Lecture 11: Bounded Linear Operator (cont'd). $B(X,Y)$. Dual Spaces

Example 2. *Let $X = C[a,b]$ with $\|\cdot\|_\infty$, where $-\infty < a < b < \infty$ and $K : [a,b]\times[a,b] \to \mathbb{R}$ be continuous. For each $u \in X$, define the integral operator*

$$Tu(x) := \int_a^b K(x,y)u(y)\,dy \quad \text{for all} \ \ x \in [a,b].$$

From previous lectures, $T : C[a,b] \to C[a,b]$ is a continuous and a compact operator.
Moreover, T is linear (prove this!) and

$$\|T\| = \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Sketch of the proof. We will compute the operator norm of T.

- Step 1: Show that

$$\|T\| \le \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Let $u \in C[a,b]$ and $x \in [a,b]$. Then

$$|Tu(x)| = \left|\int_a^b K(x,y)u(y)\,dy\right| \le \int_a^b |K(x,y)|\,|(u(y)|\,dy \le \|u\|_\infty \int_a^b |K(x,y)|dy \le \|u\|_\infty \max_{a\le x\le b}\int_a^b |K(x,y)|dy.$$

So

$$\|Tu\|_\infty = \max_{a\le x\le b} |Tu(x)| \le \|u\|_\infty \max_{a\le x\le b}\int_a^b |K(x,y)|dy.$$

Therefore

$$\|T\| \le \max_{a\le x\le b} \int_a^b |K(x,y)|dy.$$

Suppose $\max\limits_{a\le x\le b} \int_a^b |K(x,y)|dy = \int_a^b |K(x_0,y)|dy$ for some $x_0 \in [a,b]$. Since $K(x_0,y)$ is continuous on a compact set $[a,b]$, $K(x_0,y)$ is uniformly continuous on $[a,b]$. Therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|K(x_0,y_2) - K(x_0,y_1| \le \varepsilon \quad \text{for all} \ \ |y_1 - y_2| < \delta, \quad y_1, y_2 \in [a,b].$$

- Step 2: Now we will construct an $u_\varepsilon \in C[a,b]$ with $\|u_\varepsilon\|_\infty \le 1$ such that

$$Tu_\varepsilon(x_0) \ge \max_{a\le x\le b} \int_a^b |K(x,y)|dy - 4\varepsilon.$$

Let $A_\varepsilon = \{y \in [a,b] : |K(x_0,y)| \le \varepsilon\}$. Then A_ε is a closed and bounded subset in $\mathbb{R}$ (prove this). Therefore, there exists $y_1, \ldots, y_N \in A_\varepsilon$ such that

$$A_\varepsilon \subset \bigcup_{i=1}^{N} [y_i - \delta, y_i + \delta].$$

Let $V_\varepsilon = \left(\bigcup_{i=1}^{N} [y_i - \delta, y_i + \delta] \right) \cap [a,b]$ and $U_\varepsilon = [a,b] - V_\varepsilon$. Define a function on U_ε,

$$u_\varepsilon : U_\varepsilon \to \mathbb{R}, \quad u_\varepsilon(y) := \frac{K(x_0,y)}{|K(x_0,y)|}.$$

The function is well-defined and continuous since $|K(x_0,y)| > \varepsilon$ for all $y \in U_\varepsilon$ and $K(x_0,y)$ is continuous on $[a,b]$. Moreover, $|u_\varepsilon(y)| = 1$ for all $y \in U_\varepsilon$. Extend u_ε linearly, $u_\varepsilon : [a,b] \to \mathbb{R}$ so that $|u_\varepsilon(y)| \le 1$ for all $y \in [a,b]$.

Next, we will evaluate $\int_{V_\varepsilon} K(x_0,y) u_\varepsilon(y) dy$. For each $y \in V_\varepsilon$, we have $|y - y_i| \le \delta$ for some $y_i \in A_\varepsilon$, $i \in \{1, \ldots, N\}$. Therefore,

$$|K(x_0,y)| \le |K(x_0,y_i)| + |K(x_0,y) - |K(x_0,y_i)|| \le 2\varepsilon, \quad \text{for all } y \in V_\varepsilon,$$

and

$$\int_{V_\varepsilon} |K(x_0,y)| dy \le \int_{V_\varepsilon} 2\varepsilon dy \le 2(b-a)\varepsilon.$$

Also, since $|u_\varepsilon(y)| \le 1$ for all $y \in V_\varepsilon$, we have

$$K(x_0,y) u_\varepsilon(y) \ge -|K(x_0,y)|\,|u_\varepsilon(y)| = -|K(x_0,y) u_\varepsilon(y)| \ge -|K(x_0,y)|, \quad \text{for all } \ y \in V_\varepsilon.$$

Therefore,

$$\begin{aligned}
Tu_\varepsilon(x_0) &= \int_{U_\varepsilon} K(x_0,y) u_\varepsilon(y) dy + \int_{V_\varepsilon} K(x_0,y) u_\varepsilon(y) dy \\
&= \int_{U_\varepsilon} |K(x_0,y)| dy + \int_{V_\varepsilon} K(x_0,y) u_\varepsilon(y) dy \\
&\ge \int_{U_\varepsilon} |K(x_0,y)| dy - \int_{V_\varepsilon} |K(x_0,y)| dy \\
&\ge \int_a^b |K(x_0,y)| dy - 2 \int_{V_\varepsilon} |K(x_0,y)| dy \\
&\ge \int_a^b |K(x_0,y)| dy - 4(b-a)\varepsilon \\
&\ge \max_{a \le x \le b} \int_a^b |K(x,y)| dy - 4(b-a)\varepsilon.
\end{aligned}$$

Then

$$\|T\| = \sup_{u \in C[a,b], \|u\|_\infty \le 1} \|Tu\|_\infty \ge \|Tu_\varepsilon\|_\infty = \max_{x \in [a,b]} |Tu(x)| \ge Tu_\varepsilon(x_0) \ge \max_{a \le x \le b} \int_a^b |K(x,y)| dy - 4(b-a)\varepsilon.$$

Let $\varepsilon \to 0$, we have $\|T\| \ge \int_a^b |K(x,y)| dy$, which completes the proof.

□

Example 3. *Here we will show an example of a discontinuous linear operator (hence the operator is not bounded).*

Consider the differentiation operator $D = \frac{d}{dt} : X = C^1[0,1] \to Y = C[0,1]$, *where* $\|\cdot\|_\infty$ *are used for both spaces. The operator D is not continuous at 0. Here is a counter example. Consider a sequence* $\{f_n(t) = \frac{1}{n}\sin n\pi t\}_n \subset X$. *Then* $\|f_n\|_\infty = \frac{1}{n}$. *So* $\lim_{n\to\infty} \|f_n\|_\infty = 0$. *Therefore* $\lim_{n\to\infty} f_n = 0$. *On the other hand,* $Df_n = f_n' = \pi \cos n\pi t$. *So* $\|Df_n\| = \pi$ *for all* $n = 1, 2, \ldots$, *which means* $Df_n \not\to 0$.

Note: The differentiation operator is continuous (prove this) when Y is equipped with the $\|\cdot\|_\infty$ *norm and X is equipped with the following norm*

$$\|f\| := \max\{\|f\|_\infty,\ \|f'\|_\infty\}.$$

2.12 B(X,Y) and Dual Spaces

Definition 1. *Let X and Y be normed linear spaces. Define the following set*

$$B(X,Y) := \{L : X \to Y \quad \text{bounded linear operator}\}.$$

Denote $X^* = B(X,\mathbb{R})$ *(the dual space of X) and* $B(X) = B(X,X)$.

Theorem 1. *The set* $B(X,Y)$ *is a normed linear space with the operator norm.*

Proof. Exercise. □

Proposition 16. *Let X, Y and Z be normed linear spaces. If* $T \in B(X,Y)$ *and* $S \in B(Y,Z)$ *then* $ST \in B(X,Z)$ *and* $\|ST\| \le \|S\|\|T\|$.

Proof. For each $x \in X$, we have

$$\|ST(x)\| \le \|S\|\|T(x)\| \le \|S\|\|T\|\|x\|.$$

Therefore, ST is bounded and

$$\|ST\| = \sup_{\|x\|=1} \|ST(x)\| \le \sup_{\|x\|=1} \|S\|\|T\|\|x\| = \|S\|\|T\|.$$

□

Corollary 2. *Let X be a normed linear space. If* $T \in B(X)$, *then* $T^n \in B(X)$ *and* $\|T^n\| \le \|T\|^n$ *for all* $n = 1, 2, \ldots$.

Definition 2 (Convergence in Operator Norm)**.** *Let X and Y be normed linear spaces. A sequence* $\{T_n\} \subset B(X,Y)$ *is said to converge in operator norm to* $T \in B(X,Y)$ *if* $\|T_n - T\| \to 0$ *as* $n \to 0$.

Proposition 17. *Let X, Y and Z be normed linear spaces. If* $T_n, T \in B(X,Y)$ *and* $S_n, S \in B(Y,Z)$ *with* $T_n \to T$ *and* $S_n \to S$ *as* $n \to \infty$, *then* $S_nT_n \to ST \in B(X,Z)$.

Proof. We have

$$\|S_nT_n - ST\| \le \|S_nT_n - S_nT\| + \|S_nT - ST\| \le \|S_n\|\|T_n - T\| + \|S_n - S\|\|T\|.$$

Since $\lim_{n\to\infty} S_n = S$ and the norm is a continuous function, $\lim_{n\to\infty}\|S_n\| = \|S\|$. We also have $\lim_{n\to\infty}\|S_n - S\| = 0$ and $\lim_{n\to\infty}\|T_n - T\| = 0$. Therefore,

$$0 \le \lim_{n\to\infty}\|S_nT_n - ST\| \le \|S\|0 + 0\|T\| = 0,$$

So $\lim_{n\to\infty}\|S_nT_n - ST\| = 0$ and $\lim_{n\to\infty} S_nT_n = ST$. □

Theorem 2. *Let X be a normed linear space and Y be a Banach space. Then $B(X,Y)$ is a Banach space. In particular, X^* is a Banach space.*

Proof. Let $\{T_n\} \subset B(X,Y)$ be a Cauchy sequence in $B(X,Y)$. Given $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $\|T_n - T_m\| < \varepsilon$ for all $n, m \ge N_\varepsilon$.

- Step 1: Construct the limit pointwise. Indeed, for each $x \in X$ and $n, m > N_\varepsilon$, we have

$$\|T_n(x) - T_m(x)\| \le \|T_n - T_m\|\|x\| < \varepsilon\|x\|. \tag{10}$$

  Therefore, for each $x \in X$, the sequence $\{T_n(x)\}_n$ is a Cauchy sequence in Y. Since Y is Banach, the sequence $\{T_n(x)\}_n$ converges. Denote $T(x) := \lim_{n\to\infty} T_n(x)$. We have defined a function $T : X \to Y$ such that for each $x \in X$, $T(x) := \lim_{n\to\infty} T_n(x)$.

- Step 2: Show that T is linear. Indeed, let $c_1, c_2 \in \mathbb{K}$ and $x_1, x_2 \in X$. For each $n = 1, 2, \ldots$, T_n is linear, so

$$T_n(c_1x_1 + c_2x_2) = c_1T_n(x_1) + c_2T_n(x_2).$$

  Letting $n \to \infty$, we have

$$T(c_1x_1 + c_2x_2) = c_1T(x_1) + c_2T(x_2),$$

  because of the construction of T.

- Step 3: We will show that $T_n - T \in B(X,Y)$ for all n sufficiently large, $T \in B(X,Y)$, and $\|T_n - T\| \to 0$ as $n \to \infty$. From (10), letting $m \to \infty$ and keeping everything else, we get

$$\|T_n(x) - T(x)\| \le \varepsilon\|x\|, \quad \text{for all } \ x \in X \quad \text{and for all } n \ge N_\varepsilon.$$

  Therefore for every $n \ge N_\varepsilon$, $T_n - T \in B(X,Y)$ and $\|T_n - T\| \le \varepsilon$. Therefore,

$$T = T_{N_\varepsilon} - (T_{N_\varepsilon} - T) \in B(X,Y),$$

  and $T_n \to T$ as $n \to \infty$.

□