Lecture 14: Applications of Hahn-Banach Theorems. Hilbert Spaces.

Below is another version of the Hahn-Banach theorem for normed spaces.

Theorem 3 (The Hahn-Banach Theorem for normed spaces)**.** *Let X_0 be a subspace of a normed space X over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $f_0 \in X_0^*$. Then there is a linear functional $f : X \to \mathbb{R}$ such that*

$$f|_{X_0} = f_0 \quad \text{and} \quad \|f\| = \|f_0\|.$$

Proof. In class (use previous theorem with $|f_0(x_0)| \leq \|f_0\|\|x_0\|$ for all $x_0 \in X_0$. Prove that the linear functional extension also preserves the norm, that is, $\|f\| = \|f_0\|$. □

Proposition 21 (Supporting Functional)**.** *Let X be a normed space. For every $a \in X$, $a \neq 0$, there exists $f \in X^*$ such that*

$$\|f\| = 1, f(a) = \|a\|.$$

The function f is called the supporting functional of a.

Proof. Define $f_0 : X_0 = span(a) \to \mathbb{R}$, $f_0(ta) = t\|a\|$. Obviously, f_0 is a linear functional on X_0 and $|f_0(u)| = \|u\|$ for all $u \in X_0$. Applying the Hahn-Banach theorem, there exists $f \in X^*$ such that $f(a) = f_0(a) = \|a\|$ and $|f(u)| \leq \|u\|$ for all $u \in X$. So $\|f\| \leq 1$.

On the other hand, since $|f(a)| = \|a\|$, $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \geq \frac{|f(a)|}{\|a\|} = 1$. Therefore, $\|f\| = 1$. □

Example 1. *For $(\mathbb{R}^n, \|\cdot\|_2), a \in \mathbb{R}^n, a \neq 0$, a supporting functional is $f(x) = \frac{x \cdot a}{\|a\|}$.*

Proof. Exercise: Verify that $f \in (\mathbb{R}^n)^*$, $\|f\| = 1$ and $f(a) = \|a\|$. □

Recall: For a linear functional $f \in X^*$, where X is a normed linear space, the operator norm of f is

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|z\| \leq 1} |f(z)|.$$

In general, we may not be able to replace the supremum above by the maximal. That is, there exist a normed linear space $X$and a linear functional $f \in X^*$ such that $|f(x)| < \|f\|\|x\|$ for all $x \in X$, $x \neq 0$. Find an example. Note that the normed linear space in this example should be infinite dimensional. (Explain!) However, every vector $x \in X$ does attain its norm on some functional $f \in X^*$.

Corollary 3. *Let X be a normed linear space over $\mathbb{K}$. Then for all $a \in X$, we have*

$$\|a\| = \max_{g \in X^*, \|g\| \leq 1} |g(a)|.$$

Proof. If $a = 0$, $g(a) = 0$ for all $g \in X^*$. The statement holds.
If $a \neq 0$, from Proposition 21, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(a) = \|a\|$. So

$$\sup_{g \in X^*, \|g\| \leq 1} |g(a)| \geq f(a) = \|a\|.$$

On the other hand, we have

$$\sup_{g\in X^*,\|g\|\le 1}|g(a)|\le \sup_{g\in X^*,\|g\|\le 1}\|g\|\|a\|\le\|a\|.$$

Therefore,

$$f(a)=\|a\|=\sup_{g\in X^*,\|g\|\le 1}|g(a)|.$$

Note that $f\in X^*$ and $\|f\|=1$. That means

$$\|a\|=\max_{g\in X^*,\|g\|\le 1}|g(a)|.$$

□

The following theorem is useful to prove certain operator is a contraction mapping.

Theorem 4 (Generalized Mean Value Theorem)**.** *Let $F:X\to Y$ be an operator between normed linear spaces X and Y and $a,b\in X$, $a\neq b$. Suppose F is continuous on the closed segment $\{a+t(b-a),0\le t\le 1\}$ and Fréchet differentiable on the open segment $\{a+t(b-a),0<t<1\}$. Then*

$$\|F(b)-F(a)\|\le\sup_{0<t<1}\|DF(a+t(b-a))\|\,\|b-a\|.$$

Sketch of the proof. Let $g\in Y^*$ such that $g(F(b)-F(a))=\|F(b)-F(a)\|$ and $\|g\|=1$. Consider $\Phi:[0,1]\to\mathbb{R}$

$$\Phi(t)=g(F(a+t(b-a)),\ t\in[0,1].$$

Since $g\in Y^*$, $Dg(y)=g$ for all $y\in Y$. By the chain rule, the Fréchet derivative of Φ at $t\in(0,1)$ is

$$D\Phi(t):[0,1]\to X\xrightarrow{DF}Y\xrightarrow{Dg}\mathbb{R}$$

$$D\Phi(t)=Dg(F(a+t(b-a)))\Big[DF(a+t(b-a))(b-a)\Big]=g\Big[DF(a+t(b-a))\,(b-a)\Big].$$

By the mean value theorem,

$$\begin{aligned}\Phi(1)-\Phi(0)&=D\Phi(t_0)\quad\text{for some}\quad t_0\in(0,1).\\ g(F(b))-g(F(a))&=g\Big[DF(a+t_0(b-a))\,(b-a)\\ g(F(b)-F(a))&\le\|g\|\Big\|DF(a+t_0(b-a))\Big\|\|b-a\|\\ \|F(b)-F(a)\|&\le\sup_{0<t<1}\|DF(a+t(b-a))\|\|b-a\|.\end{aligned}$$

□

Theorem 5 (Separating a point from a convex set)**.** *Let K be an open convex subset of a normed space X and consider a point $x_0\notin K$. Then there exists a linear functional $f\in X^*$, $f\neq 0$ such that*

$$f(x)\le f(x_0)\quad\forall x\in K.$$

Proof. Assignment 3. □

Theorem 6 (Separating Hyperplane Theorem)**.** *Let A and B be disjoint, nonempty, convex subsets of a normed linear space X.*

1. *If A is open, then there exists a functional $f \in X^*$ and $c \in \mathbb{R}$ such that $f(a) \leq c \leq f(b)$ for all $a \in A$, $b \in B$.*
2. *If both A and B are open, then there exists a functional $f \in X^*$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$ for all $a \in A$, $b \in B$.*
3. *If A is compact and B is closed, then there is $f \in X^*$ and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$ for all $a \in A$, $b \in B$.*

Proof. Assignment 3. □

3 Inner Product Spaces

Hilbert spaces are an important and simplest class of Banach spaces, where the concept of orthogonality is defined. With a view to applications, the most important Hilbert spaces are the real and complex Lebesgue spaces $L_2(G)$ and the related Sobolev spaces $W_2^1(G)$ and $\mathring{W}_2^1(G)$, where $G \subset \mathbb{K}^N$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

3.1 Inner Product Spaces

In this chapter, the scalar field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 1 (Inner Product)**.** *Let X be a vector space over $\mathbb{K}$. An inner product on X is a function $\langle , \rangle X \times X \to \mathbb{K}$ that satisfies*

- $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \quad \forall\, x, y, z \in X$
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \quad \forall\, x, y \in X, \quad \alpha \in \mathbb{K}$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}, \quad \forall\, x, y \in X$
- $\langle x, x\rangle \geq 0, \quad \langle x, x\rangle = 0 \quad \textit{iff} \quad x = 0$

Then $(X, \langle , \rangle)$ is an inner product space.

Note: If $(X, \langle , \rangle)$ is an inner product space, then

$$\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle, \quad \forall\, x, y \in X, \quad \alpha \in \mathbb{K}$$

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle, \quad \forall\, x, y, z \in X, \forall \alpha, \beta \in \mathbb{K}.$$

$$\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle, \quad \forall\, x, y, z \in X, \forall \alpha, \beta \in \mathbb{K}.$$

Definition 2 (Orthogonality)**.** *Let $(X, \langle , \rangle)$ be an inner product space and $x, y \in X$. Then x is called orthogonal to y if $\langle x, y\rangle = 0$.*

Theorem 1 (Cauchy-Schwarz Inequality)**.** *Let X be an inner product space. Then every two vectors $x, y \in X$ satisfy*

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}.$$

Sketch of the proof. The inequality is true if $x = 0$ or $y = 0$. For fixed $x \neq 0$ and $y \neq 0$, we have

$$\langle x - \alpha y, x - \alpha y\rangle \geq 0 \quad \forall \alpha \in \mathbb{K}$$

$$\langle x, x\rangle - \overline{\alpha}\langle x, y\rangle - \alpha\left(\langle y, x\rangle - \overline{\alpha}\langle y, y\rangle\right) \geq 0$$

Choose $\overline{\alpha} = \dfrac{\langle y, x\rangle}{\langle y, y\rangle}$. Then simplifying the left hand side, we get the result. □

Corollary 4. *Let X be an inner product space. Then X is a normed space with the norm defined as*

$$\|x\| := \langle x, x\rangle^{1/2}.$$

Proof. Exercise. □

Definition 3. *Let $(X, \langle , \rangle)$ be an inner product space. X is called a Hilbert space if X is a Banach space with the normed induced by the inner product.*

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed space. The norm $\|\cdot\|$ is generated by an inner product if and only if the parallelogram equality holds:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad \forall x, y \in X$$

Sketch of the proof. ($\Rightarrow$) Suppose $\|x\| = \sqrt{\langle x, x\rangle}$ for some inner product $\langle , \rangle$ on X. Then verify that

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \quad \forall x, y \in X$$

($\Leftarrow$) Suppose the norm $\|\cdot\|$ satisfies the parallelogram equality. For $x, y \in X$, define

$$\langle x, y\rangle := \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right).$$

We will verify that $\langle , \rangle$ is an inner product on X (prove this). □

Remark 3. *Not all normed spaces are inner product spaces. For example, the space ℓ^p with $p \neq 2$ and the space $(C[a, b], \|\cdot\|_\infty)$.*