R}, \quad a_k(u) := \max\left\{\frac{1}{n} - \|Au - u_k\|, 0\right\}, \quad k = 1, \ldots, N.$$

**Claim:** $a_k : S \to \mathbb{R}$ is continuous and for each $u \in S$, $a_k(u)$ do not all vanish simultaneously. Therefore $A_n : S \to Y$ is well-defined and continuous.

- Show that $\|Au - A_n u\| \leq \frac{1}{n}$.

□