Lecture 10: Bounded Linear Operators.

2.11 Bounded Linear Operator

Recall: Let X and Y be linear spaces over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). The operator $L : X \to Y$ is called linear if for every $u, v \in X$ and $\alpha, \beta \in \mathbb{K}$, we have

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv.$$

Definition 1. *Let X and Y be normed linear spaces. A linear operator $L : X \to Y$ is called a bounded linear operator if there exists a positive constant $c > 0$ such that*

$$\|Lx\|_Y \leq c\|x\|_X, \quad \text{for all } x \in X.$$

Note: We often write $\|x\|$ and $\|Lx\|$ instead of $\|x\|_X$ and $\|Lx\|_Y$.

Proposition 13. *Let $L : X \to Y$ be a linear operator where X and Y are normed spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Then the following statements are equivalent:*

1. *L is continuous at 0.*
2. *L is continuous on X.*
3. *There is a number $c > 0$ such that $\|Lx\| \leq c$ for all $x \in X$ with $\|x\| \leq 1$.*
4. *There is a number $c > 0$ such that $\|Lx\| \leq c\|x\|$ for all $x \in X$.*

Proof. ($1 \Rightarrow 2$). Let $x \in X$ and suppose $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = x$. Then $\lim_{n\to\infty}(x_n - x) = 0$. Since L is continuous at 0, we have

$$\lim_{n\to\infty} L(x_n - x) = L(0).$$

Since L is linear, $L(0) = 0$ and $L(x_n - x) = L(x_n) - L(x)$, for all $n \in \mathbb{N}$. Therefore,

$$0 = L(0) = \lim_{n\to\infty} L(x_n - x) = \lim_{n\to\infty}(L(x_n) - L(x)) = \lim_{n\to\infty} L(x_n) - L(x).$$

Hence $\lim_{n\to\infty} L(x_n) = L(x)$, which means L is continuous at $x \in X$, for any $x \in X$. That completes the proof.

($2 \Rightarrow 3$).Suppose (3) is not true. Then there exists a sequence $\{x_n\} \subset X$ such that

$$\|x_n\| \leq 1 \quad \text{and} \quad \|L(x_n)\| \geq n, \quad \text{for all } n = 1, 2, \ldots.$$

Let $w_n = n^{-1}x_n$, then

$$\|w_n\| \leq \frac{1}{n} \quad \text{and} \quad \|Lw_n\| = \left\|L\left(n^{-1}x_n\right)\right\| = n^{-1}\|L(x_n)\| \geq 1 \quad \text{for all } n = 1, 2, \ldots$$

So $\lim_{n\to\infty} \|w_n\| = 0$ and $\lim_{n\to\infty} w_n = 0$. Since L is continuous at 0, we have $\lim_{n\to\infty} L(w_n) = L(0) = 0$, a contradiction with $\|Lw_n\| \geq 1$.

$(3 \to 4)$. If $x = 0$, then $\|L(0)\| = 0 \leq c\|0\|$.
If $x \neq 0$, let $z = \dfrac{x}{\|x\|}$. Then $\|z\| = 1$, so $c \geq \|Lz\| = \dfrac{\|Lx\|}{\|x\|}$. Therefore, $c\|x\| \geq \|Lx\|$.
In both cases, we have $\|Lx\| \leq c\|x\|$, for all $x \in X$.

$(4 \to 1)$. Given $\varepsilon > 0$. Choose $\delta = \varepsilon/c$. Then when $x \in X$ with $\|x| < \delta$, we have

$$\|Lx\| \leq c\|x\| < c\delta < \varepsilon.$$

□

So for linear operators between normed linear spaces, boundedness is equivalent to continuity.

Definition 2. *For a bounded linear operator $L : X \to Y$ where X and Y are normed linear spaces, define the operator norm*

$$\|L\| := \sup_{v \in X, \|v\| \leq 1} \|Lv\| < \infty$$

Proposition 14. *Let $L : X \to Y$ be a bounded linear operator where X and Y are normed linear spaces. Then*

1. *$\|Lu\| \leq \|L\|\,\|u\|$, for all $u \in X$.*

2. *If there is a constant $C > 0$ such that $\|Lu\| \leq C\|u\|$ for all $u \in X$, then $\|L\| \leq C$.*

3. *If $X \neq \{0\}$, then*
$$\|L\| = \sup_{v \in X, \|v\| \leq 1} \|Lv\| = \sup_{v \in X, \|v\| = 1} \|Lv\| = \sup_{v \in X, v \neq 0} \frac{\|Lv\|}{\|v\|}$$

Proposition 15 (Bounded Linear Operators Between Finite Dimensional Normed Spaces)**.** *Let X and Y be finite-dimensional normed spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) with $\dim X = N$ and $\dim Y = M$ where $N, M \geq 1$. Then any linear operator $L : X \to Y$ is bounded.*

Sketch of the Proof. Let $\{e_1, \ldots, e_N\}$ and $\{f_1, \ldots, f_M\}$ be a basis in X and Y, respectively. Suppose

$$L(e_n) = \sum_{m=1}^{M} a_{mn} f_m, \quad n = 1, \ldots, N.$$

Any $x \in X$ can be written as $x = \sum_{n=1}^{N} c_n e_n$, for some $c_1, \ldots, c_N \in \mathbb{K}$. Then

$$L\left(\sum_{n=1}^{N} c_n e_n\right) = \sum_{n=1}^{N} c_n L(e_n) = \sum_{n=1}^{N} c_n \sum_{m=1}^{M} a_{mn} f_m = \sum_{m=1}^{M} \left(\sum_{n=1}^{N} a_{mn} c_n\right) f_m$$

Recall that we have proved in previous lectures that

$$\left\|\sum_{n=1}^{N} c_n e_n\right\| := \max_{1 \leq n \leq N} |c_n|$$

is a norm on the finite dimensional normed space X.

- Show that $\|Lx\|_\infty \le \|x\|_\infty \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$.
- Using the property that any two norms on a finite dimensional normed linear spaces are equivalent, show that there is a constant $C > 0$ such that $\|Lx\| \le C\|x\|$ for all $x \in X$.

□

Example 1. *Consider a linear operator $L : \mathbb{R}^N \to \mathbb{R}^M,\ L(x) := Ax$ (matrix multiplication), where A is a matrix of real entries of size $M \times N$.*

1. *If we use the $\|\cdot\|_\infty$ norm for both $\mathbb{R}^N$ and $\mathbb{R}^M$, then $\|L\| = \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$.*
2. *If we use the $\|\cdot\|_1$ norm for both $\mathbb{R}^N$ and $\mathbb{R}^M$, then $\|L\| = \max_{1\le n\le N} \sum_{m=1}^{M} |a_{mn}|$.*
3. *If we use the $\|\cdot\|_2$ norm for both $\mathbb{R}^N$ and $\mathbb{R}^M$, then $\|L\| = \sqrt{\rho(A^TA)}$, where $\rho(B)$ is the maximum of the magnitude of the eigenvalues of the square matrix B.*

Proof. (1). For any $x \in \mathbb{R}^N$, then for any $1 \le m \le M$, we have

$$|(Lx)_m| = \left|\sum_{n=1}^{N} a_{mn}x_n\right| \le \sum_{n=1}^{N} |a_{mn}||x_n| \le \|x\|_\infty \sum_{n=1}^{N} |a_{mn}| \le \|x\|_\infty \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|.$$

Therefore,

$$\|Lx\|_\infty = \max_{1\le m\le M} |(Lx)_m| \le \|x\|_\infty \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|.$$

Therefore, $\|L\| \le \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$.

Next, we will prove that there exists $\hat{x} \in \mathbb{R}^N$ with $\|x\|_\infty = 1$ such that $\|L\hat{x}\|_\infty \ge \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$. Then

$$\|L\| = \sup_{\|z\|=1} \|Lz\|_\infty \ge \|L\hat{x}\|_\infty \ge \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$$

Therefore, $\|L\| = \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|$.

It remains to construct such $\hat{x}$. Suppose $\max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}| = \sum_{n=1}^{N} |a_{m_0 n}|$ for some $1 \le m_0 \le M$. Let

$$\hat{x}_n = \begin{cases} 1 & \text{if } a_{m_0,n} \ge 0 \\ -1 & \text{if } a_{m_0,n} < 0 \end{cases}.$$

Then $\|\hat{x}\|_\infty = 1$ and

$$\|L\hat{x}\|_\infty = \max_{1\le m\le M} |(L\hat{x})_m| \ge (L\hat{x})_{m_0} = \sum_{n=1}^{N} a_{m_0 n}\hat{x}_n = \sum_{n=1}^{N} |a_{m_0 n}| = \max_{1\le m\le M} \sum_{n=1}^{N} |a_{mn}|,$$

which completes the proof.

(2) & (3). Assignment 3 □

Example 2. *Let $X = C[a,b]$ with $\|\cdot\|_\infty$, where $-\infty < a < b < \infty$ and $K : [a,b]\times[a,b] \to \mathbb{R}$ be continuous. For each $u \in X$, define the integral operator*

$$Tu(x) := \int_a^b K(x,y)u(y)\,dy \quad \textit{for all} \;\; x \in [a,b].$$

From previous lectures, $T : C[a,b] \to C[a,b]$ is a continuous and a compact operator.
Moreover, T is linear (prove this!) and

$$\|T\| = \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Sketch of the proof. We will compute the operator norm of T.

- Step 1: Show that

$$\|T\| \le \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

  Let $u \in C[a,b]$ and $x \in [a,b]$. Then

$$|Tu(x)| = \left|\int_a^b K(x,y)u(y)\,dy\right| \le \int_a^b |K(x,y)|\,|(u(y)|\,dy \le \|u\|_\infty \int_a^b |K(x,y)|dy \le \|u\|_\infty \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

  So

$$\|Tu\|_\infty = \max_{a\le x\le b} |Tu(x)| \le \|u\|_\infty \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

  Therefore

$$\|T\| \le \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

- Step 2: (next lecture) For every $\frac{1}{4}\int_a^b |K(x,y)|dy > \varepsilon > 0$, construct an $u_\varepsilon \in C[a,b]$ with $\|u_\varepsilon\|_\infty \le 1$ such that

$$\|Tu_\varepsilon\|_\infty \ge \max_{a\le x\le b} \int_a^b |K(x,y)|dy - 4\varepsilon.$$

  Then

$$\|T\| = \sup_{u\in C[a,b], \|u\|_\infty \le 1} \|Tu\|_\infty \ge \|Tu_\varepsilon\|_\infty \ge \max_{a\le x\le b} \int_a^b |K(x,y)|dy - 4\varepsilon.$$

  Let $\varepsilon \to 0$, we have $\|T\| \ge \int_a^b |K(x,y)|dy$, which completes the proof.

□