## Lecture 09: Schauder Fixed-Point Theorem and Applications to ODEs

**Theorem 10** (Approximation Theorem for Compact Operators)**.** *Let $A : S \subset X \to Y$ be a compact operator, where $X$ and $Y$ are Banach spaces over $\mathbb{K}$ and $S$ is a bounded nonempty subset of $X$. Then for every $n = 1, 2, \ldots$, there exist a finite dimensional subspace $Y_n$ of $Y$ and a continuous operator $A_n : S \to Y_n$ such that*

$$\sup_{u \in S} \|Au - A_n u\| \leq \frac{1}{n} \quad \textit{and} \quad A_n(S) \subset co(A(S)).$$

Recall: For a set $B$ in a linear space $X$, $co(B)$ is the convex hull of $B$, $span\, B$ is the spanning set of $B$.

The idea of the proof is to use the finite $\varepsilon$-net for the set $A(S)$ in the Banach space $Y$ and use those centers to define the operator $A_n$ as a linear combination of those $Au_k$. The coefficients are chosen carefully to achieve the approximation of $\dfrac{1}{n}$.

*Sketch of the Proof.*

- Since $A$ is compact, and $S$ is bounded, $A(S)$ is relatively compact. Using the finite $\varepsilon$-net theorem, for every $n = 1, 2, \ldots$, there exists a finite $\dfrac{1}{2n}$-net for $A(S)$. That is, there are elements $Au_1, \ldots, Au_N \in A(S)$ (i.e., $u_1, \ldots, u_N \in S$) such that

$$\min_{1 \leq k \leq N} \|Au - Au_k\| \leq \frac{1}{2n}, \quad \text{for all } u \in S. \tag{7}$$

- Define the Schauder operator $A_n : S \to Y$,

$$A_n u := \frac{\sum_{k=1}^{N} a_k(u)\, Au_k}{\sum_{k=1}^{N} a_k(u)}, \quad \text{for all } u \in S, \tag{8}$$

  where

$$a_k : S \to \mathbb{R}, \quad a_k(u) := \max\left\{\frac{1}{n} - \|Au - Au_k\|, 0\right\}, \quad k = 1, \ldots, N.$$

  **Claim 1:** $A_n : S \to Y$ is well-defined and continuous. First, $a_k$ are nonnegative functions and because of Equation (7), for every $u \in S$, there is $k \in [1, N]$ such that $a_k(u) > 0$. Therefore, $A_n$ is well-defined. For each $k$, the function $a_k$ is continuous because $a_k$ is the composition of continuous functions:

$$a_k : u \longmapsto (Au - Au_k) \longmapsto \|Au - Au_k\| \longmapsto \frac{1}{n} - \|Au - Au_k\| \longmapsto \max\left\{\frac{1}{n} - \|Au - Au_k\|, 0\right\}.$$

  Therefore, $A_n$ is a continuous function on $S$. From Equation (8), we also have

$$A_n(S) \subset co(Au_1, \ldots, Au_N) \subset Y_n = Span(Au_1, \ldots, Au_N), \quad \dim Y_n < \infty$$

$$A_n(S) \subset co(Au_1, \ldots, Au_N) \subset co(A(S)).$$

**Claim 2:** Show that $\|Au - A_n u\| \leq \dfrac{1}{n}$ for any $u \in S$. Indeed, we have

$$\|Au - A_n u\| = \frac{\left\| \sum_{k=1}^{N} a_k(u)\left(Au - Au_k\right) \right\|}{\sum_{k=1}^{N} a_k(u)} \leq \frac{\sum_{k=1}^{N} a_k(u)\|Au - Au_k\|}{\sum_{k=1}^{N} a_k(u)}$$

Due to the construction of $a_k$, for any $k = 1, 2, \ldots$, we have

$$a_k(u)\|Au - Au_k\| \leq \frac{1}{n} a_k(u).$$

Hence $\|Au - A_n u\| \leq \dfrac{1}{n}$.

□

## 2.9 The Brower and Schauder Fixed-Point Theorems

Rephrased from Zeidlers'book: The Brower Fixed-Point Theorem is one of the most important existence principles in mathematics. It has interesting applications to game theory, mathematical economics, and numerical mathematics. Further important existence principles in mathematics are the Hahn-Banach theorem, the Weierstrass existence theorem for minima, and the Baire category theorem. The Schauder Fixed Point Theorem is an extension of the Brower Fixed Point Theorem. We state (without proof) the Brower Fixed-Point Theorem.

**Theorem 1** (Brower Fixed Point Theorem - Version 1)**.** *Any continuous map of a closed ball in* $\mathbb{R}^n$ *into itself must have a fixed point.*

**Example 1.** *A continuous function* $f : [a, b] \to [a, b]$ *has a fixed point* $x \in [a, b]$.

Below is another variant of the Brower Fixed-Point Theorem (in Zeidler's book).

**Theorem 2** (Brower Fixed Point Theorem - Version 2)**.** *Let* $(X, \|\cdot\|)$ *be a finite-dimensional normed space and* $S \subset X$ *is compact, convex, and nonempty. Any continuous operator* $A : S \to S$ *has at least one fixed point.*

**Example 2** (Counter Examples)**.** *The following counter examples show the essentials of each assumption in the Brower Fixed-Point Theorem (version 2).*

- $S = [0, 1]$ *compact, convex and nonempty, but* $A : S \to S$ *not continuous and the graph* $y = A(x)$ *does not cross the diagonal* $y = x$*. No fixed point.*
- $S = \mathbb{R}$ *and* $A : S \to S, A(x) = x + 1$*.* $A$ *is continuous,* $S$ *is convex, nonempty, but not compact. No fixed point.*
- *Let* $S$ *be a closed annulus and* $A : S \to S$ *is a rotation of the annulus around the center. A proper rotation is fixed-point free. In this case,* $S$ *is compact, nonempty but not convex.*

**Theorem 3** (Schauder Fixed Point Theorem - Version 1)**.** *Let* $(X, \|\cdot\|)$ *be a Banach space over* $\mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$*) and* $S \subset X$ *is closed, bounded, convex, and nonempty. Any compact operator* $A : S \to S$ *has at least one fixed point.*

The idea here is to find a fixed point for each approximation operator. Then using the compactness of the operator $A$ to show that the limit of the convergent subsequence is the fixed point of $A$.

*Proof.* From the approximation theorem for compact operators, for every $n = 1, 2, \ldots$, there exists a finite dimensional subspace $X_n$ of $X$ and a continuous operator $A_n : S \to X_n$ such that $A_n(S) \subset co(A(S))$ and

$$\|Au - A_n u\| \leq \frac{1}{n} \quad \text{for all} \quad u \in S.$$

Let $S_n = X_n \cap S$.

- **Step 1: Show that** $A_n|_{S_n} : S_n \to S_n$ **and** $S_n$ **is a compact and convex set of** $X$**.** Therefore, we can apply the Brower fixed point theorem.

  **Step 1.1: Show that** $A_n|_{S_n} : S_n \to S_n$
  Indeed,

  $$A_n(S) \subset co(A(S)) \subset co(S) \subset S,$$

  where the first inclusion comes from the construction of $A_n$, the second one is because $A : S \to S$, and the third one is derived from the convexity of $S$.

  Therefore, $A_n|_{S_n} : S_n \to S_n$.

  **Step 1.2: Show that** $S_n$ **is a compact and convex set of** $X$**.**

  - $S$ is bounded, so $S_n$ is bounded.
  - Since $X_n$ is a finite dimensional subspace of $X$, $X_n$ is a closed subset of $X$. Since the intersection of two closed subsets of $X$ is a closed subset of $X$, $S_n$ is closed.
  - Since $X_n$ is a finite dimensional space and $S_n \subset X_n$ is closed and bounded, $S_n$ must be a compact set.
  - Since $S$ and $X_n$ are convex, $S_n$ is convex.

  By the Brower fixed-point theorem, the operator $A_n : S_n \to S_n$ has a fixed point $u_n$, i.e.,

  $$A_n u_n = u_n, \quad u_n \in S_n, \quad \text{for all} \quad n = 1, 2, \ldots$$

- **Step 2: Show that** $\{u_n\}$ **and** $\{Au_n\}$ **have convergent subsequences and the limit is the fixed-point of** $A$**.**

  Since $u_n \in S_n \subset S$ and $S$ is bounded, the sequence $\{u_n\}$ is bounded. Since $A$ is compact, $\{Au_n\}_n$ is relatively compact in $X$. Therefore, there is a subsequence $\{Au_{n_k}\}_k$ of $\{Au_n\}$ such that

  $$\lim_{k\to\infty} Au_{n_k} = v \in X.$$

Since $Au_{n_k} \in S$ and $S$ is closed, $v \in S$. Moreover,

$$\|v - u_{n_k}\| \le \|v - Au_{n_k}\| + \|Au_{n_k} - u_{n_k}\| = \|v - Au_{n_k}\| + \|Au_{n_k} - A_{n_k}u_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Hence $u_{n_k} \to v$ as $k \to \infty$. Since $A$ is continuous, $Au_{n_k} \to Av$. Therefore, $Av = v$.

□

Since a continuous operator on a compact set is always a compact operator, the Schauder fixed point theorem - version 1 yields the Schauder fixed point theorem - version 2.

**Theorem 4** (Schauder Fixed Point Theorem - Version 2)**.** *Let $(X, \|\cdot\|)$ be a Banach space and $S \subset X$ is compact, convex, and nonempty. Any continuous operator $A : S \to S$ has at least one fixed point.*

### 2.10 Applications to Ordinary Differential Equations

**Theorem 1** (The Peano Theorem)**.** *Given $(x_0, u_0) \in \mathbb{R}^2$, let $F(x, w)$ be a real-valued continuous function on a rectangle*

$$S = \{(x, w) \in \mathbb{R}^2 : \ |x - x_0| \le a \text{ and } |w - u_0| \le b\},$$

*Denote $c = \max_{(x,w)\in S} |F(x, w)|$. Then for $0 < h \le a$ and $hc \le b$, the following initial value problem*

$$\begin{cases} u' &= F(x, u), \quad x_0 - h \le x \le x_0 + h \\ u(x_0) &= u_0. \end{cases} \tag{9}$$

*has at least one solution.*

*Proof.* Denote $X := C[x_0 - h, x_0 + h]$ and $M := \{u \in X : \|u - u_0\|_\infty \le b\}$.
For each $u \in M$, consider the following operator $A$

$$Au(x) := u_0 + \int_{x_0}^{x} F(y, u(y))dy, \quad \text{for } x \in [x_0 - h, x_0 + h].$$

Similar to the part of the Picard-Lindelöf theorem, we have $A : M \to M$. Next, we will prove that $A$ is continuous and $A(M)$ is bounded and equicontinuous. Since $A(M) \subset M$, the set $A(M)$ is bounded. The continuous of $A$ and the equicontinuous of $A(M)$ come from the following inequality:

$$|Au(x) - Au(z)| = \left| \int_z^x F(y, u(y))\, dy \right| \le c|z - x|.$$

By the Arzela Ascoli Theorem, the set $A(M)$ is relatively compact in $X$. Since $M$ is bounded, this implies $A : M \to M$ is a compact operator. Moreover, the closed ball $M$ is closed, bounded, convex, and nonempty. By the Schauder fixed point theorem, the equation

$$Au = u, u \in M$$

has a solution $u_* \in M$. Differentiating the integral equation with respect to $x$, we see that $u_*$ is also a solution of the IVP (9). □